COMPACT HYPERKÄHLER MANIFOLDS: GENERAL THEORY

KIERAN G. O'GRADY

"SAPIENZA"UNIVERSITÀ DI ROMA

Contents

1. Introduction 1
2. Yau's Theorem and its implications 1
3. The local period map and the B-B quadratic form 4
4. The Kähler cone 9
5. Global Torelli 12
6. Hyperkähler manifolds of type $K3^{[n]}$ 15
References 17

1. Introduction

A general theory of HK manifolds was first developed thirty years ago by Bogomolov, Fujiki and Beauville. Verbitsky and Salamon added key results on the topology of HK's, and Verbitsky developed other aspects such as the theory of hyperholomorphic sheaves. Roughly ten years ago Huybrechts made huge steps ahead in the general theory, and quite recently Verbitsky added a global Torelli Theorem. Many interesting consequences of Global Torelli have been obtained by Markman, Mongardi and Bayer-Hassett-Tshinkel. A key ingredient in many of these developments is the existence of twistor families of HK manifolds (a consequence of Yau's solution of the Calabi conjecture): these are families parametrized by $\mathbb{P}^1$, with the generic element of the family being a non-projective HK.

2. Yau's Theorem and its implications

Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$ where $c_1^{\mathbb{R}}(X)$ is the first Chern class in De Rham cohomology - equivalently the integral first Chern class $c_1(X) \in H^2(X;\mathbb{Z})$ is torsion. A *Calabi-Yau metric* on X is a Hermitian metric h such that the unique connection ∇ on K_X compatible with the holomorphic structure and the metric h (see [11]) is flat i.e. its curvature F_∇ vanishes. Vanishing of F_∇ is equivalent to vanishing of the Ricci curvature of the riemannian metric associated to h, for that reason a Calabi-Yau metric is also called a Ricci-flat metric. Below is Yau's celebrated Theorem on existence of Calabi-Yau metrics.

Theorem 2.1 (Yau [38]). *Let (X,ω) be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$. There exists a unique Calabi-Yau metric h such that the Kähler form ω_h of h is cohomologous to ω.*

Example 2.2. Let $X = \mathbb{C}^n/L$ be a compact torus. In this case the statement of **Theorem 2.1** follows from the fact that every cohomology class is represented by a form on $\mathbb{C}^n$ with constant coefficients, moreover the connection on the tangent space itself is flat.

In general Yau's theorem is a pure existence result - as far as I know no one ever wrote down a Calabi-Yau metric of a single $K3$ surface. Yau's Theorem has some very strong consequences - we will go over these results for HK manifolds.

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2.1. **Holonomy.** Let (M,g) be a Riemannian manifold. Let $p,q\in M$ and $\gamma\colon[a,b]\to M$ a piecewise-smooth path from p to q: parallel transport with respect to the Levi-Civita connection defines an isometry

$$\varphi_\gamma\colon T_pM\longrightarrow T_pM.$$

(We let T_pM be the real tangent space to M at p.) The *holonomy group* H_p at p is defined to be

$$H_p:=\{\varphi_\gamma\mid\gamma(a)=\gamma(b)=p\}<O(T_pM).$$

Forgetting about the point p we may view the holonomy group as a subgroup $H<O(n)$ (here $n:=\dim M$) well-defined modulo conjugation (we assume that M is connected).

Example 2.3. Let X be a compact Kähler manifold with Kähler metric h. One identifies Θ_pX (the holomorphic tangent space to X at p) and T_pX by mapping $v\in\Theta_pX$ to $(v+\overline{v})/2$. Given the above identification multiplication by $-\sqrt{-1}$ on Θ_pX gets identified with an endomorphism $I\colon T_pX\to T_pX$ whose square is $-\mathrm{Id}$. Since h is Kähler the endomorphism I is invariant under the holonomy group [19]: it follows that H_p is a subgroup of $\mathrm{U}(\Theta_pX,h_p)$ (again we identify T_pX with Θ_pX).

The group H_p and its representation on T_pM encodes information on the geometry of M as follows. Let $\Gamma_{par}(M;TM^{\otimes a}\otimes T^\vee M^{\otimes b})$ be the space of parallel tensors.

Holonomy Principle 2.4. *Let (M,g) be a connected riemannian manifold and $p\in M$. The evaluation map*

$$\Gamma_{par}(M;TM^{\otimes a}\otimes T^\vee M^{\otimes b})\longrightarrow T_pM^{\otimes a}\otimes T_p^\vee M^{\otimes b}$$

is injective with image the subspace of tensors invariant under the action of H_p.

Next we recall Bochner's principle [4].

Bochner's Principle 2.5. *Let X be a compact Kähler manifold and suppose that h is a Calabi-Yau metric. Let σ be a* holomorphic *tensor i.e. a global holomorphic section of $\Theta_X^{\otimes a}\otimes\Omega_X^{\otimes b}$. Then σ is parallel.*

Example 2.6. Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X)=0$ and ω be a Kähler class on X. By Yau's Theorem there exists a unique CY metric h such that ω_h is in the class of ω. Let $0\neq\alpha\in H^0(K_X)$. By Bochner's principle α is parallel: it follows that $H_p<\mathrm{SU}(\Theta_pX)$ (recall that $H_p<\mathrm{U}(\Theta_pX)$ by **Example 2.3**).

Example 2.7. Let X be a HK manifold and ω be a Kähler class on X. Let h be the unique CY metric such that ω_h is in the class of ω. Let σ be a holomorphic symplectic form. By Bochner's principle we get that $H_p<(U(\Theta_pX,h_p)\cap\mathrm{Sp}(\Theta_pX,\sigma_p))$ where $\mathrm{Sp}(\Theta_pX,\sigma_p)$ is the symplectic group of $\mathbb{C}$-linear automorphisms preserving the symplectic form σ_p on Θ_pX. Actually [4]

$$H_p=U(\Theta_pX,h_p)\cap\mathrm{Sp}(\Theta_pX,\sigma_p). \tag{2.1.1}$$

Theorem 2.8. *Let X be a HK manifold of dimension $2n$ and σ a holomorphic 2-form. Then*

$$H^0(\Omega_X^q)=\begin{cases}\mathbb{C}\sigma^i & \text{if } q=2i \text{ for } 0\le i\le n,\\ 0 & \text{otherwise.}\end{cases} \tag{2.1.2}$$

Proof. Let $p\in X$. Every holomorphic global form on X is parallel by Bochner's principle: it follows that we have an isomorphism

$$\begin{array}{ccc} H^0(\Omega_X^q) & \overset{\sim}{\longrightarrow} & (\bigwedge^q\Omega_pX)^{H_p}\\ \varphi & \mapsto & \varphi_p\end{array} \tag{2.1.3}$$

where $(\bigwedge^q\Omega_pX)^{H_p}$ is the space of H_p-invariant elements of $\bigwedge^q\Omega_pX$. The right-hand side of (2.1.3) is generated by $\bigwedge^m\sigma(p)$ if $q=2m$ and is zero otherwise: the theorem follows. □

Corollary 2.9. *Let X be a HK manifold of dimension $2n$. Then $\chi(\mathscr{O}_X)=n+1$.*

2.2. **Twistor families.** Let X be a HK manifold of dimension $2n$. Let ω be a Käbler class on X. Let h be the unique CY metric such that ω_h is in the class of ω and let g be the associated riemannian metric (the real part of h). One identifies the holonomy group $H_p = U(\Theta_p X, h_p) \cap \mathrm{Sp}(\Theta_p X, \sigma_p)$ with a group acting on $\mathbb{H}^n$ (here $\mathbb{H}$ is the algebra of quaternions) as follows. Recall that $\mathbb{H}$ is the associative real algebra with $\mathbb{R}$-basis $\{1, i, j, k\}$ such that

$$-1 = i^2 = j^2 = k^2, \qquad ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i.$$

The *conjugate* of $x = x_1 + x_2 i + x_3 j + x_4 k$ is $x = x_1 - x_2 i - x_3 j - x_4 k$; notice that $\overline{x \cdot y} = \overline{y} \cdot \overline{x}$. Multiplication on the *right* gives $\mathbb{H}^n$ the structure of an $\mathbb{H}$-module. Let $w, z \in \mathbb{H}^n$: for $s = 1, \ldots, n$ we write $w_s = a_s + j b_s$ and $z_s = c_s + j d_s$ where $a_s, b_s, c_s, d_s \in \mathbb{C}$. The *standard* hermitian quaternionic product on $\mathbb{H}^n$ is given by

$$\langle w, z \rangle := \sum_{s=1}^{n} \overline{w}_s z_s = \sum_{s=1}^{n} (\overline{a}_s c_s + \overline{b}_s d_s) + j \sum_{s=1}^{n} (a_s d_s - b_s c_s) = h_0(z, w) + j \sigma_0(z, w) \tag{2.2.1}$$

where h_0 and σ_0 are the standard hermitian and symplectic form on $\mathbb{H}^n$ viewed as complex vector-space (multiplication on the right). Notice that for every $z, w \in \mathbb{H}^n$ we have

$$h_0(z, wj) = \sigma_0(z, w). \tag{2.2.2}$$

(Notice the analogy with the decomposition of a hermitian positive definite form on a complex vector space $\langle , \rangle$ as $(g_0 - \sqrt{-1}\omega_0)$ where g_0 is a euclidean product and ω_0 is a symplectic real form such that $g_0(iv, w) = \sigma_0(v, w)$). Let $\mathrm{U}(n, \mathbb{H})$ be the group of $\mathbb{H}$-linear automorphisms $f \colon \mathbb{H}^n \to \mathbb{H}^n$ which preserve $\langle , \rangle$. Then

$$\mathrm{U}(n, \mathbb{H}) = \mathrm{U}(2n) \cap \mathrm{Sp}(2n). \tag{2.2.3}$$

In fact the left-hand side is clearly contained in the right-hand side. In order to prove that the right-hand side is contained in the left-hand side it suffices to prove that if $T \in \mathrm{U}(2n) \cap \mathrm{Sp}(2n)$ then $T(vj) = (Tv)j$ for all $v \in \mathbb{H}^n$: that follows easily from (2.2.2). Now suppose that $\mu \in \mathbb{H}$ and that $\mu^2 = -1$. Then *right* multiplication by μ, call it R_μ, defines a complex structure on $\mathbb{H}^n$. Thus we have a family of complex structures on $\mathbb{H}^n$ parametrized by

$$\{(x_1 i + x_2 j + x_3 k) \mid (x_1, x_2, x_3) \in \mathbb{R}^3, \quad x_1^2 + x_2^2 + x_3^2 = 1\} \cong S^2. \tag{2.2.4}$$

Notice that R_μ commutes with $\mathrm{U}(n, \mathbb{H})$ and that it is an isometry for the euclidean product on $\mathbb{H}^n$ defined by

$$(w, z) := \Re \langle w, z \rangle = \sum_{s=1}^{n} (\overline{w}_s z_s + \overline{z}_s w_s). \tag{2.2.5}$$

Moreover every complex structure on $\mathbb{H}^n$ is equal to R_μ for some $\mu \in \mathbb{H}$ such that $\mu^2 = -1$. Now let $p \in X$ and let h be the Calabi-Yau metric. There exists an h_p-orthonormal basis of Θ_p such that the symplectic form σ_p is in standard form, i.e. we may identify h_p and σ_p with h_0 and σ_0 of (2.2.1). By (2.1.1) and (2.2.3) we may identify H_p with the unitary quaternionic group. It follows that there is a well-defined S^2 parametrizing complex structures on $\Theta_p X$ which commute with H_p. Each such complex structure μ is an isometry and is parallel for the Levi-Civita connection of g (because it commutes with H_p), it follows that it defines an integrable complex structure X_μ and g is the real part of a (unique) Kähler hermitian metric for that complex structure. Of course the complex structure X_i is the one we started from (and the corresponding Kähler metric is h), the others are new complex structures. The complex manifolds X_μ fit together: there exist a complex manifold $\mathscr{X}(\omega)$ (diffeomorphic to $X \times S^2$) and a holomorphic map

$$\pi \colon \mathscr{X}(\omega) \to \mathbb{P}^1_{\mathbb{C}} \tag{2.2.6}$$

such thet the fiber of π over μ is isomorphic to X_μ (we identify S^2 with $\mathbb{P}^1_{\mathbb{C}}$ by the obvious procedure), see [33]. The family (2.2.6) is the *twistor fibration* associated to (X, ω) and $\mathscr{X}(\omega)$ is the *twistor space*. The remarkable feature is that we get a global deformation of X starting from the datum of a Kähler class. Given $\mu \in \mathbb{P}^1_{\mathbb{C}}$ the complex manifold has the Kähler form $\omega_\mu(v, w) = g(\mu v, w)$. Since $\mathscr{X}(\omega)$ is diffeomorphic to $X \times S^2$ it makes sense to consider the cohomology class $[\omega_\mu] \in H^2(X; \mathbb{R})$ of ω_μ: as μ varies these classes span a 3-dimensional subspace $H^2_+(X; \mathbb{R})$ and they belong to an $S^2 \subset H^2_+(X; \mathbb{R})$. Similarly we may consider a holomorphic symplectic form σ_μ on X_μ, it is well-defined up to rescaling. Their cohomology classes in $H^2(X; \mathbb{R})$ span $H^2_+(X; \mathbb{R}) \otimes \mathbb{C}$ and the image in the projectivization $\mathbb{P}H^2_+(X; \mathbb{R}) \otimes \mathbb{C}$ is a conic.

