Sobolev spaces

The Sobolev space $W^{k,p}(\Omega)$ is the space of all these distributions T on Ω that are, together with all their derivatives up to order k, represented by ~~L^p~~ functions in $L^p(\Omega)$

$$W^{k,p}(\Omega) = \{T \in \mathcal{D}'(\Omega) : \forall_{\alpha : |\alpha| \leq k} \; D^\alpha T = T_f \text{ for some } f \in L^p\}$$

Problem: Do we indeed need to control all the derivatives?

~~Theorem:~~

Def: Beppo-Levi space $BL^{k,p}(\Omega)$ is the space of all the distributions such that their derivatives of order k are in $L^p(\Omega)$:

$$BL^{k,p}(\Omega) = \{T \in \mathcal{D}'(\Omega) : \forall_{\substack{\alpha \\ |\alpha| = k}} \; D^\alpha T = T_f \text{ for some } f \in L^p\}.$$

From now on I shall identify functions in L^1_{loc} with distributions they define

<u>Theorem</u>: Let $T \in \mathcal{D}'(\Omega)$ be such that for any multiindex α with $|\alpha| = k$ $D^\alpha T \in L^p(\Omega)$. Then $T \in L^p_{loc}(\Omega)$.

<u>Sketch of a proof</u>

Choose $\mathcal{U} \subset V \subset \Omega \subset \mathbb{R}^n$ open sets, ~~contai~~ such that $\mathrm{cl}\,\mathcal{U} \Subset V$, $\mathrm{cl}\,V \Subset \Omega$.

Let $\varphi \in \mathcal{D}(\Omega)$, $\varphi \equiv 1$ on V

$\eta \in \mathcal{D}(\mathbb{R}^n)$, $\eta \equiv 1$ on a nbhd of 0,

$\operatorname{supp} \eta \subset B(0, \varepsilon)$.

~~Choose~~

Set $S = \varphi \cdot T$.

<u>Fact</u>: Let
$$\Gamma(x) = \begin{cases} c_{n,k} (-1)^k |x|^{2k-n} & \text{for } 2k < n \text{ if } n \text{ even, for all } k \text{ if } n \text{ odd} \\ c_{n,k} (-1)^{k-1} |x|^{2k-n} \log|x| & \text{for } n \leq 2k \text{ if } n \text{ even.} \end{cases}$$

Then $\underbrace{\Delta^k}_{= \sum_{|\alpha|=k} D^\alpha D^\alpha} \Gamma = \delta_0$ (fundamental solution of k-Laplacian) with $c_{n,k}$ chosen properly.

Also, $D^\alpha \Gamma \in L^1_{loc}(\mathbb{R}^n)$ for $|\alpha| \leq k$

Then, $\Delta^\ell(\eta\Gamma) = \zeta + \delta_0$, with $\zeta \in \mathcal{D}(\mathbb{R}^n)$

(calculation, derivatives of η vanish on the singularity of Γ and its derivatives; $\eta \cdot \delta_0 = \delta_0$.).

$$S * \Delta^\ell(\eta\Gamma) = S * (\zeta + \delta_0) =$$
$$= S + S * \zeta$$

on the other hand

$$S * \Delta^\ell(\eta\Gamma) = S * \sum_{|\alpha|=k} D^\alpha D^\alpha(\eta\Gamma)$$
$$= \sum_{|\alpha|=k} c_\alpha D^\alpha S * D^\alpha(\eta\Gamma)$$

thus $S = \sum_{|\alpha|=k} c_\alpha D^\alpha S * D^\alpha(\eta\Gamma) - \underbrace{S * \zeta}_{\text{this is } C^\infty, \text{ thus in } L^p_{loc}.}$

How about $D^\alpha S * D^\alpha(\eta\Gamma)$?

Note that in V $D^\alpha S = D^\alpha(\varphi T) = \varphi D^\alpha T$

thus in $\mathcal{U}$, for ε suff. small $(\varepsilon < \mathrm{dist}(\bar{\mathcal{U}}, \Omega \setminus V))$

$$D^\alpha(\eta\Gamma) * D^\alpha S = \underbrace{D^\alpha(\eta\Gamma)}_{\text{in } L^1} * \underbrace{\varphi \underbrace{D^\alpha T}_{\text{in } L^p}}_{\text{in } L^p} \in L^p(\mathcal{U}).$$

Corollary: for any $\beta : |\beta| \le k$ $D^\beta T \in L^p_{loc}$

Question: We see that it is enough to assume $D^\alpha f \in L^p$ to have sensible properties of f. Do we need to control ~~all~~ ~~norm~~ L^p- norms of all derivatives of order $\le k$ of ~~bound~~ f to know that $f \in W^{k,p}(\Omega)$?

Define: $BL^{k,p}(\Omega) = \{ f \in \mathcal{D}'(\Omega) : D^\alpha f \in L^p$ for all α s.t. $|\alpha| = k \}$

$$\dot{W}^{k,p}(\Omega) = BL^{k,p} \cap L^p(\Omega)$$

Even on bounded sets, the spaces $BL^{k,p}(\Omega)$, $\dot{W}^{k,p}(\Omega)$ and $W^{k,p}(\Omega)$ need not coincide.

(Examples later)

Key tool to prove that

$$BL^{k,p}(\Omega) = \mathring{W}^{k,p}(\Omega) = W^{k,p}(\Omega):$$

„Theorem": As long as Ω is sensible,

$\forall_{u \in BL^{k,p}(\Omega)}$ $\exists_{\substack{P - \text{polynomial} \\ \text{of order } \leq k-1}}$ such that

$$\sum_{0 \leq |\alpha| \leq k-1} \| D^\alpha (u-P) \|_{L^p} \overset{(*)}{\leq} C \sum_{|\alpha| = k} \| D^\alpha u \|_{L^p}$$

(Poincaré inequality)

Then, for „sensible" & bounded Ω, P is in $W^{k,p}(\Omega)$ and so is $u-P$, by $(*)$, thus $u \in W^{k,p}(\Omega)$.

Note: to have $(*)$ it is enough to see that $\forall_{u \in BL^{1,p}(\Omega)}$ $\exists_{a = \text{const.}}$

$$\| u - a \|_{L^p} \leq C \| \nabla u \|_{L^p}$$

+ induction.

Integral representation of Sobolev functions

Theorem: For any $u \in C_0^\infty(\mathbb{R}^n)$

$$u(x) = \frac{1}{n\,\omega_n} \int_{\mathbb{R}^n} \frac{\langle x-y, \nabla u(y)\rangle}{|x-y|^n}\, dy$$

scalar product.

ω_n: meas. of unit n-ball.

