COMPACT HYPERKÄHLER MANIFOLDS: EXAMPLES

KIERAN G. O'GRADY

"SAPIENZA" UNIVERSITÀ DI ROMA

Contents

1. Introduction

A *hyperkähler manifold* (HK) is a simply connected compact Kähler manifold whose space of global holomorphic two-forms is spanned by a symplectic form[1]. The above definition may be motivated by the following result.

Theorem 1.1 (Beauville-Bogomolov decomposition [5]). *Let X be a compact Kähler manifold with $c_1(X) = 0$. There exists an étale finite cover $\prod_{i=1}^{d} M_i \longrightarrow X$ where each of the factors M_i is either a compact complex torus, a HK manifold or a Calabi-Yau variety i.e. a compact Kähler manifold of dimension $n \geq 3$ with trivial canonical bundle and such that $h^0(\Omega^p_{M_i}) = 0$ for $0 < p < n$.*

A 2-dimensional HK manifold is nothing else but a $K3$ surface. These surfaces were known classically as complex smooth projective surfaces whose generic hyperplane section is a canonically embedded curve (example: a smooth quartic surface in $\mathbb{P}^3$) and they have proved to have a very rich geometry. Beauville [5] constructed two distinct deformation classes of HK's in every even dimension greater than 2 (a HK manifold has even complex dimension because an odd-dimensional vector-space cannot be equipped with a non-degenerate symplectic form), and the author [41, 42] constructed two extra deformation classes in dimensions 6 and 10. In these notes we will describe members of each of these deformation classes. No other deformation class of HK manifolds is known. It has proved hard to construct HK manifolds which are not deformations of the known examples, on the other hand we do not know whether the set of deformation classes of a given dimension greater than 2 is finite (see [15, 48] for work in that direction, and [47] for constraints on the topology of HK manifolds). Each of the known deformation classes has representatives which are moduli spaces of semistable sheaves on projective $K3$ surfaces or abelian surfaces (i.e. projective symplectic surfaces) or modifications of such moduli spaces. The general theory shows that a generic projective deformation of such a moduli space is not isomorphic (not even birational) to a moduli space of sheaves on a symplectic surface. Thus one is naturally led to ask for explicit models of HK varieties[2] which are not birational to Moduli spaces of sheaves on symplectic surfaces: this has been accomplished for some choices of deformation classes and polarization type [6, 19, 20, 43, 9, 27, 1]. On the other hand an intensive study of the geometry of moduli spaces of sheaves and of semistable objects in the derived category of symplectic surfaces, joined with Verbitsky's global Torelli [16, 52, 32] has produced deep results on the ample/nef/movable cone of HK manifolds in some of the known deformation classes [33, 4]. The moral is that moduli spaces of sheaves on symplectic surfaces play a fundamental rôle in the theory of HK manifolds although

Date: June 13 2014.

[1]Our terminology is at odds with the Riemannian notion of hyperkähler metric, which makes sense on a possibly non-compact manifold, but there will be no cause for confusion because we will deal exclusively with compact Kähler manifolds.

[2]A HK variety is a projective HK manifold.

the generic HK manifold is *not* birational to such a moduli space. In these notes we will sketch the construction of all the known deformation classes of HK manifolds via moduli of sheaves on symplectic surfaces.

2. Beauville's examples

2.1. **Zero-dimensional subschemes of smooth surfaces.** Let S be a smooth complex projective surface and $S^{[n]}$ be the Hilbert scheme parametrizing length-n subschemes of S. A point of $S^{[n]}$ is a subscheme $Z \subset S$ such that $H^0(\mathscr{O}_Z)$ is finite-dimensional of dimension n. It is known [11] that the generic such Z is reduced i.e. it consists of n distinct points and that $S^{[n]}$ is a smooth complex projective variety[3] of dimension $2n$. Let $S^{(n)}$ be the *symmetric* n-th power of S i.e. the quotient of S^n by the natural action of the symmetric group on n elements Σ_n. An element of $S^{(n)}$ may be written as a finite formal sum $\sum_i m_i p_i$ where $m_i \in \mathbb{N}$ for each i and $\sum_i m_i = n$. There is a regular *Hilbert-Chow map*

$$\begin{array}{ccc} S^{[n]} & \xrightarrow{\gamma} & S^{(n)} \\ Z & \mapsto & \sum_{p\in S} \ell(\mathscr{O}_{Z,p})p. \end{array} \tag{2.1.1}$$

