# Sobolev capacities $\quad \boxed{\infty > p > 1,\ n \geq 2}$

Assume $\Omega \subset \mathbb{R}^n$ is an open subset.

For any compact $K \subset \Omega$ we shall write

$$W(K,\Omega) = \{u \in C_0^\infty(\Omega) : \ u \geq 1 \text{ on } K\}.$$

Definition of $p$-capacity of $K \subset \Omega$, $K$ compact

$$\mathrm{Cap}_p(K,\Omega) = \inf_{u \in W(K,\Omega)} \int_\Omega |\nabla u|^p$$

Then, for any open $\mathcal{U} \subset \Omega$ we set

$$\mathrm{Cap}_p(\mathcal{U},\Omega) = \sup_{\substack{K \subset \mathcal{U} \\ K \text{ compact}}} \mathrm{Cap}_p(K,\Omega),$$

and for arbitrary $E \subset \Omega$

$$\mathrm{Cap}_p(E,\Omega) = \inf_{\substack{\mathcal{U} \supset E \\ \mathcal{U} \text{ open}}} \mathrm{Cap}_p(E,\Omega).$$

Note that the capacity of a compact set $E = K$ is defined ambiguously: it is not clear, whether the last method returns the same result as the original one. The same applies to $\mathcal{U}$ open. This will be resolved in a moment.

Exercise: The capacity could be equivalently defined with $W_0(K,\Omega)$ in place of $W(K,\Omega)$, where $W_0(K,\Omega) = \{u \in W_0^{1,p}(\Omega) \cap C(\Omega) : \ u \geq 1 \text{ on } K\}$.

I shall give a list of important properties of $\mathrm{Cap}_p$ in a moment; now a crucial definition:

Def: $E \subset \mathbb{R}^n$ is of $p$-capacity zero if for any ~~$\Omega \subset \mathbb{R}^n$, open,~~ open $\Omega \subset \mathbb{R}^n$
$\mathrm{Cap}_p(E \cap \Omega, \Omega) = 0$.

We shall exploit this definition ~~in a~~ shortly.

A property holds $p$-quasieverywhere if it holds everywhere but on a set of $p$-capacity arbitrarily small.
E.g. $f$ is $p$-quasicontinuous on $\Omega$ if for any $\varepsilon > 0$ there exists $X_\varepsilon \subset \Omega$, $\mathrm{Cap}_p(X_\varepsilon, \Omega) < \varepsilon$ such that $f|_{\Omega \setminus X_\varepsilon}$ is continuous.

Theorem: (Lusin type theorem for $W^{1,p}$ functions)
Any $W^{1,p}(\Omega)$ function is $p$-quasicontinuous.

Motivation

Theorem (Egorov type theorem for $W^{1,p}$ functions)
A sequence $f_n$ of continuous functions, convergent to $f$ in $W^{1,p}(\Omega)$, has a subsequence $p$-quasiuniformly convergent to $f$.

## Properties of $\mathrm{Cap}_p$

**Theorem** $E_i \subset \Omega$

Monotonicity

① If $E_1 \subset E_2$, then $\mathrm{Cap}_p(E_1, \Omega) \leq \mathrm{Cap}_p(E_2, \Omega)$

② If $\Omega_1 \subset \Omega_2$, both sets are open, then $\mathrm{Cap}_p(E, \Omega_1) \geq \mathrm{Cap}_p(E, \Omega_2)$

③ for compact $K_1, K_2 \subset \Omega$

$$\mathrm{Cap}_p(K_1 \cup K_2, \Omega) + \mathrm{Cap}_p(K_1 \cap K_2, \Omega) \leq$$
$$\leq \mathrm{Cap}_p(K_1, \Omega) + \mathrm{Cap}_p(K_2, \Omega).$$

(poor man's inclusion-exclusion formula).

④ if $K_i \subset \Omega$ are compact, $K_1 \supset K_2 \supset \ldots$, $K = \bigcap_i K_i$, then $\mathrm{Cap}_p(K, \Omega) = \lim_{i \to \infty} \mathrm{Cap}_p(K_i, \Omega)$

⑤ if $E_1 \subset E_2 \subset \ldots \subset E = \bigcup_i E_i$, then
$$\mathrm{Cap}_p(E, \Omega) = \lim_{i \to \infty} \mathrm{Cap}_p(E_i, \Omega)$$

⑥ if $E = \bigcup_i E_i$, then $\mathrm{Cap}_p(E, \Omega) \leq \sum_i \mathrm{Cap}_p(E_i, \Omega)$

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① & ② are obvious by the definition.