2.3. **Deformations are unobstructed.** Let X be a HK manifold. Let σ be a symplectic holomorphic form on X. Contraction of tangent vectors with σ defines an isomorphism of vector-bundles

$$\begin{array}{ccc} \Theta_X & \overset{L_\sigma}{\xrightarrow{\sim}} & \Omega^1_X \\ v & \mapsto & v \lrcorner \sigma. \end{array} \tag{2.3.1}$$

Thus $H^0(\Theta_X) \cong H^0(\Omega^1_X)$ and the latter space vanishes because by definition X is simpy connnected. Thus deformation theory gives that there exists a universal deformation space $\mathrm{Def}(X)$ of X.

Theorem 2.10 (Bogomolov [5])**.** *The deformation space of a HK manifold X is unobstructed.*

Explicitly **Theorem 2.10** asserts the following: There exist a submersive map $f\colon \mathscr{X} \to U$ of complex manifolds and a point $0 \in U$ such that U is a polydisc, $F^{-1}(0) \cong X$ and the Kodaira-Spencer map $\Theta_0 U \to H^1(\Theta_X)$ is an isomorphism.

Remark 2.11. Deformation theory gives that a representative of $\mathrm{Def}(X)$ is the zero-locus of an analytic obstruction map $\Phi\colon B \to H^2(\Theta_X)$ where B is a polydisc of dimension $h^1(\Theta_X)$. Thus Bogomolov's Theorem follows from general deformation theory if $H^2(\Theta_X) = 0$. Notice that by (2.3.1) we have $H^2(\Theta_X) \cong H^2(\Omega^1_X) \cong H^{1,2}(X)$. Thus $H^2(\Theta_X) = 0$ if and only if $b_3(X) = 0$ (recall that $h^{3,0}(X) = 0$ by **Theorem 2.8**). This is the case if X is a deformation of $(K3)^{[n]}$ but not if it is a deformation of a generalized Kummer. Similarly we expect that $H^2(\Theta_X) \neq 0$ if X is our 6-dimensional example of **Theorem ??**.

Proof of **Theorem 2.10** *according to Fujiki* [10]. We must prove tha the obstruction map $\Phi\colon B \to H^2(\Theta_X)$ vanishes. By (2.3.1) we have an isomorphism $H^1(\Theta_X) \cong H^1(\Omega^1_X) \cong H^{1,1}(X)$. Thus we may view $H^{1,1}_{\mathbb{R}}(X)$ as a subspace of $H^1(\Theta_X)$ and $H^1(\Theta_X)$ as the complexification of $H^{1,1}_{\mathbb{R}}(X)$: since Φ is analytic it will suffice to show that the restriction of Φ to $H^{1,1}_{\mathbb{R}}(X)$ vanishes. Let $\mathscr{K}_X \subset H^{1,1}_{\mathbb{R}}(X)$ be the Kähler cone. If $\omega \in \mathscr{K}_X$ then there is a 1-parameter deformation of X whose associated class is equal to ω: in fact this is trivial if $\omega = 0$ and if $\omega \neq 0$ such a family is provided by the twistor family (of course this needs to be proved: follow the variation of a holomorphic symplectic form on the fibers of the twistor family). It follows that the restriction of Φ to $\mathscr{K}_X$ vanishes; since $\mathscr{K}_X$ is open in $H^{1,1}_{\mathbb{R}}(X)$ we get that Φ vanishes on $H^{1,1}_{\mathbb{R}}(X)$. □

Corollary 2.12. *The deformation space of a HK manifold X has dimension equal to $(b_2(X) - 2)$.*

Proof. By **Theorem 2.10** the deformation space of X has dimension $h^1(\Theta_X)$ and the latter equals $h^1(\Omega_X)$ by (2.3.1). Now $h^1(\Omega_X) = h^{1,1}(X)$ and by Hodge Theory $b_2(X) = 2h^{2,0}(X) + h^{1,1}(X)$, thus the corollary follows from $h^{2,0}(X) = 1$. □

Remark 2.13. **Corollary 2.12** shows that if $n \geq 2$ then the generic deformation of $K(3)^{[n]}$ is not isomorphic to $K(3)^{[n]}$. In fact **Corollary 2.12** gives that a $K3$ surface has 20 moduli (the second Betti number of a $K3$ surface equals 22 by Noether's formula) while $K(3)^{[n]}$ has 21 moduli because $b_2(K(3)^{[n]}) = 23$. Similar considerations apply to the other known examples of higher-dimensional (meaning of dimension greater than 2) HK manifolds: they are all obtained starting from a (projective) $K3$ or an abelian surface but a dimension count shows that the generic deformation cannot be obtained by deforming the surface.

3. The local period map and the B-B quadratic form

3.1. **The local period map.** Let $\pi\colon \mathscr{X} \to B$ be a holomorphic submersive map of analytic spaces such that each fiber $X_t := \pi^{-1}(t)$ is a HK manifold. We assume that B is connected and hence all the X_t are deformation equivalent. In particular there exists a finitely generated torsion-free abelian group Λ such that $H^2(X_t;\mathbb{Z})$ is isomorphic to Λ for every $t \in B$. Suppose that the local system $R^2\pi_*\mathbb{Z}$ is trivial, this is the case if B is simply connected. Choose a trivialization of $F\colon R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$; it defines an isomorphism $f_t\colon H^2(X_t;\mathbb{Z}) \xrightarrow{\sim} \Lambda$ for each $t \in B$. Let $\Lambda_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$; abusing notation we denote by f_t also the map $H^2(X_t;\mathbb{C}) \xrightarrow{\sim} \Lambda_{\mathbb{C}}$ obtained by extension of scalars. The period map is defined by

$$\begin{array}{ccc} B & \xrightarrow{\mathscr{P}_\pi} & \mathbb{P}(\Lambda_{\mathbb{C}}) \\ t & \mapsto & f_t(H^{2,0}(X_t)). \end{array} \tag{3.1.1}$$

(Of course $\mathscr{P}_\pi$ depends on the trivialization chosen, our notation is somewhat imprecise.) Fundamental results of Griffiths [39] (valid for arbitrary families of Kähler manifolds) asserts that the period map is holomorphic and computes its differential as follows. Let $0 \in B$. The differential of $\mathscr{P}_\pi$ at 0 has codomain

$$\mathrm{Hom}(\mathscr{P}_\pi(0),\Lambda_{\mathbb{C}}/\mathscr{P}_\pi(0)) \cong \mathrm{Hom}(H^{2,0}(X_0),H^2(X_0)/H^{2,0}(X_0)) = \mathrm{Hom}(H^{2,0}(X_0),H^{1,1}(X_0)) \oplus \mathrm{Hom}(H^{2,0}(X_0),H^{0,2}(X_0)). \tag{3.1.2}$$

Griffiths' first result is that the image of the differential lies in $\mathrm{Hom}(H^{2,0}(X_0), H^{1,1}(X_0))$. The second result expresses the differential in terms of the Kodaira-Spencer map

$$\Theta_0 B \xrightarrow{\kappa} H^1(X_0;\Theta_{X_0}) \tag{3.1.3}$$

associated to the family $\mathscr{X}$. Let σ be a holomorphic symplectic form on X_0 and L_σ be Isomorphism (2.3.1): then

$$\langle d\mathscr{P}_\pi(v), \sigma\rangle = H^1(L_\sigma)(\kappa(v)). \tag{3.1.4}$$

Theorem 3.1 (Infinitesimal Torelli)**.** *Let X be a HK manifold and $\pi: \mathscr{X} \to B$ be a representative of* $\mathrm{Def}(X)$ *with $0 \in B$ the point such that $X_0 \cong X$ - thus B is smooth by* **Theorem 2.10**. *Suppose in addition that $R^2\pi_*\mathbb{Z}$ is trivial and let $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} \Lambda$ be a trivialization of $R^2\pi_*\mathbb{Z}$. Then $\mathscr{P}_\pi: B \to$ $\mathbb{P}(\Lambda_{\mathbb{C}})$ is an isomorphism of a neighborhood of* 0 *onto a smooth analytic hypersurface in a neighborhood of $\mathscr{P}_\pi(0)$.*

Proof. By **Corollary 2.12** we get that $\dim_0 B = b_2(X) - 2 = \dim \mathbb{P}(\Lambda_{\mathbb{C}}) - 1$. Thus it suffices to prove that the differential $d\mathscr{P}_\pi(0)$ is injective. That follows immediately from (3.1.4). □

3.2. **The Bogomolov-Beauville quadratic form.** Let X be a HK-manifold of dimension $2n$. Beauville [4] and Fujiki [10] proved that there exist an integral indivisible quadratic form

$$q_X: H^2(X) \to \mathbb{C} \tag{3.2.1}$$

(cohomology is with complex coefficients) and $c_X \in \mathbb{Q}_+$ such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X(\alpha)^n, \qquad \alpha \in H^2(X). \tag{3.2.2}$$

The above equation determines c_X and q_X with no ambiguity unless n is even. If n is even then q_X is determined up to ±1: one singles out one of the two choices by imposing the inequality

$$q_X(\sigma + \overline{\sigma}) > 0, \qquad 0 \neq \sigma \in H^{2,0}(X). \tag{3.2.3}$$

The *Beauville-Bogomolov* form and the *Fujiki constant* of X are q_X and c_X respectively. We notice that the equation in (3.2.2) is equivalent (by polarization) to

$$\int_X \alpha_1 \wedge \ldots \wedge \alpha_{2n} = c_X \sum_{\sigma \in \mathscr{R}_{2n}} (\alpha_{\sigma(1)}, \alpha_{\sigma(2)})_X \cdot (\alpha_{\sigma(3)}, \alpha_{\sigma(4)})_X \cdot \ldots \cdot (\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})_X \tag{3.2.4}$$

where $(\cdot,\cdot)_X$ is the symmetric bilinear form associated to q_X and $\mathscr{R}_{2n}$ is a set of representatives for the left cosets of the subgroup $\mathscr{G}_{2n} < \mathscr{S}_{2n}$ of permutations of $\{1,\ldots,2n\}$ generated by transpositions $(2i-1,2i)$ and by products of transpositions $(2i-1,2j-1)(2i,2j)$ - in other words in the right-hand side of (3.2.4) we avoid repeating addends which are formally equal. In defining c_X we have introduced a normalization which is not standard in order to avoid a combinatorial factor in Equation (3.2.4).