Proof: Fix $s \in S^{n-1}(0,1)$.

$$u(x) = -\int_0^\infty \frac{d}{dt} u(x+ts)\, dt$$

$$= -\int_0^\infty \langle \nabla u(x+ts), s\rangle\, dt$$

$$= -\frac{1}{|S^{n-1}(0,1)|} \int_{S^{n-1}(0,1)} \int_0^\infty \langle \nabla u(x+ts), s\rangle\, dt$$

$$= -\frac{1}{n\,\omega_n} \int_0^\infty t^{n-1} \int_{S^{n-1}} \frac{\langle \nabla u(x+ts), s\rangle}{t^{n-1}}\, ds\, dt$$

$t^{n-1}\, ds\, dt = dy$ for $y = x + ts$ $\quad t = |x-y|$

(integration in radial coordinates) $\quad s = \frac{y-x}{|x-y|}$

$$= -\frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\left\langle \nabla u(y), \frac{y-x}{|x-y|} \right\rangle}{|x-y|^{n-1}} \, dy$$

$$= \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{\langle \nabla u(y), x-y \rangle}{|x-y|^n} \, dy .$$

<u>Corollary</u>:

$$|u(x)| \leq \frac{1}{n\omega_n} \int_{\mathbb{R}^n} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy$$

Operator $(I_\alpha f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy$ is known as the Riesz potential /operator;

our Corollary reads

$$|u(x)| \leq \frac{1}{n\omega_n} I_1 (|\nabla u|)(x)$$

Do we have a similar tool for smooth functions on a ball B?

<u>Theorem</u> There exists $C = C(n)$ s.t. For any ball $B \subset \mathbb{R}^n$ and $u \in W^{1,p}(B)$ $1 \leq p \leq \infty$

$$|u(x) - u_B| \leq C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy \quad \text{a.e. in } B$$

where $u_B = \fint_B u = \frac{1}{|B|} \int_B u$.

<u>Proof</u>: Assume first $u \in C^\infty(B)$ and fix $x \in B$. For $y \neq x$, $y \in B$,

$$y = x + t \frac{y-x}{|y-x|} = \cancel{x + t\,\delta}$$

$$= x + t\lambda \qquad \lambda \in S^{n-1}(0,1)$$

Let $\delta(\lambda) = \max\{t > 0 : x + t\lambda \in B\}$

$$u(x) - u(y) = \int_0^{|y-x|} \frac{d}{ds} u(x+s\lambda) \, ds = \int_0^{|y-x|} \langle \nabla u(x+s\lambda), \lambda \rangle \, ds$$

thus

$$|u(x) - u(y)| \leq \int_0^{|y-x|} |\nabla u(x+s\lambda)| \, ds \leq \int_0^{\delta(\lambda)} |\nabla u(x+s\lambda)| \, ds. \quad (\star)$$

$$|u(x) - u_B| = \left| \fint_B (u(x) - u(y))\,dy \right|$$

$$\leq \fint_B |u(x) - u(y)|\,dy = *$$

we would like to recalculate this integral, as before, in spherical coordinates, but we need to keep track of the fact that $y \in B$:

$$* = \frac{1}{|B|} \int_{\mathbb{R}^n} |u(x) - u(y)|\, \chi_B(y)\,dy$$

$$= \frac{1}{|B|} \int_0^{\infty} t^{n-1} \int_{S^{n-1}(0,1)} |u(x) - u(x+t\lambda)|\, \chi_B(x+t\lambda)\,d\lambda\,dt$$

$$= \frac{1}{|B|} \cancel{\int_0^{\delta(\lambda)} t^{n-1}} \int_{S^{n-1}(0,1)} \int_0^{\delta(\lambda)} t^{n-1} \underbrace{|u(x) - u(x+t\lambda)|}\,dt\,d\lambda$$

$$= *$$

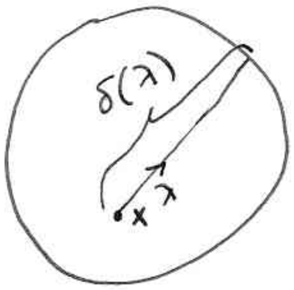
Next, we recall $(\star)$ and use it to estimate $(x + t\lambda = y)$.

$$* \leq \frac{1}{|B|} \int_{S^{n-1}} \int_0^{\delta(\lambda)} t^{n-1} \int_0^{\delta(\lambda)} |\nabla u(x+s\lambda)|\,ds\,dt\,d\lambda$$

$$\leq \frac{1}{|B|} \int_{S^{n-1}} \underbrace{\left(\int_0^{2r} t^{n-1}\,dt \right)}_{= c(n) \cdot |B|} \int_0^{\delta(\lambda)} |\nabla u(x+s\lambda)|\,ds\,d\lambda$$

$$= C(n) \int_{S^{n-1}} \int_0^{\delta(\lambda)} \frac{|\nabla u(x+s\lambda)|}{s^{n-1}} \cdot s^{n-1} \, ds \, d\lambda$$

$$= C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy .$$



We would like to extend the above result to $u \in W^{1,p}(B)$.

Fact (Exercise course, next week)
$C^\infty(\Omega)$ functions are dense in $W^{1,p}(\Omega)$ for all $p \in [1,\infty)$. For any open $\Omega \subset \mathbb{R}^n$

~~Suppose~~ $\cancel{\{u_k \in C^\infty(B)\}}$

Note that $u \in W^{1,p}(B)$, $\cancel{p \in [1,\infty}$ $1 \le p \le \infty$, then $u \in W^{1,1}(B)$.

Suppose $\underset{C^\infty(B)}{\underset{\in}{u_k}} \xrightarrow[W^{1,1}(B)]{} u$, i.e. $u_k \xrightarrow{L^1} u$

$\nabla u_k \xrightarrow{L^1} \nabla u$.

then:

- $(u_k)_B \xrightarrow{k\to\infty} u_B$

- $u_k - (u_k)_B \xrightarrow{L^1} u - u_B$

  passing to a subsequence we can have also $u_k - (u_k)_B \xrightarrow[a.e.]{} u - u_B$.

How about RHS ?

<u>Exercise</u>: Prove that there exists $C = C(n)$ such that $\displaystyle\int_E \frac{dy}{|x-y|^{n-1}} \leq C(n)\, |E|^{1/n}$.

Let us write the RHS using local Riesz potential: $\cancel{I_\alpha^\Omega = \int}$

$$\cancel{I_\alpha^B} \quad I_\alpha^\Omega f(x) = \int_\Omega \frac{f(y)}{|x-y|^{n-\alpha}}\, dy$$

$$\text{RHS} = C(n) \cdot \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy = C(n)\, I_1^B (|\nabla u|)(x)$$

I_1^B is a linear mapping on $L^1(B)$,

$$\| I_1^B g \|_{L^1(B)} = \int_B \left| \int_B \frac{g(y)}{|x-y|^{n-1}} dy \right| dx$$

$$\leq \int_B \int_B \frac{|g(y)|}{|x-y|^{n-1}} dy\, dx$$

$$\overset{\text{Fubini}}{=} \int_B |g(y)| \int_B \frac{dx}{|x-y|^{n-1}} \overset{\text{Exercise}}{\leq} C(n)|B|^{1/n} \|g\|_{L^1},$$

continuous. $L^1 \to L^1$.