To prove ③, we need the following

**Exercise** If $u \in W^{1,p}(\Omega)$, then $\underbrace{\max\{u, 0\}}_{= u^+} \in W^{1,p}(\Omega)$

$$\nabla u^+ = \begin{cases} \nabla u & u > 0 \\ 0 & u \leq 0 \end{cases}.$$

**Corollary** (Another exercise): If $u, v \in W^{1,p}(\Omega)$, then $\max\{u, v\}$ and $\min\{u, v\} \in W^{1,p}(\Omega)$

$$\nabla \max\{u, v\} = \begin{cases} \nabla u & u \geq v \\ \nabla v & v \geq u \end{cases}.$$

Proof of 3 If $u_1 \in W(K_1, \Omega)$
$u_2 \in W(K_2, \Omega)$
then $\max\{u_1, u_2\} \in W_0(K_1 \cup K_2, \Omega)$
$\min\{u_1, u_2\} \in W_0(K_1 \cap K_2, \Omega)$

and

$$\mathrm{Cap}_p(K_1 \cup K_2, \Omega) + \mathrm{Cap}_p(K_1 \cap K_2, \Omega) \leq$$

$$\leq \int_\Omega \cancel{\max} |\nabla \max\{u_1, u_2\}|^p + \int_\Omega |\nabla \min\{u_1, u_2\}|^p$$

$$= \int_\Omega |\nabla u_1|^p + \int_\Omega |\nabla u_2|^p$$

and this holds for any $u_1, u_2$.
This proves the desired inequality.

4 By 1, $(\mathrm{Cap}_p(K_i, \Omega))_i$ is a non-increasing sequence $\Rightarrow$ it converges.

Let $b = \lim_{i \to \infty} \mathrm{Cap}_p(K_i, \Omega)$.

For any $i$, $\mathrm{Cap}_p(K, \Omega) \leq \mathrm{Cap}_p(K_i, \Omega)$

$\Rightarrow \mathrm{Cap}_p(K, \Omega) \leq b$.

Fix $\varepsilon > 0$ and take $u \in W(K, \Omega)$ s.t

$$\int_\Omega |\nabla u|^p < \mathrm{Cap}_p(K, \Omega) + \varepsilon.$$

$u \in C_0^\infty$, thus $A_\varepsilon = \{u \geq 1 - \varepsilon\}$ is a compact subset of $\Omega$, and $K \subset \operatorname{int} A_\varepsilon$.
Therefore for large $i$ $K_i \subset A_\varepsilon$.

$$b \leq \mathrm{Cap}_p(K_i, \Omega) \leq \mathrm{Cap}_p(A_\varepsilon, \Omega) = (*)$$

Note that $\tilde{u} = \frac{u}{1-\varepsilon} \in W(A_\varepsilon, \Omega)$, thus

$$b \le (*) \le \int_\Omega |\nabla \tilde{u}|^p = (1-\varepsilon)^{-p} \int_\Omega |\nabla u|^p$$

$$< \frac{\mathrm{Cap}_p(K,\Omega) + \varepsilon}{(1-\varepsilon)^p}$$

Taking $\varepsilon \to 0$ we see that

$$b \le \mathrm{Cap}_p(K,\Omega).$$

(5) and (6) are far more technical and I will skip these proofs.

Any set-function defined on $2^\Omega$ that enjoys properties (1), (4) and (5) is called a Choquet capacity. The following useful theorem holds:

<u>Theorem</u> (Choquet): For any Borel $E \subset \Omega$

$$\mathrm{Cap}_p(E,\Omega) = \sup_{K \subset E} \mathrm{Cap}_p(K,\Omega)$$

On the exercise course we shall calculate the capacity of a point and a ball relative to a larger ball.

Now we shall concentrate on sets of capacity zero.

Motivation for studying sets of p-capacity zero:

<u>Theorem</u>: Suppose $E \subset \Omega$ is a closed (in a subspace topology of $\Omega$) subset of $\Omega$. Then

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus E)$$

if and only if $E$ is of p-capacity zero.

(i.e. every function in $W_0^{1,p}(\Omega)$ can be approximated in $W^{1,p}$-norm by a sequence of smooth functions with supports in $\Omega \setminus E$).

Thus null-p-capacity-sets are negligible in the Sobolev context.

<u>Observation</u>: Every set of p-capacity zero is contained in a Borel (even $G_\delta$) set ~~of the~~ with the same property.
(Proof – exercise).

By property (6), any countable union of sets of p-cap. zero is again of p-cap. zero.