Here the sum is a *formal* sum. Let $\sum_i m_i p_i \in S^{(n)}$ where the points p_i are *pairwise distinct*: by a result of Iarrobino [18]

$$\dim \gamma^{-1}(\sum_i m_i p_i) = \sum_i (m_i - 1). \tag{2.1.2}$$

Let $\Delta_n \subset S^{[n]}$ be the subset of *non-reduced* subschemes. By (2.1.2) Δ_n is the exceptional set of γ. Since $S^{(n)}$ is $\mathbb{Q}$-factorial it follows that Δ_n has pure codimension 1: by (2.1.2) we get that Δ_n is irreducible. We will give an explicit description of the subset $U_n \subset S^{[n]}$ defined by

$$U_n := \{Z \in S^{[n]} \mid |\operatorname{supp} Z| \geq (n-1)\}. \tag{2.1.3}$$

In other words $Z \in U_n$ if either Z is reduced or it is the disjoint union of $(n-2)$ reduced points and a subscheme of length 2. Since $\gamma(U_n)$ is open and $U_n = \gamma^{-1}(\gamma(U_n))$ the subset U_n is open. Equation (2.1.2) gives that

$$\dim(S^{[n]} \setminus U_n, S^{[n]}) = 2. \tag{2.1.4}$$

An explicit description of U_n goes as follows. Let $V_n \subset S^n$ be defined by

$$V_n := \{(x_1, \ldots, x_n) \in S^n \mid |\{x_1, \ldots, x_n\}| \geq (n-1)\}. \tag{2.1.5}$$

In other words $(x_1, \ldots, x_n) \in V_n$ if there exists at most one couple $1 \leq i < j \leq n$ such that $x_i = x_j$. The tautological closed subset $\mathscr{W} \subset V_n \times S$ consisting of couples $((x_1, \ldots, x_n), y)$ such that $y = x_i$ for some $1 \leq i \leq n$ is *not* a flat family of length-n subschemes of S unless $n = 1$. In fact an explicit computation shows that the length of the fiber of $\mathscr{W} \to V_n$ over a point $(x_1, \ldots, x_n)$ with a repetition (i.e. $x_i = x_j$ for some $1 \leq i < j \leq n$) is equal to $(n+1)$ instead of n; the point is that as x_i gets close to x_j the subscheme $\{x_1, \ldots, x_n\}$ approaches different subschemes, they are parametrized by the limiting direction of $(x_i - x_j)$ (if x_i, x_j belong to a coordinate patch in the classical topology). In order to get a flat family we must blow-up the large diagonal. More precisely for $1 \leq i < j \leq n$ let

$$D_{ij} := \{(x_1, \ldots, x_n) \in S^n \mid x_i = x_j\}. \tag{2.1.6}$$

The *large diagonal* $D_n \subset S^n$ is the union of all the D_{ij}. We let

$$f\colon \widetilde{V}_n \to V_n \tag{2.1.7}$$

be the blow-up of $D_n \cap V_n$. There is a subscheme $\mathscr{Z} \subset \widetilde{V_n} \times S$ flat over $\widetilde{V_n}$ such that $Z_t := \mathscr{Z} \cap (\{t\} \times S)$ is a length-n subscheme for every $t \in \widetilde{V_n}$ and if $f(t) = (x_1, \ldots, x_n)$ where the x_i's are pairwise distinct then Z_t is the reduced scheme $\{x_1, \ldots, x_n\}$. Thus $\mathscr{Z}$ induces a regular surjective map