Thus if ~~...~~ $\nabla u_k \to \nabla u$ in L^1,

~~so does~~ $\Rightarrow |\nabla u_k| \to |\nabla u|$ — " —

$$\Rightarrow I_1^B |\nabla u_k| \to I_1^B |\nabla u| \text{ in } L^1$$

and passing, again, to subsequences, we can have convergence a.e.

Therefore

$$|u - u_B| \leq C(n) I_1^B |\nabla u| \quad \text{a.e.}$$

for all $u \in W^{1,p}(B)$.

__Exercise__: Prove the following generalization:

Let $\Omega \subset \mathbb{R}^n$ be bounded, open and convex, $S \subset \Omega$ measurable, $|S| > 0$.

Then for any $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$

$$|u(x) - u_S| \leq \frac{(\operatorname{diam} \Omega)^n}{n\,|S|} \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy \quad \text{a.e.}$$

__Corollary__ (Poincaré inequality on a ball)

For any $B = B(x_0, r)$, $u \in W^{1,p}(B)$, $1 \leq p < \infty$,

$$\left(\fint_B |u - u_B|^p \right)^{1/p} \leq c(n,p) \cdot r \left(\fint_B |\nabla u|^p \right)^{1/p}$$

__Proof__:

$|u(x) - u_B| \leq C(n)\, I_1^B(|\nabla u|)(x)$, thus

$$\int_B |u(x) - u_B|^p \leq C^p \int_B \left(I_1^B |\nabla u| \right)^p$$

$$\leq \underbrace{\tilde{C}}_{C(n,p)} \cdot |B|^{p/n} \int_B |\nabla u|^p$$

thus

$$\left(\fint_B |u - u_B|^p \right)^{1/p} \leq \tilde{C} \cdot |B|^{1/n} \left(\fint_B |\nabla u|^p \right)^{1/p}$$

$$= C(n,p)\, r \left(\fint_B |\nabla u|^p \right)^{1/p}.$$

Another exercise

A domain (bounded, open, connected) Ω is star-shaped with respect to $x \in \Omega$ if any ray starting at x intersects $\partial\Omega$ at exactly one point.

Ω is star-shaped with respect to $A \subset \Omega \iff \Omega$ is star-shaped w.r to x for all $x \in A$.

Prove that if Ω is star-shaped with respect to a ball $B \subset \Omega$, then for any $u \in W^{1,p}(\Omega)$, $1 \le p \le \infty$,

$$|u(x) - u_B| \le C(n) \underbrace{\frac{(\operatorname{diam}\Omega)^n}{|B|}}_{\text{anyway, } C(n, \cancel{\operatorname{diam}}\, \Omega, |B|)} \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}}\, dy$$

a.e. in Ω.

A useful version of Poincaré inequality:

Thm Let Ω be open, bounded, convex, central symmetric, and suppose $u \in W^{1,p}(\Omega)$ for $1 \le p < \infty$.

Then for every measurable $S \subset \Omega$, $|S| > 0$

$$\int_\Omega |u - u_S|^p \le 2^n (\operatorname{diam}\Omega)^p \frac{|\Omega|}{|S|} \int_\Omega |\nabla u|^p$$

Direct proof:

1. Simplifications: note that both sides are insensitive to translations: we may assume Ω is central-symmetric w.r. to the origin.

Both sides are also continuous in $W^{1,p}(\Omega)$, thus it is enough to prove the inequality for $u \in C^\infty(\Omega)$ and then argue by density.

2. As usual, $u(x) - u(y) = \int_0^1 \langle \nabla u(y + t(x-y)), x - y \rangle \, dt$

We integrate y over S:

$$|u(x) - u_S| = \left| \fint_S (u(x) - u(y))\, dy \right| \le \fint_S |u(x) - u(y)|$$

$$\le \fint_S \int_0^1 |\nabla u(y + t(x-y))| \, |x-y| \, dt\, dy \le \operatorname{diam}\Omega \int_0^1 \fint_S |\nabla u(y + t(x-y))| \, dy\, dt$$

By Hölder's inequality $\left(\fint_A f\right)^p \leq \fint_A f^p$

(trivial exercise)

thus

$$|u(x) - u_S|^p \leq (\operatorname{diam} \Omega)^p \int_0^1 \fint_S |\nabla u(y + t(x-y))|^p \, dy \, dt,$$

integrating x over Ω

$$\int_\Omega |u - u_S|^p \leq \frac{(\operatorname{diam} \Omega)^p}{|S|} \int_0^1 \int_{\Omega \times S} |\nabla u(y + t(x-y))|^p \, dx \, dy \, dt$$

3. Change of variables: for t fixed,

$$(x, y) \longrightarrow (\xi, \zeta)$$

$$\xi = y + t(x-y) \in \Omega \qquad \zeta = y - x \in \Omega - \Omega = 2\Omega$$

<u>Exercise</u>: check that the Jacobian det. of this change of variables is equal to 1.

$dx\,dy = d\xi\,d\zeta$

Then

$$\int_\Omega |u - u_S|^p \leq \frac{(\operatorname{diam} \Omega)^p}{|S|} \int_0^1 \int_{\Omega \times 2\Omega} |\nabla u(\xi)|^p \, d\xi \, d\zeta \, dt$$

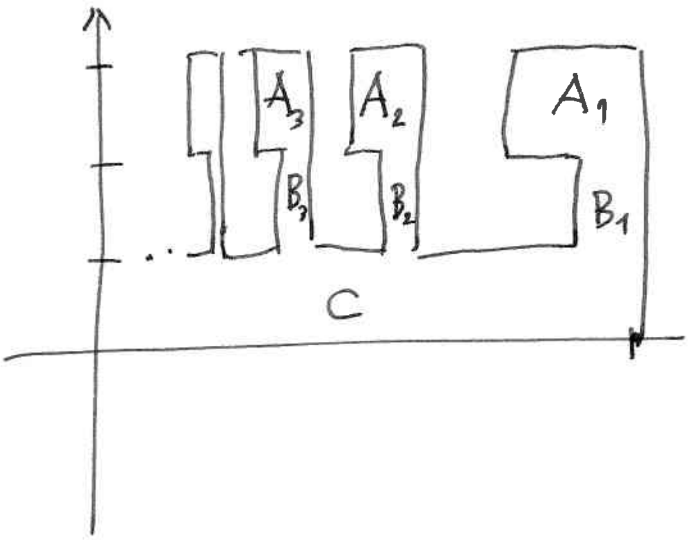
$$= \frac{(\operatorname{diam} \Omega)^p \cdot |2\Omega|}{|S|} \int_\Omega |\nabla u(\xi)|^p \, d\xi$$