Proof of existence of q_X and c_X. Let $\pi: \mathscr{X} \to B$ be a deformation of X representing $Def(X)$ with $0 \in B$ and $X_0 \xrightarrow{\sim} X$. By **Theorem 2.10** we know that B is smooth at 0. We may assume that B is contractible and hence there exists a trivialization $F: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} B \times \Lambda$ where Λ is a finitely generated torsion-free abelian group. Let $\mathscr{P}_\pi$ be the period map (3.1.1). By Infintesimal Torelli, see **Theorem 3.1**, $\mathrm{Im}\,\mathscr{P}_\pi$ is an analytic hypersurface in an open (classical topology) neighborhood of $\mathscr{P}_\pi(0)$ and hence its Zariski closure $V = \overline{Im\pi}$ is either all of $\mathbb{P}(H^2(X))$ or a hypersurface. One shows that the latter holds by considering the (non-zero) degree-$2n$ homogeneous polynomial

$$\begin{array}{ccc} H^2(X) & \xrightarrow{G} & \mathbb{C} \\ \alpha & \mapsto & \int_X \alpha^{2n} \end{array} \tag{3.2.5}$$

In fact if $\sigma_t \in H^{2,0}(X_t)$ then

$$\int_{X_t} \sigma_t^{2n} = 0 \tag{3.2.6}$$

by type consideration. It follows by Gauss-Manin parallel transport that G vanishes on V. Thus $I(V) = (F)$ where F is an irreducible homogeneous polynomial. By considering the derivative of the period map (3.1.1) one checks easily that V is not a hyperplane and hence $\deg F \geq 2$. On the other hand type consideration gives something stronger than (3.2.6), namely

$$\int_{X_t} \sigma_t^{n+1} \wedge \alpha_1 \cdots \wedge \alpha_{n-1} = 0 \qquad \alpha_1, \ldots, \alpha_{n-1} \in H^2(X_t). \tag{3.2.7}$$

It follows that all the derivatives of G up to order $(n-1)$ included vanish on V. Since $\deg G = 2n$ and $\deg F \geq 2$ it follows that $G = c \cdot F^n$ and $\deg F = 2$. By integrality of G there exists $\lambda \in \mathbb{C}^*$ such that $c_X := \lambda c$ is rational positive, $q_X := \lambda \cdot F$ is integral indivisible and (3.2.2) is satisfied. □

Remark 3.2. Let X be a HK manifold of dimension $2n$ and $\omega \in H^{1,1}_{\mathbb{R}}(X)$ be a Kähler class.

(1) Equation (3.2.2) gives that with respect to $(,)_X$ we have

$$H^{p,q}(X) \bot H^{p',q'}(X) \text{ unless } (p',q') = (2-p, 2-q). \tag{3.2.8}$$

(2) $q_X(\omega) > 0$. In fact let σ be generator of $H^{2,0}(X)$; by Equation (3.2.4) and Item (1) above we have

$$0 < \int_X \sigma^{n-1} \wedge \overline{\sigma}^{n-1} \wedge \omega^2 = c_X (n-1)! (\sigma, \overline{\sigma})_X q_X(\omega). \tag{3.2.9}$$

Since $c_X > 0$ and $(\sigma, \overline{\sigma})_X > 0$ we get that $q_X(\omega) > 0$ as claimed.

(3) The index of q_X is $(3, b_2(X) - 3)$ (i.e. that is the index of its restriction to $H^2(X;\mathbb{R})$). In fact applying Equation (3.2.4) to $\alpha_1 = \ldots = \alpha_{2n-1} = \omega$ and arbitrary α_{2n} we get that $\omega^{\bot}$ is equal to the primitive cohomology $H^2_{pr}(X)$ (primitive with respect to ω). On the other hand Equation (3.2.4) with $\alpha_1 = \ldots = \alpha_{2n-2} = \omega$ and $\alpha_{2n-1}, \alpha_{2n} \in \omega^{\bot}$ gives that a positive multiple of $q_X|_{\omega^{\bot}}$ is equal to the standard quadratic form on $H^2_{pr}(X)$. By the Hodge index Theorem it follows that the restriction of q_X to $\omega^{\bot} \cap H^2(X;\mathbb{R})$ has index $(2, b_2(X) - 3)$. Since $q_X(\omega) > 0$ it follows that q_X has index $(3, b_2(X) - 3)$.

(4) Let D be an effective divisor on X; then $(\omega, D)_X > 0$. In fact the inequality follows from the inequality $\int_D \omega^{2n-1} > 0$ together with (3.2.4) and Item (2) above.

(5) Let $f\colon X \dashrightarrow Y$ be a birational map where Y is a HK manifold. Since X and Y have trivial canonical bundle f defines an isomorphism $U \xrightarrow{\sim} V$ where $U \subset X$ and $V \subset Y$ are open sets with complements of codimension at least 2. It follows that f induces an isomorphism $f^*\colon H^2(Y;\mathbb{Z}) \xrightarrow{\sim} H^2(X;\mathbb{Z})$; f^* is an isometry of lattices, see Lemma 2.6 of [14].

Of course if X is a $K3$ then q_X is the intersection form of X (and $c_X = 1$). In general q_X gives $H^2(X;\mathbb{Z})$ a structure of lattice just as in the well-known case of $K3$ surfaces. Suppose that X and Y are deformation equivalent HK-manifolds: it follows from (3.2.2) that $c_X = c_Y$ and the lattices $H^2(X;\mathbb{Z}), H^2(Y;\mathbb{Z})$ are isometric (see the comment following (3.2.2) if n is even). The Fujiki constant and B-B quadratic form of the known HK manifolds of dimension greater than 2 are given in Table (1). A word about notation: H is the hyperbolic lattice i.e. $H \cong \mathbb{Z}^2$ with a basis e, f such that $0 = (e,e) = (f,f)$ and $(e,f) = 1$, $E_8(-1)$ is the unique negative definite even unimodular lattice of rank 8 (a root system of type E_8 gives a basis of $E_8(-1)$, provided we change sign to every product of roots), $A_2(-1)$ is given by the root system A_2 with signs changed, and for $d \in \mathbb{Z}$ we let (d) be the rank-1 lattice with generator of square d. For $(K3)^{[n]}$ and $K^{[n]}(T)$ the result is folklore, for the 6 and 10-dimensional examples of O'Grady the proofs are due to Rapagnetta [32].

Remark 3.3. Let X be a HK manifold of dimension $2n$. Existence of the B-B quadratic form and Fujiki constant is a rather strong topological condition. Salamon [34] proved the following relation between Betti numbers of X:

$$n b_{2n}(X) = 2 \sum_{i=1}^{2n} (-1)^i (3i^2 - n) b_{2n-i}(X). \tag{3.2.10}$$

Is it possible to obtain other topological constraints on HK manifolds? In particular: can we bound rank, discriminant of the B-B quadratic form and Fujiki constant in a given dimension? That would give that the number of deformation classes of a given dimension is finite, see [16] for related work. Salamon's relation (3.2.10) gives (Beauville (unpublished) and Guan [13]) that if X is a HK 4-fold then $b_2(X) \leq 23$ (notice that $b_2(K3^{[2]}) = 23$) and that if equality holds then cup-product defines an isomorphism $\mathrm{S}^2 H^2(X;\mathbb{Q}) \xrightarrow{\sim} H^4(X;\mathbb{Q})$. Guan [13] has obtained other restrictions on $b_2(X)$ for a HK four-fold X: for example either $b_2(X) \leq 8$ or $b_2(X) = 23$.

The proof of existence of q_X and c_X may be adapted to prove the following useful generalization of (3.2.2).

TABLE 1. Fujiki constant and B-B form of the known examples of dim > 2.

X	$\dim(X)$	$b_2(X)$	c_X	$H^2(X,\mathbb{Z})$
$(K3)^{[n]}$	$2n$	23	1	$H^3 \oplus E_8(-1)^2 \oplus (-2(n-1))$
$K^{[n]}(T)$	$2n$	7	$(n+1)$	$H^3 \oplus (-2(n+1))$
$\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$	10	24	1	$H^3 \oplus E_8(-1)^2 \oplus A_2(-1)$
$\widetilde{\mathscr{M}}_T(2\mathbf{v}_0)^0$	6	8	4	$H^3 \oplus (-2)^2$

Proposition 3.4. *Let X be a HK manifold of dimension $2n$. Let $\mathscr{X} \to T$ be a representative of the deformation space of X. Suppose that $0 \neq \gamma \in H^{p,p}(X)$ is a class which remains of type (p,p) under Gauss-Manin parallel transport (e.g. the Chern class $c_p(X)$).*

(1) *If p is odd then*
$$\int_X \gamma \wedge \alpha^{2n-p} = 0 \qquad \forall \alpha \in H^2(X). \tag{3.2.11}$$
(2) *If p is even there exists $c_\gamma \in \mathbb{R}$ such that*
$$\int_X \gamma \wedge \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2} \qquad \forall \alpha \in H^2(X). \tag{3.2.12}$$

3.3. **Matsushita's Theorem.** Matsushita's **Theorem 3.5** is a perfect illustration of how one may use the B-B quadratic form in order to get strong geometric results. Let X be a HK manifold. A subvariety $Y \subset X$ is *lagrangian* if $2\dim Y = \dim X$ and the restriction to the smooth locus of Y of a holomorphic symplectic form on X is zero.

Theorem 3.5 (Matsushita [25, 26])**.** *Suppose that X is a HK manifold and that $f\colon X \to B$ is a surjective map with connected fibers to a Kähler manifold B such that $0 < \dim B < \dim X$. Then $2\dim B = \dim X$ and the generic fiber of f is a lagrangian torus in X. Moreover $b_2(B) = 1$.*

Proof. Suppose that $0 \neq \alpha \in H^0(\Omega^2_B)$. Then $f^*\alpha$ is a non-zero degenerate holomorphic 2-forms on X, that contradicts the definition of HK manifold. Thus $H^{2,0}(B) = 0$ and hence B is a smooth projective variety. Let $\dim X = 2n$ and $\dim B = m$. If $\alpha \in H^2(B)$ then $\int_X f^*\alpha^{2n} = 0$ because $m < 2n$ and hence $q_X(f^*\alpha) = 0$ by (3.2.2). Let H be an ample divisor on B and $\alpha := c_1(\mathscr{O}_B(H))$. Let ω be a Kähler form on X: then
$$\int_X (f^*\alpha)^m \wedge \omega^{2n-m} > 0. \tag{3.3.1}$$
Suppose that $m > n$: since $q_X(\alpha) = 0$ Equation (3.2.2) gives that the left-hand side of (3.3.1) vanishes, that contradicts (3.3.1). This proves that $m \le n$. Next notice that $f^*\alpha \neq 0$ because H is ample (and $m > 0$). Since the B-B form is non-degenerate there exists $\beta \in H^2(X)$ such that $0 \neq (f^*\alpha, \beta)_X$. Since $q_X(f^*\alpha) = 0$ Equation (3.2.2) gives that
$$\int_X (f^*\alpha)^n \wedge \omega^n = n!(f^*\alpha, \beta)^n_X \neq 0. \tag{3.3.2}$$
It follows that $m \ge n$. Thus $m = n$. Let $b \in B$ be generic and $X_b := f^{-1}(b)$. Then $2\dim X_b = \dim X$ because $2\dim B = \dim X$. Let's prove that X_b is lagrangian. Letσ be a holomorphic symplectic form on X: it suffices to prove that
$$\int_{X_b} \sigma \wedge \overline{\sigma} \wedge \omega^{n-2} = 0 \tag{3.3.3}$$
where ω is a Kähler form on X. Let H be an ample divisor on B and $\alpha := c_1(\mathscr{O}_B(H))$: then
$$\int_X (f^*\alpha)^n \wedge \sigma \wedge \overline{\sigma} \wedge \omega^{n-2} = \deg(\underbrace{H \cdot \ldots \cdot H}_{n}) \int_{X_b} \sigma \wedge \overline{\sigma} \wedge \omega^{n-2}. \tag{3.3.4}$$
On the other hand Equation (3.2.2) gives that the left-hand side of (3.3.4) vanishes because $q_X(f^*\alpha) = 0$ and $0 = (f^*\alpha, \sigma)_X = (f^*\alpha, \overline{\sigma})_X$ (see **Remark 3.2**). This proves (3.3.3), thus X_b is lagrangian. Since X_b is lagrangian the symplectic form defines an isomorphism between the tangent bundle of X_b and the conormal of X_b in X: the latter is trivial because X_b is a regular fiber of f. Hence the tangent bundle of X_b is trivial and therefore X_b is a torus. Lastly we notice that since B is a smooth projective variety

the pull-back $H^2_{\mathbb{R}}(f)\colon H^2(B;\mathbb{R})\to H^2(X;\mathbb{R})$ is injective. We have proved that the image is isotropic for the B-B quadratic form. The image of $H^2_{\mathbb{R}}(f)$ lies in $H^{1,1}_{\mathbb{R}}(X)$. By **Remark 3.2** the restriction of q_X to $H^{1,1}_{\mathbb{R}}(X)$ has signature $(1,b_2(X)-3)$ and hence the maximum dimension of an isotropic subspace is 1: it follows that $b_2(B)=1$. □