Our definition of p-cap. zero engages all open $\Omega \subset \mathbb{R}^n$; for bounded $E$ it is enough to find one:

<u>Lemma</u>: If $E$ is bounded and for some bounded and open $\Omega \supset E$ we have $\mathrm{Cap}_p(E, \Omega) = 0$ then $E$ is of p-cap. zero.

Proof: Take any open $\tilde{\Omega}$. We want to prove that
$$\mathrm{Cap}_p(\tilde{E} \cap \tilde{\Omega}, \tilde{\Omega}) = 0.$$
By Choquet's theorem it is enough to prove that for any $K \subset \tilde{E} \cap \tilde{\Omega}$, $K$ compact
$$\mathrm{Cap}_p(K, \tilde{\Omega}) = 0.$$

Take $u \in W(K, \Omega)$ and $v \in W(K, \tilde{\Omega})$.
Then $uv \in W(K, \tilde{\Omega})$

$$\mathrm{Cap}_p(K, \tilde{\Omega}) \leq \int_{\tilde{\Omega}} |\nabla(u \cdot v)|^p = \int_{\Omega} |\nabla u \cdot v + u \cdot \nabla v|^p$$

we can put $\Omega$ here
$\mathrm{supp}\, uv \subset \Omega \cap \tilde{\Omega}$

$$\leq \int_{\Omega} 2^p \left( \max_{\Omega} |\nabla v|^p \cdot |u|^p + \max_{\Omega} |v|^p \cdot |\nabla u|^p \right)$$

$$\leq C(p, v) \left( \int_{\Omega} |u|^p + \int_{\Omega} |\nabla u|^p \right)$$

$$\leq C(p, v, \Omega) \int_{\Omega} |\nabla u|^p$$

Poincare ineq. ($u \in C_0^\infty(\Omega)$)

this can be made arbitrarily small by proper choice of $u$.

Thus $\mathrm{Cap}_p(K, \tilde{\Omega}) = 0$. $\square$

Theorem: If $E$ is of $p$-capacity zero, then $|E|=0$.

Proof: Take any bounded $\Omega$.

$$\mathrm{Cap}_p(E\cap\Omega,\Omega)=0 \Rightarrow \exists_{\mathcal{U}\text{ open}} : E\cap\Omega\subset\mathcal{U}\subset\Omega,$$

s.t. $\mathrm{Cap}_p\,\mathcal{U}<\varepsilon$.

Fix $K\subset\mathcal{U}$ compact, then $\mathrm{Cap}_p(K,\Omega)<\varepsilon$ as well.

We can choose $\varphi\in W(K,\Omega)$ such that

$$\int_\Omega|\nabla\varphi|^p\le \mathrm{Cap}_p(K,\Omega)+\varepsilon<2\varepsilon.$$

Then $|K|\le\int_\Omega|\varphi|^p\underset{\text{Poincaré ineq.}}{\le} c\int_\Omega|\nabla\varphi|^p<2c\varepsilon$

Thus $|\mathcal{U}|=\sup_{K\subset\mathcal{U}}|K|\le 2c\varepsilon\Rightarrow|E|=0$. □.

A much stronger theorem holds:

Theorem: Suppose $1<p\le n$ and $E\subset\mathbb{R}^n$ is of $p$-capacity zero. Then the Hausdorff dimension of $E$ is at most $n-p$.

and almost converse: for $p\in(1,n)$ if $\mathcal{H}^{n-p}(E)<\infty$, then $E$ is of $p$-capacity zero.

Proof is not beyond our abilities, but beyond our time resources.

<u>Proof</u> that $W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus E)$ iff $E$ is of p-cap. zero. (for bounded $\Omega$).

If $E$ is of p-capacity zero, then ~~there~~ ~~exists~~ $C_p(\Omega \cap E, E) = 0 \Rightarrow$ there exists a sequence of ~~$C_0^\infty(\Omega)$-functions $\varphi$~~ functions $u_j \in W(\Omega \cap E, E)$ such that

$$\int_\Omega |\nabla u_j|^p \xrightarrow[j \to \infty]{} 0 .$$

Take any $\varphi \in C_0^\infty(\Omega)$, then $(1-u_j)\varphi \in W_0^{1,p}(\Omega \setminus E)$;

$(1-u_j)\varphi \to \varphi$ in $W^{1,p}(\Omega \setminus E)$

thus $W_0^{1,p}(\Omega \setminus E) \supseteq W_0^{1,p}(\Omega)$

The reverse inclusion is trivial.

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Let now $K \subset E$ be compact. We assume $W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus E)$ and need to prove that $\mathrm{Cap}_p(K, \Omega) = 0$.