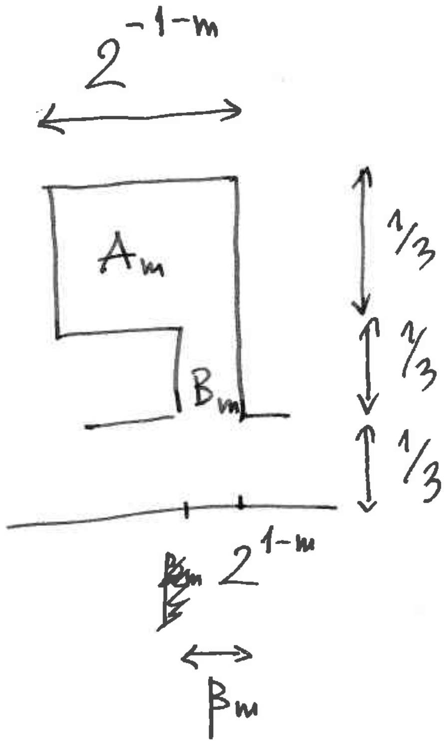
$$= \frac{(\operatorname{diam} \Omega)^p \cdot 2^n |\Omega|}{|S|} \int_\Omega |\nabla u|^p$$

Examples when $BL^{n,k}(\Omega) \neq W^{n,k}(\Omega)$

or ~~BL~~ $\mathring{W}^{n,k}(\Omega) \neq W^{n,k}(\Omega)$.

Nikodým, 1933





Function:

$$f = \begin{cases} \alpha_m & \text{on } A_m \\ 0 & \text{on } C \\ \text{linear} & \text{on } B_m \end{cases}$$ to have f continuous

on $\Omega = C \cup \bigcup_m (A_m \cup B_m)$.

with $\alpha_m > 0$ chosen so that $\sum \alpha_m |A_m| =$

$$= \sum_{m=1}^{\infty} \alpha_m \frac{1}{3 \cdot 2^{m+1}} = +\infty$$

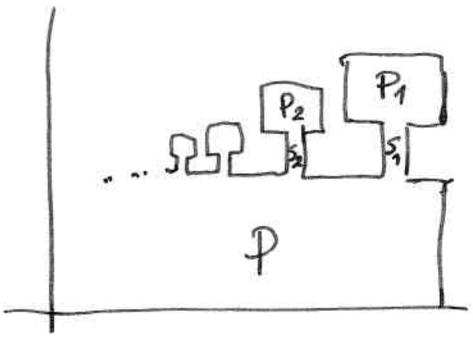
Then, obviously, $f \notin L^2(\Omega)$. However, $Df \neq 0$ only on B_m, $|Df| = 3\alpha_m$

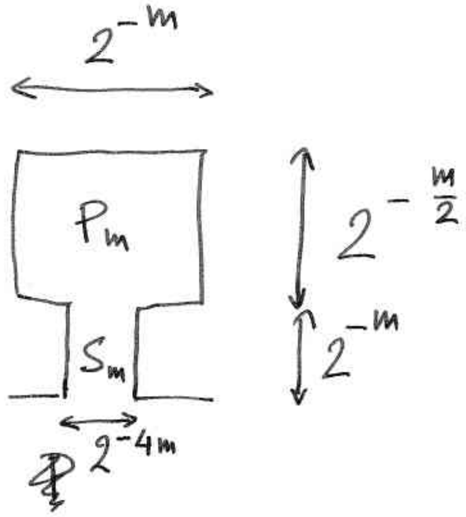
$\int_\Omega |Df|^2 = \sum_{m=1}^{\infty} 9\alpha_m^2 \cdot \frac{1}{3} \beta_m$; we can choose β_m small enough to have this converge

$\beta_m \leq \frac{1}{2^m \alpha_m^2}$

Then $f \in BL^{1,2}(\Omega)$, but $f \notin L^2(\Omega) \Rightarrow$
$f \notin W^{1,2}(\Omega)$.

Example 2 (Maz'ya)





Function:

$$f(x,y) = \begin{cases} 0 & \text{on } P \\ 2^{m-1}(y-1)^{\cancel{2}} - 1 & \text{on } P_m \\ 4^m (y-1)^2 & \text{on } S_m \end{cases}$$

Exercise: prove that $|D^2 f| \in L^2(\Omega)$,
$|f| \in L^2$,
but $|\nabla f| \notin L^2(\Omega)$

thus $f \in \dot{W}^{2,2}(\Omega)$, but not to $W^{2,2}(\Omega)$.

Recall <u>Exercise</u>: Integral representation

$$|u(x) - u_B| \le C(n, \Omega, |B|) \int_\Omega \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy$$

(and thus Poincaré inequality) holds for any domain Ω star-shaped with respect to the ball B.

A rather general (but not exhaustive) class of ~~spaces~~ bounded domains on which Poincaré inequality holds (and thus $BL = \dot{W} = W$): domains with ~~cone~~ (interior) cone property.

Ω has interior cone property, if there exists a cone: $C_{\alpha,\beta} = \{x_1^2 + x_2^2 + \dots + x_{n-1}^2 \le \alpha x_n^2, \; 0 < x_n < \beta\}$ such that every point in Ω is a vertex (tip) of an isometric image of $C_{\alpha,\beta}$ <u>contained</u> in Ω.
(with its closure)

Sketch of the argument:

<u>Theorem</u>: If Ω has cone property, then it is a finite ~~sum~~ union of domains $\Omega_1, \dots, \Omega_r$, each star-shaped w.r. to a ball.

<u>Proof</u>:

0. Let C_x be the cone with vertex x.

1. $\Omega = \bigcup_{x \in \Omega} C_x$ $\quad$ ($\overline{C_x} \subset \Omega \Rightarrow$ we can move C_x a little in all directions w-out leaving Ω).

2. All C_x are star-shaped w. r. to balls B_x, with fixed radius R and centers in ~~[illegible]~~ ~~[illegible]~~ $\sigma(x)$.

3. Choose one cone C_x. Add all these cones C_y for which $|\sigma(y) - \sigma(x)| \leq R/2$

Union of these cones is Ω_1;

all these cones are star-shaped w.r. to a ball B_1, contained in the intersection of ~~all~~ B_x and all B_y, with center at $\sigma(x)$. So is Ω_1.

Take any $z \in \Omega \setminus \Omega_1$.