Example 3.6. Let S be a $K3$ surface and $\mathbf{v}\in\widetilde{H}(S)$ be a Mukai vector as in (**??**) with $r=0$, i.e. $\mathbf{v}=\ell+\eta$. Assume that $\mathscr{M}_S(\mathbf{v})$ is not empty and that $\mathscr{M}_S(\mathbf{v})=\mathscr{M}_S(\mathbf{v})^{st}$ so that $\mathscr{M}_S(\mathbf{v})$ is a HK variety. Let's suppose also that $\ell=c_1(\mathscr{O}_S(D))$ where D is an ample divisor on S. Then $\mathscr{M}_S(\mathbf{v})$ has a lagrangian fibration $f\colon\mathscr{M}_S(\mathbf{v})\to|D|$ defined by mapping $[E]\in\mathscr{M}_S(\mathbf{v})$ to the schematic support of E. Let us check that $2\dim|D|=\dim\mathscr{M}_S(\mathbf{v})$. Let g be the arithmetic genus of curves in $|D|$. Then $\dim|D|=g$. On the other hand if $C\in|D|$ is smooth the fiber of f over C is identified with $\mathrm{Pic}^d(C)$ where d is determined by the Mukai vector $\mathbf{v}$. Thus we see that $\dim f^{-1}(C)=g=\dim|D|$.

Remark 3.7. In the examples above the base of the lagrangian fibration is isomorphic to a projective space. Hwang [18] has proved that if X is projective then B is indeed isomorphic to a projective space. It is conjectured (e.g. [35]) that if X is a HK manifold and $0\neq\gamma\in H^{1,1}_{\mathbb{Z}}(X)$ has square 0 and is nef then there exists a lagrangian fibration $X\to B$ such that $\gamma\in f^*H^2(B)$. See [24, 36, 23] for work on this conjecture.

3.4. **The period domain.** Existence of the Beauville-Bogomolov quadratic form gives that periods of HK manifolds belong to certain open subsets of smooth quadric hypersurfaces. In the present subsection we give the relevant details and we point out a consequence noticed by Beauville, namley that HK's are deformations of projective varieties. Let Λ be a lattice i.e. a finitely generated torsion-free abelian group equipped with a non-degenerate symmetric bilinear form $(,)_\Lambda$. We let q_Λ be the associated quadratic form. For a commutative ring R let $\Lambda_R:=\Lambda\otimes_{\mathbb{Z}}R$. Then $(,)_\Lambda$ extends to an R-valued bilinear symmetric form on Λ_R; abusing notation we will denote it by $(\alpha,\beta)_\Lambda$ or simply (α,β) if there is no risk of confusion. We let q_Λ be the associated quadratic form on Λ_R. Now assume that $\Lambda_{\mathbb{R}}$ has signature $(3,\mathrm{rk}\,\Lambda-3)$. We let

$$\Omega_\Lambda:=\{[\alpha]\in\mathbb{P}(\Lambda_{\mathbb{C}})|\quad q_\Lambda(\alpha)=0,\quad q_\Lambda(\alpha+\bar{\alpha})>0\}. \tag{3.4.1}$$

Thus Ω_Λ is an open subset (in the classical topology) of a smooth quadric of dimension $(\mathrm{rk}\,\Lambda-2)$ and hence it is naturally a complex manifold: it is the *period domain* associated to Λ. Up to isomorphism Ω_Λ depends only on the rank of Λ - on the other hand the orthogonal group $O(\Lambda)$ acts naturally on Ω_Λ and the different realizations of Ω_Λ correspond to different group actions. Now let X be a HK manifold and Λ a lattice isometric to $H^2(X;\mathbb{Z})$ equipped with the B-B quadratic form. By **Remark 3.2** the signature of the B-B quadratic form on $H^2(X;\mathbb{R})$ is $(3,b_2(X)-3)$ and hence the period domain Ω_Λ is defined.

Definition 3.8. Let X be a HK manifold deformation equivalent to X_0. A *marking* of X consists of an isometry $f\colon H^2(X;\mathbb{Z})\xrightarrow{\sim}\Lambda$. A *marked pair* is a couple (X,f) where X is a HK manifold (deformation equivalent to X_0) and f is a marking of X. An isomorphism between marked pairs (X,f) and (Y,g) is an isomorphism $\varphi\colon X\xrightarrow{\sim}Y$ such that $f\circ H^2(\varphi)=fg$.

Let (X,f) be a marked pair. We denote by f the linear map $H^2(X;\mathbb{C})\xrightarrow{\sim}\Lambda_{\mathbb{C}}$ obtained by extension of scalars. Then $f(H^{2,0}(X))\in\mathbb{P}(\Lambda_{\mathbb{C}})$. By (3.2.2) we have that $H^{2,0}(X)$ is an isotropic line for the B-B quadratic form and moreover $(\sigma,\overline{\sigma})_X>0$ for $0\neq\sigma\in H^{2,0}(X)$ by (3.2.3). Since f is an isometry it follows that

$$P(X,f):=f(H^{2,0}(X))\in\Omega_\Lambda. \tag{3.4.2}$$

The point $P(X,f)$ is the *period point* associated to the marked pair (X,f). Now let $\pi\colon\mathscr{X}\to B$ be a holomorphic submersive map of analytic spaces such that each fiber $X_b:=\pi^{-1}(b)$ is a HK manifold deformation equivalent to X_0. Suppose that the local system $R^2\pi_*\mathbb{Z}$ is trivial. In order to define the period map (3.1.1) we choose a trivialization $F\colon R^2\pi_*\mathbb{Z}\xrightarrow{\sim}B\times\Lambda$ defining an isometry $f_b\colon H^2(X_b;\mathbb{Z})\xrightarrow{\sim}\Lambda$ for each $b\in B$. Then $\mathscr{P}_\pi(B)\subset\Omega_\Lambda$. **Theorem 3.1** gives the following result.

Theorem 3.9 (Infinitesimal Torelli + Local surjectivity)**.** *Keep notation as above and suppose that $\pi\colon\mathscr{X}\to B$ is a representative of* $\mathrm{Def}(X)$ *with* $0\in B$ *the point such that* $X_0\cong X$ *- thus B is smooth by* **Theorem 2.10**. *Then $\mathscr{P}_\pi\colon B\to\Omega_\Lambda$ is an isomorphism of a neighborhood of* 0 *onto an open neighborhood of $\mathscr{P}_\pi(0)$ in Ω_Λ.*

Corollary 3.10. *A HK manifold is deformation equivalent to a HK variety.*

Proof. Let X be a HK manifold and $\pi: \mathscr{X} \to B$ be a representative of $\mathrm{Def}(X)$ with $0 \in B$ the point such that $X_0 \cong X$. There exist points $[\sigma] \in \Omega_\Lambda$ arbitrarily close to $\mathscr{P}_\pi(0)$ such that the span of $\{\sigma, \overline{\sigma}\}$ in $\Lambda_\mathbb{C}$ is defined over $\mathbb{Q}$ i.e. it is spanned by its intersection with $\Lambda_\mathbb{Q}$. By **Theorem 3.9** there exists $b \in B$ such that $\mathscr{P}_\pi(b)$ is such a $[\sigma]$. Then $H^{1,1}(X_b;\mathbb{R})$ is also defined over $\mathbb{Q}$ and hence $H^{1,1}(X_b;\mathbb{Q})$ is dense in $H^{1,1}(X_b;\mathbb{R})$; since the Kähler cone is open in $H^{1,1}(X_b;\mathbb{R})$ it follows that there exists a Kähler integral class and hence X_b is projective by Kodaira. □

4. The Kähler cone

4.1. **The main result.** We recall that the Kähler cone of a $K3$ surface S is described as follows. Let $\omega \in H^{1,1}_\mathbb{R}(S)$ be one Kähler class and $\mathscr{N}_S$ be the set of nodal classes

$$\mathscr{N}_S := \{\delta \in H^{1,1}_\mathbb{Z}(X) \mid q_S(\delta) = -2, \quad (\delta,\omega)_S > 0\} \tag{4.1.1}$$

(Every class in $\mathscr{N}_S$ is represented by an effective divisor - this follows from the (Atiyah-Singer) Riemann-Roch formula.) The Kähler cone $\mathscr{K}_S$ is given by

$$\mathscr{K}_S := \{\alpha \in H^{1,1}_\mathbb{R}(S) \mid q_S(\alpha) > 0, \quad (\alpha,\delta)_S > 0 \quad \forall \delta \in \mathscr{N}_S\}. \tag{4.1.2}$$

In other words we have a Hodge-theoretic description of $\mathscr{K}_S$. In fact Hodge isometries of $H^2(X)$ act transitively on the set of connected components of the complement of the union of walls $\delta^\perp \cap H^{1,1}_\mathbb{R}(X)$ where $\delta \in H^{1,1}_\mathbb{Z}(X)$ has square -2, hence the choice of ω is needed only to pin-down which is the open chamber containing Kähler classes. If X is a general HK manifold there exists a characterization of the Kähler cone $\mathscr{K}_X$ due to Huybrechts and Boucksom which is not purely Hodge-theoretic. In order to state the result by Huybrechts and Boucksom we must introduce the positive cone. Let $\widetilde{\mathscr{C}}_X \subset H^{1,1}_\mathbb{R}(X)$ be the set of classes of strictly positive square; it has two connected components because the restriction to $H^{1,1}_\mathbb{R}(X)$ of the B-B quadratic form has signature $(1, b_2(X) - 3)$. The Kähler cone $\mathscr{K}_X$ is an open convex subset of $\widetilde{\mathscr{C}}_X$ (see Item (2) of **Remark 3.2**), thus there is a well-determined connected component of $\widetilde{\mathscr{C}}_X$ containing $\mathscr{K}_X$: this is the *positive cone* $\mathscr{C}_X \subset H^{1,1}_\mathbb{R}(X)$.

Theorem 4.1. [Huybrechts [15]+Boucksom [6]] *Let X be a HK manifold. A class $\alpha \in H^{1,1}_\mathbb{R}(X)$ is Kähler if and only if it belongs to the positive cone $\mathscr{C}_X$ and moreover $\int_C \alpha > 0$ for every rational curve C (a curve is rational if it is irreducible and its normalization is rational).*

One spectacular offspring of the methods leading to the proof of **Theorem 4.5** is the following result.