Take any $\varphi \in W(K, \Omega)$. Since $\varphi \in C_0^\infty(\Omega) \subseteq W_0^{1,p}(\Omega)$ we can approximate $\varphi$ with a sequence $\varphi_j \in C_0^\infty(\Omega \setminus E)$, $\varphi_j \xrightarrow[W^{1,p}]{} \varphi$.

Then $\varphi - \varphi_j \in W(K, \Omega)$,

$$\mathrm{Cap}_p(K, \Omega) \leq \int_\Omega |\nabla \varphi - \nabla \varphi_j|^p \xrightarrow[j \to \infty]{} 0 \qquad \square .$$

A similar theorem, but only one-way holds for $W^{1,p}(\Omega)$:

<u>Theorem</u>: If $E \subset \Omega$ is closed (rel to $\Omega$) and of p-cap. zero, then $W^{1,p}(\Omega) = W^{1,p}(\Omega \setminus E)$.

<u>Proof</u>: Exercise.

<u>Theorem</u>: $W^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ iff $\mathbb{R}^n \setminus \Omega$ is of p-cap. zero

<u>Proof</u>: $\Leftarrow$

$$W^{1,p}(\Omega) = W^{1,p}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega)) = W^{1,p}(\mathbb{R}^n)$$
$$= W_0^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega)) = W_0^{1,p}(\Omega).$$

$\Rightarrow$

$$W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n) \subset W^{1,p}(\Omega) =$$
$$= W_0^{1,p}(\Omega) \subset W_0^{1,p}(\mathbb{R}^n)$$

thus $W_0^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\Omega) = W_0^{1,p}(\mathbb{R}^n \setminus (\mathbb{R}^n \setminus \Omega))$
which implies $\mathbb{R}^n \setminus \Omega$ is of p-cap. zero.

<u>Theorem</u>: If $E$ is of p-cap zero, then $\mathbb{R}^n \setminus E$ is connected.

Exercise $\quad Cap_p(\{x_0\}, B(x_0,1)) \;?\quad x_0 \in \mathbb{R}^n$

We can of course assume $x_0 = 0$.

Take radial function: $f(x) = \begin{cases} 0 & |x| \geq 1 \\ 1-|x| & |x|<1 \end{cases}$

This is a Lipschitz, thus a $W^{1,p}$ function for any $p$.

$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right)$.

For any $\varepsilon \in (0,1)$ $Cap_p(\{0\}, B(x_0,1)) \leq \int_{B(0,1)} |\nabla f_\varepsilon|^p = (*)$

$\nabla f_\varepsilon = \nabla f\left(\frac{x}{\varepsilon}\right)\cdot\frac{1}{\varepsilon}$, $|\nabla f_\varepsilon|^p = \frac{1}{\varepsilon^p}|\nabla f(x/\varepsilon)|^p$

$(*) = \int_{\mathbb{R}^n} |\nabla f(x/\varepsilon)|^p \cdot \frac{1}{\varepsilon^p}\cdot dx = \int_{\mathbb{R}^n} |\nabla f(y)|^p \frac{1}{\varepsilon^p}\varepsilon^n dy =$

$= \varepsilon^{n-p}\|\nabla f\|_p^p$

If $n>p$, this estimate goes to $0$ as $\varepsilon \to 0$.

$\Rightarrow$ for $n>p$ $\quad Cap_p(\{0\}, B(0,1)) = 0$.

For $n<p$ we know that $W^{1,p} \subset L^\infty$ and if $f \in W(\{0\}, B(0,1))$, then

$1 \leq \|f\|_\infty \leq \|f\|_{W^{1,p}} \underset{\uparrow}{\leq} C\cdot\|\nabla f\|_{L^p}$

Poincaré ineq.

which proves that $Cap(\{0\}, B(0,1)) \geq \frac{1}{C}$.

How about $p=n$?

$W^{1,n}(\mathbb{R}^n)$ does contain unbounded functions, (with c. supp) e.g. $f(x) = \begin{cases} \log \log \frac{1}{|x|} & |x| < \frac{1}{e} \\ 0 & |x| \geq \frac{1}{e} \end{cases}$

~~f~~ By convergence of the improper integral

$$\int_{B(0,r)} |\nabla f|^p \xrightarrow[r \to 0]{} 0 .$$

Let ~~f~~ $\varphi_n(x) = \cancel{\min\{f} \max\{\min\{f(x) - n, 1\}, 0\}$

$$\nabla \varphi_n(x) = \begin{cases} \nabla f(x) & \text{if } f(x) \in (n, n+1) \\ 0 & \text{elsewhere.} \end{cases}$$