Then $|\sigma(z) - \sigma(x)| > R/2$, we repeat the construction for z in place of x, obtaining Ω_2 star-shaped with respect to $B_2 = B(\sigma(z), R/2)$, and so on. Subsequent balls have centers at distance $\geq \frac{R}{2}$, and radii $= \frac{R}{2}$; by boundedness of Ω there can fit only a finite ~~no~~ numbers of such balls $\Rightarrow$

$\Rightarrow$ the construction is finite.

Writing out Poincaré inequality for each Ω_i and summing up we get Poincaré ineq. for Ω, with constant additionally dependent on m.

Our next goal: another characterization of Sobolev spaces:

Theorem: Let $1 < p < \infty$

f is in $W^{1,p}(\mathbb{R}^n)$ if and only if

$(\cdot)\ f \in L^p$

and

$(\cdot\cdot)\ \exists_{C>0}\ \forall_{h \in \mathbb{R}^n \setminus \{0\}}\quad \left\| \frac{u(\cdot+h) - u(\cdot)}{h} \right\|_{L^p} \leq C$

(equivalently, $(\cdot\cdot)$ can be written as

$\exists_{C>0}\ \forall_{h \in \mathbb{R}^n}\quad \| u(\cdot+h) - u(\cdot) \|_{L^p} \leq C \cdot |h|$).

To prove the theorem, we need to recall / introduce several definitions and facts from functional analysis:

1. Observe, that if X is a Banach space (or, more generally, ~~linear to~~ topological vector space), the space X embeds in its bidual space X^{**} (the space of continuous ~~linear~~ functionals on the dual space X^*)

The embedding is given by the evaluation mapping:

$$\begin{array}{ccc} X & & X^{**} \\ \Psi & & \Psi \\ x & \overset{i}{\longmapsto} & \hat{x} \end{array}$$ such that

for any $\varphi \in X^*$ $\quad \hat{x}(\varphi) = \varphi(x)$

If X is a normed space (standing assumption), then

$$x \longrightarrow \hat{x}$$ is an isometry (onto image)

(by Hahn-Banach, there exists $\varphi \in X^*$, $\|\varphi\|_{X^*} = 1$ s.t $\varphi(x) = \hat{x}(\varphi) = \|x\|_X$; on the other hand $\forall_{\substack{\psi \in X^* \\ \|\psi\|_{X^*} \le 1}}$ $|\hat{x}(\psi)| = |\psi(x)| \le \|\psi\|_{X^*} \|x\|_X = \|x\|_X$

$$\Longrightarrow \sup_{\|\psi\|_{X^*} \le 1} |\hat{x}(\psi)| = \|x\|_X$$

$$\|\hat{x}\|_{X^{**}}$$

If $i(X) = X^{**}$, then we say that X is a reflexive space

(this is more than asking X isometric to X^{**}; there are examples when $X \underset{\text{isom}}{\approx} X^{**}$, but $i(X) \subsetneq X^{**}$)

R.C. James

If X is reflective, then $X = (X^*)^* = Y^*$

weak convergence on X is $*$-weak convergence on Y^* for $Y = X^*$, applying B-A theorem we get 1.

Application of 2.

$$W^{1,p}(\underset{\mathbb{R}^n}{\overset{\cup}{\Omega}}, \mathbb{R}) \hookrightarrow L^p(\Omega, \mathbb{R}^{n+1})$$

$$u \longmapsto (u, \nabla u)$$

Exercise: prove that the image of $W^{1,p}$ is closed in $L^p(\Omega, \mathbb{R}^{n+1})$.

This shows that $W^{1,p}(\Omega)$ is reflexive ($L^p(\Omega, \mathbb{R}^{n+1}) = L^p(\Omega) \times L^p(\Omega) \times \ldots \times L^p(\Omega)$; $L^p(\Omega)$ is reflexive).

Def: A sequence (x_n) in a topological vector space X is <u>weakly convergent</u> to $x \in X$ if for any $\varphi \in X^*$ we have $\varphi(x_n) \to \varphi(x)$.
(convergence of distributions is $*$-weak convergence in $\mathcal{D}^*$)

Facts: 1. In a reflexive space, every bounded sequence (x_n) contains a weakly convergent subsequence.

2. Closed subspace of a reflexive space is again a reflexive space

2. is abstract nonsense; 1. is a consequence of <u>Banach-Alaoglu theorem</u>:

If X is a normed linear space, then the ~~ball~~ B_{X^*} unit ball in X^* is compact in the $*$-weak topology.

$*$-weak topology: $\underset{\overset{\Downarrow}{X^*}}{\varphi_n} \xrightarrow{w-*} \varphi \Leftrightarrow \forall_{x \in X} \; \varphi_n(x) \to \varphi(x)$

(pointwise convergence);

in other words $X^* \subset C(X, \mathbb{R}) \subset \mathbb{R}^X$

$*$-weak topology is subspace topology $\longleftarrow$ this has product (Alexander) topology

one can easily check $B_{X^*} \subset [-1,1]^X \leftarrow$ compact,

B_{X^*} is closed $\Rightarrow$ compact.

Proof of the theorem

$\Longrightarrow$

Assume first that $u \in C^\infty(\mathbb{R}^n)$.

$$u(x+h) - u(x) = \int_0^1 \frac{d}{dt} u(x+th)\,dt =$$

$$= \int_0^1 \langle \nabla u(x+th), h \rangle\,dt$$

$$\Rightarrow \int_{\mathbb{R}^n} |u(x+h) - u(x)|^p dx = \int_{\mathbb{R}^n} \left| \int_0^1 \langle \nabla u(x+th), h \rangle dt \right|^p dx$$

$$\leq \int_{\mathbb{R}^n} \int_0^1 |\nabla u(x+th)|^p |h|^p \,dt\,dx$$

$$= |h|^p \cdot \int_0^1 \underbrace{\int_{\mathbb{R}^n} |\nabla u(x+th)|^p \,dx}_{\|\nabla u\|_p^p} \,dt = |h|^p \|\nabla u\|_p^p$$

$$\Rightarrow \left\| \frac{u(\cdot+h) - u(\cdot)}{|h|} \right\|_p \leq \|\nabla u\|_p .$$

$\Leftarrow$

We shall prove that, for any $i \in \{1, 2, \dots, n\}$, the distributional partial derivative $\partial_{x_i} u$ $= D^{(0,0,\dots,\overset{i}{1},\dots,0)} u$ is in L^p.

Choose i. By assumption, taking $\mathbb{R} \ni h_k \xrightarrow[k\to\infty]{} 0$,

$$\left\| \frac{u(\cdot + h_k e_i) - u(\cdot)}{h_k} \right\|_p \leq C \text{ for all } k.$$

Thus the above sequence $\left(\frac{u(\cdot + h_k e_i) - u(\cdot)}{h_k} \right)_{k \in \mathbb{N}}$ is bounded in $L^p(\mathbb{R}^n)$. We choose a weakly-convergent subsequence, that (weakly) converges to some $v \in L^p(\mathbb{R}^n)$.