Theorem 4.2. *Let X and Y be bimeromorphic HK manifolds. Then X and Y are deformation equivalent.*

We will sketch part of the arguments which lead to the proof of **Theorem 4.1**.

4.2. **The Kähler cone of a general deformation.** Let X be a HK. We will prove that for a general deformation of X the Kähler cone equals the positive cone. Let $\pi: \mathscr{X} \to B$ be a representative of the deformation space of X; for $t \in B$ we let $X_t := \pi^{-1}(t)$ and we assume that $0 \in B$ is the base point, hence $X_0 \cong X$. We assume that B is simply connected and hence the Gauss-Manin connection gives an identification

$$H^*(X;\mathbb{Q}) \cong H^*(X_t;\mathbb{Q}) \qquad \forall t \in B. \tag{4.2.1}$$

Given $\gamma \in H^{2p}(X)$ we let

$$B_\gamma := \{t \in B \mid \ \gamma \in H^{p,p}(X_t;\mathbb{Q})\}. \tag{4.2.2}$$

(The above makes sense by (4.2.1).) Then B_γ is a an analytic subset of B. Let

$$E(B) := \bigcup_{B_\gamma \neq B} B_\gamma. \tag{4.2.3}$$

Thus $E(B)$ is a countable union of *proper* analytic subsets of B.

Proposition 4.3 (Huybrechts, Erratum of [14])**.** *If $t \in (B \setminus E(B))$ then $\mathscr{K}_{X_t} = \mathscr{C}_{X_t}$.*

Proof. This is a consequence of **Proposition 3.4** and the Kähler version of Nakai-Moishezon's ampleness criterion proved by Demailly and Paun [8]. In fact Demailly and Paun tell us that $\mathscr{K}_{X_t}$ is a connected component of the set

$$\mathscr{P}_{X_t} := \left\{\alpha \in H^{1,1}_\mathbb{R}(X) \mid \ \int_\Gamma \alpha^m > 0 \qquad \Gamma \subset X_t \text{ a closed subset of dim} = m\right\}.$$

Thus it suffices to prove that $\mathscr{C}_{X_t} \subset \mathscr{P}_{X_t}$. Let $\omega \in H^2(X_t)$ be a Kähler class. Let $\Gamma \subset X_t$ be a closed subset of $\dim = m$. Then

$$\int_\Gamma \omega^m > 0. \tag{4.2.4}$$

Now let $\alpha \in H^2(X_t)$; since $t \notin E(B)$ we have

$$\int_\Gamma \alpha^m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ c_\Gamma q_{X_t}(\alpha)^{m/2} & \text{if } m \text{ is even} \end{cases}$$

by **Proposition 3.4**. Comparing with (4.2.4) we get that m is even and that $c_\Gamma > 0$. Thus $\int_\Gamma \alpha^m > 0$ for every $\alpha \in \mathscr{C}_{X_t}$. □

Example 4.4. Let S be a $K3$ surface and $\mathscr{X} \to B$ be a representative of the deformation space of S. Let $\mathscr{N}(B) \subset B$ be the set of $t \in B$ such that $H^{1,1}_{\mathbb{Z}}(X_t)$ contains a nodal class i.e. such that there exists $\delta \in H^{1,1}_{\mathbb{Z}}(X_t)$ whose square is (-2). Then $\mathscr{N}(B)$ is a countable union of proper closed subsets of B and by the description of the Kähler cone of a $K3$ given in **Subsection 4.1** we get that $\mathscr{K}_{X_t} = \mathscr{C}_{X_t}$ if and only if $t \in (B \setminus \mathscr{N}(B))$.

Proposition 4.3 is the main ingredient in Huybrechts' proof of the following result.

Theorem 4.5 (Projectivity criterion [14])**.** *A HK manifold X is projective if and only if there exists a class $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ of strictly positive square.*

Notice that if X is projective and $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ is an ample class then $q_X(\alpha) > 0$ by (3.2.2). The non-trivial statement of **Theorem 4.5** is that if there exists $\alpha \in H^{1,1}_{\mathbb{Z}}(X)$ of strictly positive square then X is projective - of course this does not mean that one among $\pm\alpha$ is an ample class (unless $h^{1,1}_{\mathbb{Z}}(X) = \mathbb{Z}$).

4.3. **Fake twistor families.** Let X be a HK and $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ a Kähler class: then we have the twistor family $\mathscr{X}(\alpha) \to \mathbb{P}^1_{\mathbb{C}}$. One can associate an analogue of the twistor family to an arbitrary class $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ of strictly positive square: we may call it a *fake twistor family.* A key ingredient in the work of Huybrechts [14, 15] is to compare a fake twistor family to an actual twistor family parametrized by the same base with the property that they are isomorphic over a non-empty open subset of the base. A fake twistor family associated to $\alpha \in H^{1,1}_{\mathbb{R}}(X)$ of strictly positive square is defined as follows. Let $\pi\colon \mathscr{X} \to B$ be a representative of the deformation space of X and keep notation and assumptions of the previous subsection. Let Λ be a lattice isometric to $H^2(X;\mathbb{Z})$: we have the local period map

$$\mathscr{P}_\pi\colon B \longrightarrow V(q_\Lambda) \subset \mathbb{P}(\Lambda_{\mathbb{C}}). \tag{4.3.1}$$

Let $F(\alpha) \subset \Lambda_{\mathbb{C}}$ be defined by

$$F(\alpha) := \mathbb{C}\mathscr{P}(0) \oplus \mathbb{C}\overline{\mathscr{P}(0)} \oplus \mathbb{C}\alpha. \tag{4.3.2}$$

The B-B form on $F(\alpha)$ is non-degenerate and hence $\mathbb{P}(F(\alpha)) \cap V(q_\Lambda)$ is a smooth conic. Let

$$T(\alpha) := \mathscr{P}^{-1}(\mathbb{P}(F(\alpha)) \cap V(q_\Lambda)). \tag{4.3.3}$$

Thus $T(\alpha)$ is a smooth curve in B. Let $\mathscr{X}(\alpha) \to T(\alpha)$ be the restriction of $\pi\colon \mathscr{X} \to B$ to $T(\alpha)$ (a.k.a. the fibered product...). We identify Λ with $H^2(X_t;\mathbb{Z})$ for arbitrary $t \in B$ via the Gauss-Manin connection; in particular it makes sense to speak of the positive cone $\mathscr{C}_t \subset (\mathscr{P}(0)^\perp \cap \overline{\mathscr{P}(0)}^\perp \cap \Lambda_{\mathbb{R}})$: this is the cone which gets identified with the positive cone $\mathscr{C}_{X_t}$. Let $\omega_t \in H^{1,1}_{\mathbb{R}}(X_t)$ be a class such that

$$0 \neq \omega_t \in \{\xi \in F(\alpha) \cap \mathscr{C}_t \mid \ \xi \perp \mathscr{P}(t),\ \xi \perp \overline{\mathscr{P}(t)}\}. \tag{4.3.4}$$

Notice that ω_t is well-defined up to multiplication by a positive scalar and that ω_0 is a multiple of α. If α is a Kähler class then $\mathscr{X}(\alpha) \to T(\alpha)$ is obtained by base change from the twistor family associated to α and ω_t is a corresponding Kähler class on X_t. If α is not Kähler then $\mathscr{X}(\alpha) \to T(\alpha)$ is a fake twistor family. (As a matter of terminology we call $\mathscr{X}(\alpha) \to T(\alpha)$ a fake twistor family even if α is a Kähler class.)

Proposition 4.6. *Keep notation and assumptions as above and suppose that there exists $\bar{t} \in T(\alpha)$ such that $\omega_{\bar{t}}$ is a Kähler class. Then there exists a countable subset $J(\alpha) \subset T(\alpha)$ such that ω_t is Kähler for $t \in (T(\alpha) \setminus J(\alpha))$.*

Proof. Suppose that $0 \neq \gamma \in H^{p,p}(X)$ is a class which remains of type (p,p) under Gauss-Manin parallel transport *along* $T(\alpha)$. Then for all $t \in T(\alpha)$ we have

(1) if p is odd

$$\int_{X_t} \gamma \wedge \alpha^{2n-p} = 0 \qquad \forall \alpha \in H^2(X_t), \tag{4.3.5}$$

(2) if p is even there exists $c_\gamma \in \mathbb{R}$ such that

$$\int_{X_t} \gamma \wedge \alpha^{2n-p} = c_\gamma q_{X_t}(\alpha)^{n-p/2} \qquad \forall \alpha \in H^2(X_t). \tag{4.3.6}$$

In fact the proof is similar to the proof of the existence of the B-B quadratic form and of **Proposition 3.4**. Arguing as in the proof of **Proposition 4.3** we get that if in addition $\gamma \in H^{p,p}_{\mathbb{Q}}(X)$ and is effective *for all* $t \in T(\alpha)$ then p is even and

$$\int_{X_t} \gamma \wedge \alpha^{2n-p} > 0 \qquad \forall \alpha \in \mathscr{C}_t. \tag{4.3.7}$$

In fact the case of p odd is ruled out because on one hand $\int_{X_{\overline{t}}} \gamma \wedge \omega_{\overline{t}}^{2n-p} = 0$ by Item (1) and on the other hand $\int_{X_{\overline{t}}} \gamma \wedge \omega_{\overline{t}}^{2n-p} > 0$ because γ is effective and $\omega_{\overline{t}}$ is a Kähler class. A similar argument shows that if p is even then $c_\gamma > 0$ and hence (4.3.7) holds. Now let $J(\alpha) \subset T(\alpha)$ be the subset of points such that there exists an *effective* $\gamma \in H^{p,p}_{\mathbb{Q}}(X_t)$ which does not extend to an *effective* class for all $t \in T(\alpha)$. Then $J(\alpha)$ is a discrete subset of the curve $T(\alpha)$ and hence is countable. By the argument above ω_t is Kähler for $t \in (T(\alpha) \setminus J(\alpha))$. □

Example 4.7. Let S be a $K3$ surface. Let $W_S \subset \mathscr{C}_S$ be the union of all walls $\delta^\perp \cap \mathscr{C}_S$ for $\delta \in H^{1,1}_{\mathbb{Z}}(S)$ of square (-2). Let $\alpha \in \mathscr{C}_S$: there exists $t \in T(\alpha)$ such that $\omega_{\overline{t}}$ is a Kähler class if and only if $\alpha \notin W_S$.

Starting from the Projectivity Criterion i.e. **Theorem 4.5** Huybrechts proved the following result.

