19.04.2013

Sobolev spaces - exercise course

Exercise 1: Let

(DONE)

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy$$ non-centered maximal function

and $$\tilde{M}(f)(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$ centered maximal function

Prove that there exist constants $C_1 = C_1(n)$, $C_2 = C_2(n)$ such that

$$C_1(n) Mf(x) \leq \tilde{M}f(x) \leq C_2(n) Mf(n)$$

Exercise 2: Construct an example in $\mathbb{R}^n$, $n \geq 2$ to show that $f \in L^1(\mathbb{R}^n)$ does not imply $Mf \in L^1(\mathbb{R}^n)$

(TO BE DONE)

Exercise 3: . Prove that if f has compact support

(TO BE DONE) contained in a ball $B \subset \mathbb{R}^n$ and

$$\int_B |f| \log(2 + |f|) dx < \infty, \text{ then } Mf \in L^1(B).$$

Exercises connected to the Hardy-Littlewood-Sobolev theorem (fractional integration theorem).

Exercise 4: Show that the condition

(DONE)

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$$

is a necessary condition to have $\| I_\alpha f \|_q \leq C \|f\|_p$

Hint: Consider dilations $\tau_\delta f(x) = f(\delta x)$.

(DONE)

Exercise 5: Show that for $p = \frac{n}{\alpha}$ the Riesz potential I_α is not a bounded operator ~~operator~~ from $L^{\frac{n}{\alpha}}$ to L^∞.

Hint: Consider a function

$$f(x) = \begin{cases} |x|^{-\alpha} \left(\log \frac{1}{|x|}\right)^{-\frac{\alpha}{n}(1+\varepsilon)} & , \ |x| \leq \frac{1}{2} \\ 0 & , \ |x| > \frac{1}{2} \end{cases}$$

for ε- sufficiently small. ($I_\alpha f \notin L^\infty$).

(DONE)

Exercise 6: Show that the Riesz potential I_α is not a bounded operator from L^1 to $L^{\frac{n}{n-\alpha}}$

i.e. one cannot have constant $C>0$ s.t.

$$\| I_\alpha f \|_{\frac{n}{n-\alpha}} \leq C \| f \|_1$$

Hint: Consider a function

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Exercise 7 (TO BE DONE): Prove the second statement of the H-L-S theorem:

If $f \in L^1(\mathbb{R}^n)$, then for all $\lambda > 0$

$$\left| \{ x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda \} \right| \leq \left(\frac{C}{\lambda} \| f \|_1 \right)^{\frac{n}{n-\alpha}}$$

with $C = C(\lambda, n)$

Hint: use connection between the Riesz potential and the maximal function (see lecture) and the weak-type estimates for the maximal function (Hardy-Littlewood-Wiener theorem).

$\searrow$ see lecture.