We want to prove that $v = \partial_{x_i} u$ (as distributions):

For any $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} v \varphi = \hat{\varphi}(v) = \lim_{k\to\infty} \hat{\varphi}\left(\frac{u(\cdot + h_{k_\ell} e_i) - u(\cdot)}{h_{k_\ell}} \right)$$

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{u(x + h_{k_\ell} e_i) - u(x)}{h_{k_\ell}} \varphi(x)$$

$$= \lim_{k\to\infty} \int_{\mathbb{R}^n} \frac{\varphi(x - h_{k_\ell} e_i) - \varphi(x)}{h_{k_\ell}} u(x)$$

$$= \int_{\mathbb{R}^n} (-\partial_{x_i}\varphi)\cdot u = -\int_{\mathbb{R}^n} u\cdot \partial_{x_i}\varphi$$

$$T_v(\varphi) = -T_u(\partial_{x_i}\varphi) \iff \underset{\substack{\uparrow \\ \text{distributional} \\ \text{derivative.}}}{\partial_{x_i} u} = v$$

This shows that $\partial_{x_i} u \in L^p$; since, by assumption, $u \in L^p$, we have $u \in W^{1,p}(\mathbb{R}^n)$.

<u>Exercise</u>: Prove that if $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p < \infty$, then

$$\frac{u(\cdot + h\cdot e_i) - u(\cdot)}{h} \xrightarrow[\text{in } L^p]{h\to 0} \partial_{x_i} u$$

<u>Exercise</u>: Let $1 < p < \infty$. Suppose $\Omega \subset \mathbb{R}^n$ is open. Prove that

$u \in W^{1,p}_{loc}(\Omega)$ iff $(\cdot)$ $u \in L^p_{loc}(\Omega)$

and $(\cdot\cdot)$ for every $\mathcal{U} \Subset \Omega$ (open) there exists $C_{\mathcal{U}} > 0$

s.t. $$\left\| \frac{u(\cdot + h) - u(\cdot)}{h} \right\|_{L^p(\mathcal{U})} \le C_{\mathcal{U}}.$$

for all h: $|h| < \min\{\frac{1}{2}\operatorname{dist}(\mathcal{U}, \partial\Omega), 1\}$.

Aim: Sobolev embedding theorem:

Suppose $u \in W^{1,p}(\Omega)$. Clearly $u \in L^p(\Omega)$, but can we do better? Does it depend on the smoothness of $\partial\Omega$?

Standard PDE course: ~~for~~ $1 \le p < n$, $u \in W^{1,p}(\mathbb{R}^n) \Rightarrow$ ~~we have~~ $u \in L^{p^*}(\mathbb{R}^n)$ with $p^* = \frac{np}{n-p}$.

Proof: E.g. Evans

To obtain similar results for bounded Ω, we need extension theorems (to extend $u \in W^{1,p}(\Omega)$ to $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$, with $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \lesssim \|u\|_{W^{1,p}(\Omega)}$).

We shall follow another approach.

Def: ~~For~~ For any $f \in L^1_{loc}(\mathbb{R}^n)$ we may define Hardy-Littlewood maximal function

$$Mf(x) = \sup_{\cancel{x \ni B}\; B \ni x} \fint_B |f(y)|\,dy \quad \left(\in [0,\infty]\right)$$

Theorem $\Big(\overbrace{\text{Hardy - Littlewood}}^{1930,\text{ on } \mathbb{R}} - \overbrace{\text{Wiener}}^{1939,\text{ on } \mathbb{R}^n}\Big)$

~~for $f \in L^1(\mathbb{R}^n)$~~

(1) if $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, then $Mf(x) < \infty$ for a.e. x

(2) if $f \in L^1(\mathbb{R}^n)$, then ~~$Mf(x)$~~

$$|\{x \in \mathbb{R}^n : Mf(x) > t\}| \le \frac{3^n}{t} \|f\|_{L^1(\mathbb{R}^n)}$$

(3) if $f \in L^p(\mathbb{R}^n)$, then $Mf \in L^p(\mathbb{R}^n)$ and

$$\cancel{\le C\|f\|_{L^p}} \quad \|Mf\|_{L^p} \le C(n,p)\,\|f\|_{L^p}$$

(In fact, $C(n,p)$ can be taken independent on the dimension n – this is a deep result of Stein, 1982)

For the proof we need perhaps the simplest of Vitali's covering lemmata:

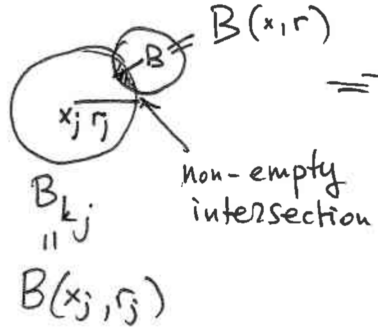
~~Suppose~~ <u>Lemma</u>: Suppose Ω is covered by a finite family $\mathcal{B}=\{B_k\}_{k=1,\dots,b}$ of balls: $\Omega \subset \bigcup_{k=1}^{b} B_k$. Then we can choose a ~~fi~~ subfamily $B_{k_1},\dots,B_{k_m}$ of disjoint balls out of $\mathcal{B}$, such that $\Omega \subset \bigcup_{j=1}^{m} 3B_{k_j}$. $(3B(x_0,R) := B(x_0,3R))$

(i.e. Ω is covered by balls concentric with B_{k_j}, but with tripled radii).

<u>Proof:</u> Very simple: choose B_{k_1} to be the biggest ball in $\mathcal{B}$. B_{k_2} is the biggest ball in $\mathcal{B}$ that is disjoint with B_{k_1}, ~~and so on. $\mathcal{B}$ is finite, so thus is our subfamily $B_{k_1},\dots,B_{k_m}$. Any ball $B\in\mathcal{B}$ intersects with one of B_{k_i}~~ and B_{k_3} – the biggest among all balls in $\mathcal{B}$ that are disjoint with B_{k_1} and B_{k_2} and so on.

~~E~~Every $B\in\mathcal{B}$ intersects with (at least) one of $B_{k_1},\dots,B_{k_m}$; let j be the smallest index s.t. $B_{k_j}\cap B \neq \emptyset$.

Then the radius of B is not greater than that of B_{k_j}



$$\Rightarrow B \subset B(x_j, r_j + 2r) \subset \underbrace{B(x_j, 3r_j)}_{= 3B_{k_j}}.$$

Since $\Omega \subset \bigcup_{B \in \mathcal{B}} B$, we have $\Omega \subset \bigcup_{j=1}^{m} 3B_{k_j}$.