Proposition 4.8 (p.97 of [14])**.** *Let X be a HK and $\alpha \in \mathscr{C}_X$ a general point i.e. belonging to the complement of a countable union of closed nowhere dense subsets of $\mathscr{C}_X$. Then there exists $\overline{t} \in T(\alpha)$ such that $\omega_{\overline{t}}$ is a Kähler class.*

Now let $\alpha \in \mathscr{C}_X$ be a general point as in **Proposition 4.8**, and hence there exists $\overline{t} \in T(\alpha)$ such that $\omega_{\overline{t}}$ is a Kähler class. Let $U(\alpha) \subset T(\alpha)$ be the open subset of t such that ω_t is a Kähler class. The complement of $U(\alpha)$ (in $T(\alpha)$) is countable by **Proposition 4.6**; it follows that $0 \in \overline{U(\alpha)}$. Now let $\overline{t} \in U(\alpha)$ and let $\mathscr{X}(\omega_{\overline{t}}) \to \mathbb{P}^1$ be the (true) twistor family associated to the Kähler class $\omega_{\overline{t}}$ on $X_{\overline{t}}$. The period map identifies an open subset of the base $\mathbb{P}^1$ with $T(\alpha)$, taking the inverse image of that open set we get a twistor family $\mathscr{X}(\omega_{\overline{t}})' \to T(\alpha)$ with the property that $X_t \cong X'_t$ for every $t \in U(\alpha)$. Denote X'_0 by X' and let $\Gamma_0 \subset X \times X'$ be the limit for $t \to 0$ of the graph of an isomorphism $X_t \xrightarrow{\sim} X'_t$ for $t \in U(\alpha)$ (one must choose the isomorphism to be compatible with the chosen trivializations of the local systems with fibers $H^2(X_t)$ and $H^2(X'_t)$). Huybrechts proved that $\Gamma_0 = Z + \sum_i Y_i$ where Z is the graph of a bimeromorphic map $X \dashrightarrow X'$, and the images of the Y_i's under the projections to X and X' are proper (closed) subsets of X and X' respectively. Moreover by construction we have that

$$\Gamma_{0,*}(\alpha) = \omega_0 \in \mathscr{K}_{X'} \tag{4.3.8}$$

i.e. $\Gamma_{0,*}(\alpha)$ is a Kähler class. We have just described the main "twistor tryck" that enters into the proof of **Theorem 4.1** and **Theorem 4.2**

Example 4.9. Let S be a $K3$ surface and $W_S \subset \mathscr{C}_S$ be as in **Example 4.7**. Let $\alpha \in (\mathscr{C}_S \setminus W_S)$. There exists a set $\{\delta_1, \ldots, \delta_m\}$ of classes $\delta_i \in H^{1,1}_{\mathbb{Z}}(S)$ of square (-2) such that the composition of the reflections in $\delta_1, \ldots, \delta_m$ takes α to a Kähler class. It follows that the sum of the diagonal $\Delta_S \subset S \times S$ and a suitable collection of 2-cycles built out of curves $C_1, \ldots, C_m$ representing $\pm\delta_i$ is a 2-cycle Γ_0 such that (4.3.8) holds. The simplest non-trivial case is that of α separated from the Kähler chamber by a single wall $\delta^\perp$: in that case $\Gamma_0 = (\Delta_S \pm C \times C)$ (with a suitable sign) where C is a curve representing $\pm\delta$.

Below is a particular case of **Theorem 4.1** which is sufficient to prove the Global Torelli Theorem for HK manifolds.

Proposition 4.10. *Keep notation and assumptions as above. If $H^{1,1}_{\mathbb{Z}}(X) = 0$ then $\mathscr{K}_X = \mathscr{C}_X$.*

Proof. Since the Kähler cone is convex it suffices to prove that a general $\alpha \in \mathscr{C}_X$ is Kähler. Thus the discussion above applies and (4.3.8) holds. Write

$$\Gamma_{0,*}(\alpha) = Z_*(\alpha) + \sum_i Y_{i,*}(\alpha). \tag{4.3.9}$$

Since $H^{1,1}_{\mathbb{Z}}(X) = 0$ the image of each Y_i under the projection to X' has codimension at least 2: it follows that $Y_{i,*}(\beta) = 0$ for any $\beta \in H^2(X)$. Thus $\Gamma_{0,*}(\alpha) = Z_*(\alpha)$. On the other hand X and X' do not contain any curves because $H^{1,1}_{\mathbb{Z}}(X) = 0$, and hence Z is the graph of an isomorphism. Thus $X \cong X'$ and $\Gamma_{0,*}(\alpha) = \alpha$; by (4.3.8) we get that α is Kähler. □

5. Global Torelli

5.1. **Introduction.** Below is the celebrated Global Torelli Theorem for $K3$ surfaces.

Theorem 5.1 (Shafarevich and Pjateckii-Shapiro [30], Burns and Rapaport [7]). *Let X, Y be $K3$ surfaces. Then the following hold:*

1. *(Weak Global Torelli) X is isomorphic to Y if and only if there exists an integral Hodge isometry $\varphi\colon H^2(X) \to H^2(Y)$.*
2. *(Strong Global Torelli) Let $\varphi\colon H^2(X) \to H^2(Y)$ be an integral Hodge isometry. There exists an isomorphism $f\colon Y \xrightarrow{\sim} X$ such that $f^* = \varphi$ if and only if $f^*\mathscr{K}_X = \mathscr{K}_Y$.*

Remark 5.2. Two non-trivial results on $K3$ surfaces which are proved using the Global Torelli Theorem are the classification of automorphisms of $K3$ surfaces and irreducibility of the moduli space of polarized $K3$ surfaces of a given degree.

How does one extend Global Torelli to higher-dimensional HK's? A bimeromorphic map $f\colon Y \dashrightarrow X$ between HK manifolds induces an integral Hodge isometry $f^*\colon H^2(X) \xrightarrow{\sim} H^2(Y)$. Starting from dimension 4 there exist bimeromorphic non-regular maps between HK's, hence a Weak Global Torelli for higher-dimensional HK's will aim to prove that if there exists an integral Hodge isometry $\varphi\colon H^2(X) \to H^2(Y)$ (maybe satisfying extra hypotheses) then X is bimeromorphic to Y. In each even dimension greater than 2 there is more than one deformation class of HK's; since bimeromorphic HK's are deformation equivalent by Huybrecht's **Theorem 4.2** we should assume from the start that X and Y are deformation equivalent. Lastly we should (following Markman) not forget the importance of the monodromy group. More precisely suppose that X, Y are HK manifolds which are deformation equivalent: an isometry $\psi\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ is a *parallel-transport operator* if there exist

1. a family $\pi\colon \mathscr{X} \to B$ of HK manifolds,
2. a path $\sigma\colon [0,1] \to B$,
3. and isomorphisms $Z_{\sigma(0)} \cong X$, $Z_{\sigma(1)} \cong Y$

such that ψ is identified with the isometry $H^2(Z_{\sigma(0)};\mathbb{Z}) \xrightarrow{\sim} H^2(Z_{\sigma(1)};\mathbb{Z})$ defined by Gauss-Manin parallel transport along σ. Below we will give a formulation of Global Torelli for HK manifolds which is as close as possible to the classical Global Torelli for $K3$ surfaces: the key ingredient in its proof is provided by Verbitsky's Global Torelli i.e. **Theorem 5.6**.

Theorem 5.3 (Markman [22]). *Let X, Y be HK manifolds which are deformation equivalent. The following hold:*

1. *(Weak Global Torelli) X is bimeromorphic to Y if and only if there exists a parallel-transport operator $\varphi\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ defining a Hodge isomorphism.*
2. *(Strong Global Torelli) Let $\varphi\colon H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ be a parallel-transport operator defining a Hodge isomorphism. There exists an isomorphism $\alpha\colon Y \xrightarrow{\sim} X$ such that $\alpha^* = \varphi$ if and only if $\alpha^*\mathscr{K}_X = \mathscr{K}_Y$.*

Remark 5.4. The Global Torelli Theorem for HK's, joined with results on parallel-transport operators, has given results for higher-dimensional HK's analogous to those for $K3$'s, see **Remark 5.2**: see [27, 28] for results on automorphisms/bimeromorphisms groups and [12, 1] for results on moduli of polarized HK's.

In the following subsection we will state Verbitsky's Global Torelli and we will show how to obtain **Theorem 5.3** from Verbitsky's Theorem. In the other subsections we will sketch Huybrechts' proof of Verbitsky's Global Torelli. We refer to Markman's paper [22] for a detailed discussion of the mathematics surrounding the Global Torelli for HK's.

5.2. **The global period map.** Verbitsky's Global Torelli for HK manifolds [37] (in Huybrechts' formulation) is a statement regarding the global period map from the moduli space of marked pairs to the period space defined in **Subsection 5.2**. Let X_0 be a HK manifold and Λ a lattice isometric to $H^2(X_0;\mathbb{Z})$ (equipped with the B-B quadratic form). The moduli space $\mathfrak{M}$ of marked pairs of type X_0 is the set of isomorphism classes of marked pairs (X,f) such that X is a HK deformation equivalent to X_0. Deformation theory equips $\mathfrak{M}$ with a structure of analytic space. Since the deformation space of a HK manifold is unobstructed $\mathfrak{M}$ is a complex manifold: a neighborhood containing (X,f) is given by a simply-connected representative of $\mathrm{Def}(X)$, say B, with family $\pi\colon\mathscr{X}\to B$ and trivialization $F\colon R^2\pi_*\mathbb{Z}\xrightarrow{\sim} B\times\Lambda$ extending f. Already in the case of $K3$ surfaces the complex manifold $\mathfrak{M}$ is *not* Hausdorff, as shown by the example below.

Example 5.5. Let S be a $K3$ surface containing a smooth rational curve with Poincarè dual δ and let $r\colon H^2(S;\mathbb{Z})\to H^2(S;\mathbb{Z})$ be the reflection

$$r(\alpha):=\alpha+(\alpha,\delta)\delta.$$

Given any marking $f\colon H^2(S;\mathbb{Z})\to H^3\oplus E_8(-1)^2$ the points $[(S,f)]$ and $[(S,f\circ r)]$ are distinct because $\pm r$ is *not* induced by an automorphism of S. On the other hand $[(S,f)]$ and $[(S,f\circ r)]$ cannot be separated. In fact let $\pi\colon\mathscr{S}\to B$ be a representative of $\mathrm{Def}(S)$ with B simply connected and let $F\colon R^2\pi_*\mathbb{Z}\xrightarrow{\sim}B\times\mathbb{Z}$ be the trivialization such that $F_0=f$. Open neighborhood of (S,f) and $[(S,f\circ r)]$ are defined by the couples $(\pi\colon\mathscr{S}\to B,F)$ and $(\pi\colon\mathscr{S}\to B,F\circ r)$. Let $B_0\subset B$ be the open dense subset of $t\in B$ such that the class of δ (this makes sense because of the trivialization of $R^2\pi_*\mathbb{Z}$) is *not* of type $(1,1)$. Then B_0 defines open subsets $B_0(F)$ and $B_0(F\circ r)$ of the open neighborhoods of (S,f) and $[(S,f\circ r)]$ defined above. We have $B_0(F)=B_0(F\circ r)$. Varying B we get a fundamental system of neighborhoods of $[(S,f)]$; it follows that (S,f) and $[(S,f\circ r)]$ cannot be separated.

Verbitsky's Gobal Torelli is a statement about the global period map

$$\begin{array}{ccc}\mathfrak{M} & \xrightarrow{\mathfrak{p}} & \Omega_\Lambda\\ (X,f) & \mapsto & P(X,f)\end{array}\tag{5.2.1}$$

Theorem 5.6 (Verbitsky [37], Huybrechts [17])**.** *Keep notation as above. Let $\mathfrak{M}^0$ be a connected component of $\mathfrak{M}$. The restriction of $\mathfrak{p}$ to $\mathfrak{M}^0$ is surjective. Let $(X,f),(Y,g)\in\mathfrak{M}^0$; then $\mathfrak{p}(X,f)=\mathfrak{p}(Y,g)$ if and only if there exists a bimeromorphic map $\alpha\colon Y\dashrightarrow X$ such that $g\circ H^2(\alpha)=f$ i.e. if and only if there is bimeromorphic map between the marked pairs (X,f) and (Y,g).*

We will sketch the proof of **Theorem 5.6** in **Subsection 5.3** and **Subsection 5.4**. Here we will grant **Theorem 5.6** and we will show how to prove **Theorem 5.3**. We must prove the following statements:

(1a) If X is bimeromorphic to Y there exists a parallel-transport operator $\varphi\colon H^2(X;\mathbb{Z})\to H^2(Y;\mathbb{Z})$ defining a Hodge isomorphism.

(1b) If there exists a parallel-transport operator $\varphi\colon H^2(X;\mathbb{Z})\to H^2(Y;\mathbb{Z})$ defining a Hodge isomorphism then X is bimeromorphic to Y.