~~$\int |\nabla$~~ $\varphi_n \in W_0(\{0,\}, B(0,1))$

$$\mathrm{Cap}_p(\{x_0\}, B(0,1)) \leq \int_{B(0,1)} |\nabla \varphi_n|^p = \int_{B(0,e^{-e^n}) \setminus B(0,e^{-e^{n+1}})} |\nabla f|^n \leq$$

$$\leq \int_{B(0,e^{-e^n})} |\nabla f|^n \xrightarrow[n \to \infty]{} 0 .$$

$\log \log 1/|x| = n$

$\log \frac{1}{|x|} = e^n$

$\frac{1}{|x|} = e^{e^n}$

$|x| = e^{-e^n}$

for $0 \le r \le R \le \infty$

Exercise: Prove that $Cap_p(\overline{B}(x_0,r), B(x_0,R)) =$

$$= \begin{cases} n\omega_n \left(\frac{|n-p|}{p-1}\right)^{p-1} \left|R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right|^{1-p} & p \neq n \\ n\omega_n \left(\log R/r\right)^{1-n} & p = n \end{cases}$$

Proof: Suppose $u \in W(\overline{B}(x_0,r), B(x_0,R))$, i.e. $u \in C_0^\infty(B(x_0,R))$, $u \geq 1$ on $\overline{B}(x_0,r)$.

$X_0 = 0$

Then for any $z \in S^{n-1}$

$$\int_r^R \frac{d}{dt}[u(tz)]\,dt = -1, \quad \text{thus}$$

$$1 \leq \int_r^R \left|\frac{d}{dt}[u(tz)]\right| dt = \int_r^R |\langle \nabla u(tz), z\rangle|\,dt$$

$$\leq \int_r^R |\nabla u(tz)|\,dt = \int_r^R |\nabla u(tz)|\, t^{\frac{n-1}{p}} \cdot t^{\frac{1-n}{p}}\,dt$$

$$\underset{\text{Hölder}}{\leq} \left(\int_r^R |\nabla u(tz)|^p t^{n-1}\,dt\right)^{1/p} \cdot \underbrace{\left(\int_r^R t^{\frac{1-n}{p}\cdot\frac{p}{p-1}}\,dt\right)^{\frac{p-1}{p}}}_{A(p,n,r,R)}$$

For $p \neq n$

$$A(p,n,r,R) = \left(\int_r^R t^{\frac{1-n}{p-1}}\,dt\right)^{\frac{p-1}{p}} =$$

$$= \left(\frac{p-1}{p-n} t^{\frac{p-n}{p-1}}\Big|_r^R\right)^{\frac{p-1}{p}} = \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}} \left(R^{\frac{p-n}{p-1}} - r^{\frac{p-n}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\left(\ln t\Big|_r^R\right)^{\frac{p-1}{p}} = A \qquad p = n$$

We have $\quad 1 \leq A^p \int_r^R |\nabla u(tz)|^p t^{n-1}\,dt$

integrating this over all $z \in S^{n-1}$ we get

$$n\omega_n = |S^{n-1}| \leq A^p \int_{S^{n-1}} \int_r^R |\nabla u(tz)|^p t^{n-1}\,dt\,dz$$

$$= A^p \int_{S^{n-1}} \int_0^R |\nabla u(tz)|^p t^{n-1}\,dt\,dz = A^p \int_{B(x_0,R)} |\nabla u(x)|^p\,dx$$

$(|\nabla u| = 0$ on $B(x_0,r))$

~~therefore~~ $\cancel{\mathrm{Cap}_p(\overline{B}(x_0,r), B(x_0,R)) \leq}$

$$\cancel{\leq \int_{B(x_0,R)} |\nabla u|^p \leq}$$

for any $u \in W(\overline{B}(x_0,r), B(x_0,R))$

$$\int_{B(x_0,R)} |\nabla u|^p \geq A^{-p} n\omega_n$$

$$\Rightarrow \mathrm{Cap}_p(\overline{B}(x_0,r), B(x_0,R)) \geq$$

$$\geq n\omega_n A^{-p}.$$

Reverse inequality: for

$$u(x) = \begin{cases} \int_{\cancel{|x-x_0|}\,|x|}^{R} t^{\frac{1-n}{p-1}}\,dt \Big/ \int_r^R t^{\frac{1-n}{p-1}}\,dt & \text{on } B(0,R)\setminus \overline{B}(0,r) \\ 1 & \text{on } B(0,r) \\ 0 & \text{outside } B(0,R) \end{cases}$$

is in $W(\overline{B}(0,r), B(0,R))$ and

$$\int_{B(0,R)} |\nabla u|^p = n\omega_n A^{-p}$$