Before we use the lemma to prove H-L-W theorem, note that (1) follows from (2) and (3):

if $p>1$, then $Mf \in L^p(\mathbb{R}^n) \Longrightarrow Mf$ is finite a.e.

if $p=1$, $\{Mf = \infty\} \subset \{Mf > t\}$ for any t,

thus $\displaystyle |\{Mf=\infty\}| \le \frac{3^n \|f\|_{L^1}}{t} \xrightarrow[t\to\infty]{} 0$.

We now prove (2).

~~Denote~~

If $x \in \{Mf > t\}$, then, by definition, $\exists_{B \ni x}$ s.t. $\fint_B |f| > t$

$\displaystyle \Rightarrow |B| < \frac{1}{t}\int_B |f| \le \frac{1}{t}\|f\|_{L^1(\mathbb{R}^n)}$. ~~then~~

Let K be any compact subset of $\{Mf > t\}$.

We cover K with balls B_x with the above property:

$\cancel{x \in B}(\cdot)\ \forall_{x\in K}\quad x \in B_x$

$\displaystyle (\cdot\cdot)\ |B_x| \le \frac{1}{t} \cancel{\|f\|_{L^1(\mathbb{R}^n)}} \cdot \int_{B_x} |f|$

K is compact $\Rightarrow$ we can choose a finite subcovering $\{B_{x_1}, \dots, B_{x_\ell}\} = \mathcal{B}$ $\underset{\text{Vitali}}{\Longrightarrow}$

we choose ~~$B_{x_{k_1}}$~~ out of $\mathcal{B}$ balls (disjoint!) $B_1, \dots, B_m$ such that $3B_1, \dots, 3B_m$ cover K

and $\forall_j \; |B_j| \le \frac{1}{t} \int_{B_j} |f|$.

Then
$$|K| \le \sum_{j=1}^{m} |3B_j| = 3^n \sum_{j=1}^{m} |B_j|$$
$$\le 3^n \sum_{j=1}^{m} \frac{1}{t} \int_{B_j} |f| \underset{\substack{B_j \text{ are} \\ \text{disjoint}}}{=} \frac{3^n}{t} \int_{\bigcup_{j=1}^m B_j} |f| \le \frac{3^n}{t} \|f\|_{L^1(\mathbb{R}^n)}.$$

This holds for any compact $K \subset \{Mf > t\}$, $|A| = \sup$ of measures of compacts in A

$$\Rightarrow |\{Mf > t\}| \le \frac{3^n}{t} \|f\|_{L^1(\mathbb{R}^n)}.$$

Now, we conclude (3) from (2):

~~Let~~ ~~For~~ Let us write
$$f^{[t]} = f \cdot \chi_{\{|f| > t\}}$$
$$f_{[t]} = f \cdot \chi_{\{|f| \le t\}};$$

$f = f^{[t]} + f_{[t]}$, thus
$$Mf \le Mf^{[t]} + Mf_{[t]}$$
$$\le Mf^{[t]} + t$$

Then also $Mf \le Mf^{[t/2]} + t/2$, and if $Mf > t$, then $Mf^{[t/2]} + t/2 > t \Rightarrow Mf^{[t/2]} > t/2$

In other words $\{Mf > t\} \subset \{Mf^{[t/2]} > t/2\}$,

~~$\{Mf > t\}$~~

$$|\{x : Mf(x) > t\}| \leq |\{x : Mf^{[t/2]} > t/2\}|$$

$$\underset{(2)}{\leq} \frac{3^n}{t/2} \|f^{[t/2]}\|_{L^1} = \frac{2 \cdot 3^n}{t} \int_{\{|f| > t/2\}} |f|$$

Now, by Cavalieri's principle: (Exercise, if you don't remember)

$$\left(\int_{\mathbb{R}^n} g^p = p \int_0^\infty t^{p-1} |\{g > t\}| \, dt \quad \text{for any measurable } g \geq 0, \ p \geq 1 \right)$$

$$\left(\|Mf\|_{L^p}\right)^p = \int_{\mathbb{R}^n} Mf^p = p \int_0^\infty t^{p-1} |\{Mf > t\}| \, dt$$

$$\leq p \int_0^\infty t^{p-1} \cdot \frac{2 \cdot 3^n}{t} \int_{\{|f| > t/2\}} |f| = 2 \cdot 3^n p \int_0^\infty \int_{\mathbb{R}^n} t^{p-2} |f(x)| \chi_{\{(x,t) : |f(x)| > t/2\}}(x,t) \, dx \, dt$$

$$= 2 \cdot 3^n \cdot p \int_{\mathbb{R}^n} |f(x)| \int_0^\infty t^{p-2} \chi_{\{(x,t) : t < 2|f(x)|\}}(x,t) \, dt \, dx$$

$$= 2 \cdot 3^n \cdot p \int_{\mathbb{R}^n} |f(x)| \int_0^{2|f(x)|} t^{p-2} \, dt \, dx = \underbrace{2^p \frac{3^n p}{p-1}}_{C(n,p)} \int_{\mathbb{R}^n} |f|^p$$

$$= 2^p 3^n \frac{p}{p-1} \|f\|_{L^p}^p$$

Note that the constant $C(n,p)$ explodes when $p \to 1^+$.
Indeed, for $p = 1$ we have $\int_0^{2|f|} t^{p-2}\,dt = +\infty$!

$\square$.

Simple examples show that ~~the~~ (2) cannot be improved to $\|Mf\|_{L^1} \leq C(n)\, \|f\|_{L^1}$:
for ~~f~~ $n=1$, $f = \chi_{[0,1]}$ and $x > 0$

$$Mf(x) \geqslant \frac{1}{2x} \int_0^x \chi_{[0,1]}(x)\,dx = \underbrace{\begin{cases} \frac{1}{2} & x \leq 1 \\ \frac{1}{2x} & x > 1 \end{cases}}_{\text{this is not in } L^1 \,!}$$

Riesz potentials

Recall: For any $a \in (0,n)$ we define Riesz potential

$$I_a f(x) = \frac{1}{\gamma(a)} \int_{\mathbb{R}^n} |x-y|^{a-n} f(y)\,dy$$

we add a constant here to get better properties; $\gamma(a) = 2^a \pi^{n/2}\, \Gamma\left(\frac{a}{2}\right) \Big/ \Gamma\left(\frac{n}{2} - \frac{a}{2}\right)$

then $\Delta(I_a f) = I_a(\Delta f) = -I_{a-2}(f)$. $(a \geq 2)$

$I_a = (-\Delta)^{-a/2}$

Theorem:

(Hardy-Littlewood-Sobolew theorem on fractional integration).