(2a) Let $\varphi\colon H^2(X;\mathbb{Z})\to H^2(Y;\mathbb{Z})$ be a parallel-transport operator defining a Hodge isomorphism. If there exists an isomorphism $\alpha\colon Y\xrightarrow{\sim}X$ such that $H^2(\alpha)=\varphi$ then $\varphi(\mathscr{K}_X)=\mathscr{K}_Y$.

(2b) Let $\varphi\colon H^2(X;\mathbb{Z})\to H^2(Y;\mathbb{Z})$ be a parallel-transport operator defining a Hodge isomorphism. If $\varphi(\mathscr{K}_X)=\mathscr{K}_Y$ there exists an isomorphism $\alpha\colon Y\xrightarrow{\sim}X$ such that $H^2(\alpha)=\varphi$.

Proof of (1a). This follows from Huybrechts' proof of **Theorem 4.2**. In fact let $\alpha\colon Y\dashrightarrow X$ be a bimeromorphic map. The proof of Theorem 2.5 of [15] gives that there exist families $\mathscr{X}\to S$ and $\mathscr{Y}\to S$ over a smooth curve S, a point $\mathbf{0}\in S$ such that $X_{\mathbf{0}}\cong X$, $Y_{\mathbf{0}}\cong Y$, and a cycle Γ on $\mathscr{Y}\times_S\mathscr{X}$ of relative dimension equal to $\dim X_s=\dim Y_s$, whose restriction to $Y_s\times X_s$ for $s\notin\{\mathbf{0}\}$ is the graph of an isomorphism $f_s\colon Y_s\xrightarrow{\sim}X_s$, and such that

$$\Gamma_{\mathbf{0}}=Z+\sum_i W_i\tag{5.2.2}$$

where Z is the graph of α and each W_i is mapped by the projections of $Y\times X$ to subsets of codimension at least 2. Thus the action of each W_i on $H^2(Y)$ is trivial and hence $H^2(\alpha)$ is identified via parallel transport with $H^2(f_s)$. Now let $s_0\in(S\smallsetminus\{\mathbf{0}\})$ and let $S\cup_{s_0}S$ be the curve obtained by gluing two copies of S at s_0. Let $\mathscr{Z}\to(S\cup_{s_0}S)$ be the family of HK's obtained by gluing together $\mathscr{X}\to S$ and $\mathscr{Y}\to S$ along $X_{s_0}\cong Y_{s_0}$. Letting $\sigma\colon[0,1]\to(S\cup_{s_0}S)$ be a path joining the point $\mathbf{0}$ in the "first" copy of S to the point $\mathbf{0}$ in the "second" copy of S we get that the isomorphism of Hodge structures $H^2(\alpha)$ is a parallel-transport operator.

Proof of (1b). Let g be a marking of Y and f the marking of X defined by $f := g \circ \varphi$. Since φ is a parallel transport operator the marked pairs (X, f) and (Y, g) belong to the same connected component of $\mathfrak{M}$, call it $\mathfrak{M}^0$. On the other hand since φ defines an isomorphism of Hodge structures we have $P(X, f) = P(Y, g)$ and hence by **Theorem 5.6** there exists a bimeromorphic map $\alpha: Y \dashrightarrow X$ such that $g \circ H^2(\alpha) = f$. In particular we get that X is bimeromorphic to Y.

Proof of (2a). An isomorphism takes Kähler classes to Kähler classes.

Proof of (2b). The proof of (1b) gives that there exists a bimeromorphic map $\alpha: Y \dashrightarrow X$ such that $H^2(\alpha) = \varphi$. Thus $H^2(\alpha)(\mathscr{K}_X) = \mathscr{K}_Y$ and hence α is regular by Corollary 3.3 of [9], see also Prop. 2.1 of [15].

5.3. **Hausdorffization of the moduli space of marked pairs.** The global period map $\mathfrak{p}: \mathfrak{M} \to \Omega_\Lambda$ is a local homeomorphism by **Theorem 3.9**. We would like to prove that $\mathfrak{p}$ is a topological covering, but this is certainly not true since $\mathfrak{M}$ is (in general) not Hausdorff while Ω_Λ is. Thus one replaces $\mathfrak{M}$ by its maximal Hausdorff quotient. First we discuss this kind of construction for a general topological space Z. We say that $x, y \in Z$ are *inseparable* if given open sets $x \in U \subset Z$ and $y \in V \subset Z$ the intersection $U \cap V$ is not empty: in symbols $x \sim y$ and we name $\sim$ the *Hausdorff* relation on Z. Let $\Delta_Z \subset Z \times Z$ be the diagonal: we have

$$\overline{\Delta}_Z = \{(x, y) \mid x \sim y\}. \tag{5.3.1}$$

Clearly $\sim$ is reflexive and symmetric. The example below shows that $\sim$ is not necessarily transitive i.e. it need not be an equivalence relation.

Example 5.7 (Verbitsky [37]). Let $\mathscr{R}$ be the equivalence relation on $\mathbb{R} \coprod \mathbb{R} \coprod \mathbb{R}$ defined as follows. We denote a point of $\mathbb{R} \coprod \mathbb{R} \coprod \mathbb{R}$ as $a_i \in \mathbb{R}$ for $i = 1, 2, 3$ meaning that it belongs to the i-th copy of $\mathbb{R}$. Then $\mathscr{R}$ is generated by the relations $a_1 \mathscr{R} b_2$ if $a_1 < 0$ and $a_1 = b_2$ and $b_2 \mathscr{R} c_3$ if $b_2 > 0$ and $b_2 = c_3$. Let $X := (\mathbb{R} \coprod \mathbb{R} \coprod \mathbb{R})/\mathscr{R}$. The points $[0_1], [0_2], [0_3] \in X$ are distinct and $0_1 \sim 0_2$, $0_2 \sim 0_3$ but $0_1 \not\sim 0_3$.

Proposition 5.8. *Keep notation as above and suppose that the following hold:*

1. *The Hausdorff relation $\sim$ is an equivalence relation and hence the quotient topological space $\overline{Z} := Z/\sim$ exists.*
2. *The Hausdorff relation is open i.e. the quotient map $\pi: Z \to \overline{Z}$ is open.*

Then $\overline{Z}$ is Hausdorff.

Proof. It suffices to prove that the diagonal $\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z}$ is closed. Let $\pi: Z \to \overline{Z}$ be the quotient map. Let

$$\begin{array}{ccc} Z \times Z & \xrightarrow{\varphi} & \overline{Z} \times \overline{Z} \\ (z_1, z_2) & \mapsto & (\pi(z_1), \pi(z_2)) \end{array}$$

The map φ is the set-theoretic quotient map for the equivalence relation $\mathscr{R}$ defined by declaring $(z_1, z_2) \mathscr{R} (z_1', z_2')$ if $z_1 \sim z_1'$ and $z_2 \sim z_2'$. Moreover φ is continuous. Now let $\mathscr{U} \subset \overline{Z} \times \overline{Z}$ and suppose that $\varphi^{-1}\mathscr{U}$ is open: then $\mathscr{U} = \varphi(\varphi^{-1}\mathscr{U})$ is open because by hypothesis $\sim$ is open. This proves that φ is the quotient map in the category of topological space and hence $\Delta_{\overline{Z}} \subset \overline{Z} \times \overline{Z}$ is closed if $\varphi^{-1}(\Delta_{\overline{Z}})$ is closed in $Z \times Z$: the latter holds by (5.3.1). □

If the hypotheses of **Proposition 5.8** are satisfied the quotient $\overline{Z}$ is the *Hausdorffization of Z*. The following universal property holds: if W is a Hausdorff topological space and $f: Z \to W$ is a (continuous) map then there exists a unique continuous map $\overline{f}: \overline{Z} \to W$ such that $f = \overline{f} \circ \pi$. One shows that the Hausdorff relation $\sim$ on $\mathfrak{M}$ satisfies the hypothesis of **Proposition 5.8**: the key ingredients are Huybrechts' results on non-separated points of $\mathfrak{M}$, namely **Theorem 4.2** and the following generalization of the Burns-Rapaport Main Lemma [7].

Theorem 5.9 (Theorem 4.3 of [14])**.** *Let X_0 be a HK manifold and $\mathfrak{M}$ the moduli space of marked pairs (X, f) where X is a deformation of X_0. If $[(X, f)]$, $[(Y, g)]$ are non-separated points of $\mathfrak{M}$ then X is bimeromorphic to Y.*

For details of the proof that ~ on $\mathfrak{M}$ we refer to Huybrechts' Bourbaki talk [17]. Here we simply indicate how one proves that ~ is transitive (and hence is an equivalence relation). If $[(X,f)] \sim [(Y,g)]$ and $[(Y,g)] \sim [(Z,h)]$ then by **Theorem 5.9** X is bimeromorphic to Y and Y is bimeromorphic to Z. It follows that X is bimeromorphic to Z "compatibly" with the markings f and h. Then Theorem 2.5 of [15] (see the proof of Item (1a) in **Subsection 5.2**) gives that $[(X,f)]$ and $[(Z,h)]$ are non-separated points of $\mathfrak{M}$. Thus we have the Hausdorffization $\overline{\mathfrak{M}}$ and the period map $\mathfrak{p}\colon\mathfrak{M} \to \Omega_\Lambda$ descends to a continuous map

$$\overline{\mathfrak{p}}\colon\overline{\mathfrak{M}} \longrightarrow \Omega_\Lambda. \tag{5.3.2}$$

which is a *local isomorphism* by **Theorem 3.9**.

5.4. **The descended period map is a topological covering.** The period map Ω_Λ is simply connected: it follows that in order to prove **Theorem 5.6** it suffices to show that the descended period map $\overline{\mathfrak{p}}$ is a topological covering. The proof breaks up in two steps. First there is a result in general topology which gives a sufficient condition for a local homeomorphism between manifolds to be a topological covering: this was proved by Verbitsky (we will present Markman's proof). The second step is the proof that Verbitsky's Criterion is satisfied by the decsended period map $\overline{\mathfrak{p}}$: here the key ingredients are Huybrechts' results on the Kähler cone (**Proposition 4.10** suffices) and the existence of twistor families, we refer to Huybrechts' Bourbaki talk [17] for details. Before proving the general topology result we recall the following well-known result.

Lemma 5.10. *Let $f\colon M \to N$ be a local homeomorphism of topological spaces and suppose that M is Hausdorff. Let X be a connected topological space and $x_0 \in X$. Suppose that $\sigma,\tau\colon X \to M$ are continuous maps such that $\sigma(x_0) = \tau(x_0)$ and $f\circ\sigma = f\circ\tau$. Then $\sigma = \tau$.*

In order to state Verbitsky's Criterion we give two definitions Let M be a topological manifold. A *closed ball* in M is a closed $D \subset M$ contained in a coordinate chart (U,φ) (here $\varphi\colon U \xrightarrow{\sim} \mathbb{R}^n$ is a homeomorphism) such that $\varphi(D)$ is a closed ball $\overline{D}_R(a)$ of strictly positive radius R (and center a). We let $B = D_R(a)$ be the interior of D (an open ball) and we denote D by $\overline{B}$. If M is a smooth manifold a *smooth closed ball* in M is defined as above - of course (U,φ) must belong to C^∞-atlas.