Let $a \in (0,n)$, $1 \leq p < q < \infty$ satisfy

$$\left.\frac{1}{q} = \frac{1}{p} - \frac{a}{n}.\right\} \Rightarrow q = \frac{np}{n-ap}$$

Then, ~~(1) if $f \in L^p(\mathbb{R}^n)$ the~~ $\cancel{\|I_a f\|_q \leq C(a,p,q,n)\|f\|_p}$

(1) If $f \in L^p(\mathbb{R}^n)$ $\boxed{p>1}$, then $I_a f \in L^q(\mathbb{R}^n)$

and $\|I_a f\|_q \leq C(a,p,q,n)\|f\|_p$

(2) if $p=1$, $f \in L^1(\mathbb{R}^n)$, then

$$|\{I_a f > t\}| \leq \left(\frac{C(a,n)\|f\|_{L^1}}{t}\right)^{\frac{n}{n-a}}$$

Proof: Since $C_0^\infty(\mathbb{R}^n)$ is dense in ~~L^p~~ $L^p(\mathbb{R}^n)$ for any p, it is enough to assume $f \in C_0^\infty(\mathbb{R}^n)$. ($f$ compactly supported, $|x-y|^{a-n}$ locally integrable).

$$|I_a f(x)| \leq \int_{\mathbb{R}^n} |x-y|^{a-n} |f(y)| dy =$$

$$= \int_{B(x,\varepsilon)} |x-y|^{a-n} |f(y)| dy + \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} |x-y|^{a-n} |f(y)| dy$$

$$= \cancel{A + B}\ S_1 + S_2$$

To deal with S_1, we divide $B(x,\varepsilon)$ into annuli $B(x,\varepsilon) = \bigcup_{k=1}^{\infty} A_k$ $\quad A_k = \left\{ y : |x-y| \in \left[\frac{\varepsilon}{2^k}, \frac{\varepsilon}{2^{k-1}}\right) \right\}$

Writing $B_k = B(x, \frac{\varepsilon}{2^{k-1}})$ we have,

for $x \in A_k$: $\quad |x-y| \approx \frac{\varepsilon}{2^k} \Rightarrow |x-y|^{-n} \approx \underbrace{|B_k|^{-1}}_{\frac{2^{nk}}{\varepsilon^n}} \approx |A_k|^{-1}$

up to a constant

thus $S_1 \leq \sum_{k=1}^{\infty} \int_{A_k} |x-y|^{a-n} |f(y)| dy$

$$\lesssim \sum_{k=1}^{\infty} \left(\frac{\varepsilon}{2^k}\right)^a \cdot \frac{1}{|B_k|} \int_{B_k} |f(y)| dy \lesssim Mf(x) \sum_{k=1}^{\infty} \left(\frac{\varepsilon}{2^k}\right)^a$$

$$\approx \varepsilon^a Mf(x)$$

S_2 is easier

$$S_2 = \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} |x-y|^{a-n} |f(y)| \, dy$$

$p, \frac{p}{p-1}$ are Hölder conjugates

$$\leq \left(\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} |x-y|^{(a-n)\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} |f|^p \right)^{1/p}$$

$$\leq \underbrace{\left(\int_{\mathbb{R}^n \setminus B(x,\varepsilon)} |x-y|^{(a-n)\frac{p}{p-1}} \right)^{\frac{p-1}{p}}} \|f\|_{L^p(\mathbb{R}^n)}$$

Exercise: Prove that this equals $\left[\frac{n\omega_n(p-1)}{n-ap}\right]^{\frac{p-1}{p}} \cdot \varepsilon^{a-\frac{n}{p}}$

Altogether, we have

$$|I_a f(x)| \leq \varepsilon^a \cdot Mf(x) + \varepsilon^{a-\frac{n}{p}} \cdot C(n,a,p) \cdot \|f\|_{L^p}$$

$$\leq C(n,a,p) \left[\varepsilon^a Mf(x) + \varepsilon^{a-\frac{n}{p}} \|f\|_{L^p} \right]$$

Fix $\varepsilon = \left[\|f\|_p / Mf(x) \right]^{p/n}$ (minimum of this)

$$|I_a f(x)| \leq C(a,n,p) \cdot \|f\|_p^{\frac{ap}{n}} \cdot Mf(x)^{\cancel{1-\frac{ap}{n}}\; 1-\frac{ap}{n}}$$

By H-L-W, $f \in L^p \Rightarrow Mf \in L^p \Rightarrow (Mf)^{1-\frac{ap}{n}} \in L^{\frac{p}{1-\frac{ap}{n}}} = L^q$

$$p \big/ \left(1-\tfrac{ap}{n}\right) = \frac{np}{n-ap}$$

Proof of ②: Exercise.

Recall now that for any ball $B \subset \mathbb{R}^n$ we have the representation formula:

for any $u \in W^{1,1}_{loc}(B)$

$$|u(x) - u_B| \leq C(n) \int_B \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \quad \text{for a.e. } x \in B$$

$$= C(n)\, I_1(|\nabla u| \cdot \chi_B)(x). \quad (*)$$

By H-L-S, if $u \in W^{1,p}(B)$, we obtain, for $p > 1$

$$\|I_1(|\nabla u| \cdot \chi_B)\|_{L^q(\mathbb{R}^n)} \leq C \cdot \|\nabla u \cdot \chi_B\|_{L^p(\mathbb{R}^n)} = C \cdot \|\nabla u\|_{L^p(B)},$$

with $q = \frac{np}{n-p}$

Including $(*)$, we get

$$\|u(x) - u_B\|_{L^q(B)} \leq C(n,p) \|\nabla u\|_{L^p(B)}.$$

Equivalently

$$\left(\fint_B |u - u_B|^q \right)^{1/q} \lesssim r \left(\fint_B |\nabla u|^p \right)^{1/p}.$$

Note that the same proof works for any Ω bounded, open convex, star-shaped w.r. to a ball or having the cone property – all we need is the representation formula:

~~Sobolev~~

<u>Sobolev inequality</u>

$$\|u - u_\Omega\|_{L^q(\Omega)} \le C(n,p,\Omega)\|\nabla u\|_{L^p(\Omega)}.$$

~~To distinguish~~

How about the case $p = 1$?

H-L-S argument fails!

Here we shall resort to an argument resembling the ~~PDE course~~ one in Evans:

~~Sobo~~

<u>Lemma</u> (Sobolev inequality for $p=1$, on cubes)

For any $f \in W^{1,1}(Q)$, $Q \subset \mathbb{R}^n$ a cube we have

$$\|f\|_{L^{n/n-1}(Q)} \le \|\nabla f\|_{L^1(Q)} + \|f\|_{L^1(Q)} \cdot |Q|^{-1/n}$$

(for $n=1$ it reads: $\|f\|_{L^\infty} \le \ldots$)