Proposition 5.11 (Verbistky [37])**.** *Let $f\colon M \to N$ be a local homeomorphism of topological (smooth) manifolds and suppose that M is Hausdorff. Then f is a topological covering if and only if the following holds for each closed ball (respectively smooth closed ball) $\overline{B} \subset N$: if C is a connected component of $f^{-1}\overline{B}$ then $f(C) = \overline{B}$.*

Proof. It is clear that the condition is necessary, the point is to prove that it is sufficient. Since N is covered by open sets of an atlas we may assume that $N = \mathbb{R}^n$. We will prove that if M is connected then $f\colon M \to \mathbb{R}^n$ is a homeomorphism: the proposition follows by restricting f to the connected components of the domain. Let $m \in M$ and $a := f(m)$. Let $I \subset [0,+\infty)$ be the set of R such that there exists a continuous section $s_R\colon\overline{D}_R(a) \to M$ through m i.e. $s_R(a) = m$ and $f\circ s_R = \mathrm{Id}_{\overline{D}_R(a)}$. Clearly $0 \in I$ and I is an interval. An easy compactness argument shows that I is open (use **Lemma 5.10**) - here we do not use our hypothesis $(f(C) = \overline{B})$, one only needs that f is a local homeomorphism. Thus it suffices to prove that $\sup I = +\infty$. Suppose the contrary and let $R_0 := \sup I$. There is a section $t_0\colon D_{R_0}(a) \to M$ through m. Let $C_0 := \overline{\operatorname{Im} t_0} \cap f^{-1}\overline{D}_{R_0}$. We claim that $f|_{C_0}$ is injective and that C_0 is open in $f^{-1}\overline{D}_{R_0}$. In fact let $x,y \in C_0$ such that $f(x) = f(y)$. If $f(x) = f(y) \in D_{R_0}(a)$ then $x = t_0(f(x)) = t_0(f(y)) = y$ because M is Hausdorff. Next suppose that If $f(x) = f(y) \in \partial D_{R_0}(a)$. Let $x \in U \subset M$ and $y \in V \subset M$ be open connected neighborhoods such that $f|_U$ and $f|_V$ are homeomorphisms onto their images. Let $\{z_n\}$ be a sequence in $f(U) \cap f(V) \cap D_{R_0}(a)$ converging to $f(x) = f(y)$. Then $t_0(z_n) \in U \cap V$ and $t_0(z_n) \to x$, $t_0(z_n) \to y$, since M is Hausdorff $x = y$. Moreover the inverse of $f|_{U\cap V}$ gives a section which coincides with t_0 on $U \cap V \cap D_{R_0}(a)$ by **Lemma 5.10**: this proves that C_0 is open. Since C_0 is closed by construction (and non-empty) it is a connected component of $f^{-1}\overline{D}_{R_0}$. By hypothesis $f(C_0) = \overline{D}_{R_0}$, i.e. $f|_{C_0}$ is bijective. The argument given above shows that $(f|_{C_0})^{-1}$ is continuous and hence it is the desired section $s_{R_0}\colon\overline{D}_{R_0} \to M$. □

6. Hyperkähler manifolds of type $K3^{[n]}$

6.1. **The monodromy group and parallel-transport operators.** A HK manifold is *of type* $K3^{[n]}$ if it is deformation equivalent to $S^{[n]}$ where S is a $K3$ surface. Let X and Y be HK manifolds of type $K3^{[n]}$. Markman [20, 21] has described the set of parallel transport operators $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$.

First we describe the set of parallel transport operators $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$; this is a group denoted $\mathrm{Mon}^2(X)$. Given $\delta \in H^2(X;\mathbb{Z})$ of square ± 2 let $r_\delta\colon H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ be the reflection

$$r_\delta(\alpha) := \begin{cases} \alpha + (\alpha,\delta)\delta & \text{if } \delta^2 = -2, \\ -\alpha + (\alpha,\delta)\delta & \text{if } \delta^2 = 2. \end{cases} \tag{6.1.1}$$

Theorem 6.1 (Markman, Thm. 1.2 of [21]). *If X is a HK manifold of type $K3^{[n]}$ then $\mathrm{Mon}^2(X)$ is generated by the reflections r_δ for $\delta \in H^2(X;\mathbb{Z})$ of square ± 2.*

Now let X be a HK of type $K3^{[n]}$ where $n \geq 2$. Then $H^2(X;\mathbb{Z}) \cong H^3 \oplus E_8(-1)^2 \oplus (-2(n-1))$ and hence there exists a primitive embedding of lattices

$$H^2(X;\mathbb{Z}) \subset H^4 \oplus E_8(-1)^2. \tag{6.1.2}$$

Since $H^4 \oplus E_8(-1)^2$ has discriminant 1 we have

$$H^2(X;\mathbb{Z})^\perp = \mathbb{Z}v, \qquad v^2 = 2(n-1).$$

The group of isometries $\mathrm{O}(H^2(X;\mathbb{Z}))$ acts on the set of Embeddings (6.1.2) as follows. View (6.1.2) as given by a unimodular overlattice of $H^2(X;\mathbb{Z}) \oplus (-2(n-1))$ and hence as a subgroup of $H^2(X;\mathbb{Q}) \oplus \mathbb{Q}v$ where v is a generator of $(-2(n-1))$: then realize $\mathrm{O}(H^2(X;\mathbb{Z}))$ as a subgroup of the isometries of $H^2(X;\mathbb{Q}) \oplus \mathbb{Q}v$ by mapping $g \in \mathrm{O}(H^2(X;\mathbb{Z}))$ to $(g, \mathrm{Id}_{\mathbb{Q}v})$. In general (depending on n) such an embedding is not invariant under $\mathrm{O}(H^2(X;\mathbb{Z}))$, but it is invariant under the reflections r_δ given by (6.1.1). Thus **Theorem 6.1** gives that we can choose an embedding

$$H^2(X;\mathbb{Z}) \subset \widetilde{\Lambda}_X \cong H^4 \oplus E_8(-1)^2 \tag{6.1.3}$$

for every hyperkähler X of type $K3^{[n]}$ so that it is covariant under parallel transport. In other words every parallel-transport operator $H^2(X;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ lifts to an isometry $\widetilde{\Lambda}_X \to \widetilde{\Lambda}_Y$.

Example 6.2. Let S be a $K3$ surface, $\mathbf{v} \in \widetilde{H}(S)$ an indivisible Mukai vector with $\mathbf{v}^2 > 0$ and H a $\mathbf{v}$-generic polarization of S. Then $X = \mathscr{M}_S(\mathbf{v})$ is a HK variety of type $K3^{[n]}$ where $2n = (2 + \mathbf{v}^2)$. An obvious choice for $\widetilde{\Lambda}_X$ and v as above is $\widetilde{H}(S)$ and $\mathbf{v}$ respectively. This is behind the proof of **Theorem 6.1**. We may interpret the existence of a well-defined $\widetilde{\Lambda}_X$ and vector v for each X of type $K3^{[n]}$ as stating that even though X in general will not be a moduli space of sheaves on a $K3$, a ghost of a $K3$ has survived.

Markman proved that the set of parallel-transport operators is almost equal to the set of isometries which lift. In order to give the precise statement we must introduce the natural orientation of $H^2(X;\mathbb{R})$ for a HK manifold X. The set $\mathrm{Gr}(3, H^2(X;\mathbb{R}))_+$ parametrizing 3-dimensional (real) subspaces $W \subset H^2(X;\mathbb{R})$ such that q_X is positive-definite on W (such a space is *positive definite*) is contractible: it follows that the set $\mathrm{Gr}(3, H^2(X;\mathbb{R}))^{or}_+$ parametrizing couples (W,τ) with $W \in \mathrm{Gr}(3, H^2(X;\mathbb{R}))_+$ and τ an orientation of W has two connected components. Now let σ be a generator of $H^{2,0}(X)$ and ω be a Kähler form of X: then

$$W = \mathbb{R}\omega \oplus \{t\sigma + \overline{t\sigma} \mid t \in \mathbb{C}\}$$

is a positive definite subspace of $H^2(X;\mathbb{R})$ and it has the orientation given by its identification with $\mathbb{R} \oplus \mathbb{C}$. Since the set of Kähler classes is a cone and rescaling σ does not change the orientation we have determined a well-defined connected component of $\mathrm{Gr}(3, H^2(X;\mathbb{R}))^{or}_+$: this is the *orientation class* of X. Now let Y be another HK manifold and $\varphi \in H^2(X;\mathbb{R}) \to H^2(X;\mathbb{R})$ an isometry. Then φ maps $\mathrm{Gr}(3, H^2(X;\mathbb{R}))^{or}_+$ to $\mathrm{Gr}(3, H^2(Y;\mathbb{R}))^{or}_+$: we say that φ is *orientation preserving* if it maps the orientation class of X to the orientation class of Y. An isometry $\varphi \in H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ is orientation preserving if the isometry $H^2(X;\mathbb{R}) \to H^2(X;\mathbb{R})$ obtained by extension of scalars is orientation preserving.

Theorem 6.3 (Markman, Thm. 1.2 of [21]). *Let X and Y be HK manifolds of type $K3^{[n]}$. An isometry $\varphi\colon H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ is a parallel-transport operator if and only if it lifts to an isometry $\widetilde{\Lambda}_X \to \widetilde{\Lambda}_Y$ and it is orientation preserving.*

6.2. **The cone of curves.** Bayer-Hassett-Tschinkel have given a numerical description of the cone of curves on projective HK's of type $K3^{[n]}$ - some of the main ingredients of the proof are Markman's results on parallel-transport operators for HK's of type $K3^{[n]}$ and Bayer-Macrì's determination of the cones of curves of moduli spaces of sheaves on $K3$'s, see [3]. (See also [29] for similar results.) Let $n \geq 2$ and X be a HK of type $K3^{[n]}$. We equip $\widetilde{\Lambda}_X \otimes \mathbb{C}$ with a weight-2 Hodge structure by requiring that on $H^2(X) \subset \widetilde{\Lambda}_X \otimes \mathbb{C}$ we get the obvious Hodge structure and requiring that v is of type $(1,1)$. Thus we have the lattice $\widetilde{\Lambda}_X^{(1,1)}$ of integral $(1,1)$-classes. Identify $H_2(X;\mathbb{Z})$ with $H^2(X;\mathbb{Z})^\vee$; we have a map

$$\begin{array}{ccc} \widetilde{\Lambda}_X & \xrightarrow{\theta^\vee} & H_2(X;\mathbb{Z}) \\ \alpha & \mapsto & \beta \mapsto (\alpha,\beta) \end{array} \tag{6.2.1}$$

If $\alpha \in \widetilde{\Lambda}_X^{(1,1)}$ then $\theta^\vee(\alpha)$ is represented by an algebraic 1-cycle (modulo multiplication by a non-zero scalar); viceversa the homology class of an algebraic 1-cycle is represented by $\theta^\vee(\alpha)$ for a certain $\alpha \in \widetilde{\Lambda}_X^{(1,1)}$. In order to state the result of Bayer-Hassett-Tschinkel we recall that the *Mori cone* of X is the cone in $H_2(X;\mathbb{R})_{alg} = H^{1,1}_{\mathbb{R}}(X)^\vee$ spanned by homology classes of algebraic curves. The positive cone in $H_2(X;\mathbb{R})_{alg}$ is defined as follows. The B-B quadratic form defines an isomorphism $H_2(X;\mathbb{R})_{alg} \xrightarrow{\sim} H^{1,1}_{\mathbb{R}}(X)$ and hence also a (dual) quadratic form $q^\vee$ on $H_2(X;\mathbb{R})_{alg}$: the *positive cone* in $H_2(X;\mathbb{R})_{alg}$ is the cone of α such that $q^\vee(\alpha) > 0$ and $(\alpha,h) > 0$ for an ample class h on X (the last condition is independent of the choice of h).

Theorem 6.4 (Bayer-Hassett-Tschinkel [2]). *Let $n \geq 2$ and X be a HK variety of type $K3^{[n]}$ and $h \in H^{1,1}_{\mathbb{Z}}(X)$ be an ample class. Let $v \in \widetilde{\Lambda}_X$ be a generator of $H^2(X;\mathbb{Z})^\perp$. The cone of effective curves on X is generated by the positive cone and the classes $\theta^\vee(\alpha)$ for $\alpha \in \widetilde{\Lambda}_X^{(1,1)}$ such that*

$$-2 \leq \alpha^2, \qquad |(\alpha,v)| \leq v^2/2, \qquad (\alpha,h) > 0.$$

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