$$g\colon \widetilde{V_n} \longrightarrow U_n. \tag{2.1.8}$$

The group Σ_n on n elements acts on $\widetilde{V}_n$ and on $\mathscr{Z}$: it follows that g is invariant under the action of Σ_n on $\widetilde{V}_n$ and hence it descends to a regular map

$$h\colon \Sigma_n \backslash \widetilde{V}_n \longrightarrow U_n. \tag{2.1.9}$$

Since h is injective and $S^{[n]}$ is smooth it follows that h is an isomorphism.

[3] A variety is integral i.e. reduced and irreducible.

2.2. **Topology of $S^{[n]}$.** We will study the topology of $S^{[n]}$ for $n \geq 2$. We start by analyzing the fundamental group. Let $p_1, \ldots, p_n \in S$ be pairwise distinct and

$$\begin{array}{ccc} S \smallsetminus \{p_1, \ldots, p_{n-1}\} & \overset{h}{\hookrightarrow} & S^{[n]} \\ p & \mapsto & \{p_1, \ldots, p_{n-1}, p\}. \end{array} \tag{2.2.1}$$

One proves easily that the homomorphism

$$\pi_1(S \smallsetminus \{p_1, \ldots, p_{n-1}\}; p_n) \xrightarrow{h_\#} \pi_1(S^{[n]}; \{p_1, \ldots, p_n\})$$

is surjective and the image is a commutative subgroup (the latter holds because $n \geq 2$). Since $H_1(S \smallsetminus \{p_1, \ldots, p_{n-1}\}) \cong H_1(S; \mathbb{Z})$ we get that $h_\#$ induces a surjective homomorphism

$$H_1(S; \mathbb{Z}) \longrightarrow \pi_1(S^{[n]}; \{p_1, \ldots, p_n\}). \tag{2.2.2}$$

Proposition 2.1. *Keep assumptions and notation as above, in particular $n \geq 2$. Then* (2.2.2) *is an isomorphism.*

Proof. Let $\mathrm{Alb}(S) = H^0(\Omega^1_S)^\vee / H_1(S; \mathbb{Z})$ be the Albanese variety of S. Choose a base point $p_0 \in S$ and let

$$\begin{array}{ccc} S & \xrightarrow{u} & \mathrm{Alb}(S) \\ p & \mapsto & (\omega \mapsto \int_{p_0}^{p} \omega) \end{array} \tag{2.2.3}$$

be the Albanese map. Let

$$\begin{array}{ccc} S^{[n]} & \xrightarrow{s_n} & \mathrm{Alb}(S) \\ Z & \mapsto & \sum_{p \in S} \ell(\mathscr{O}_{Z,p}) u(p) \end{array} \tag{2.2.4}$$

where the sum is *not* a formal sum, it is the sum in the group $\mathrm{Alb}(S)$ (the notation is consistent with that of (2.4.2)). Composing (2.2.2) and the map

$$\pi_1(S^{[n]}; \{p_1, \ldots, p_n\}) \xrightarrow{s_{n,\sharp}} \pi_1(\mathrm{Alb}(S); u(p_1) + \ldots + u(p_n))$$

we get a homomorphism $H_1(S; \mathbb{Z}) \to \pi_1(Alb(S); u(p_1) + \ldots + u(p_n)) \cong H_1(S; \mathbb{Z})$ which is the identity. It follows that (2.2.2) is injective; since it is surjective it is an isomorphism. □

Next we will describe the low-dimensional cohomology groups of $S^{[n]}$. The rational cohomology of $S^{(n)}$ is naturally identified with the Σ_n-invariant summand of the rational cohomology of S^n i.e.

$$H^p(S^{(n)}; \mathbb{Q}) \cong H^p(S^n; \mathbb{Q})^{\Sigma_n}. \tag{2.2.5}$$

Thus Poincarè duality for S^n gives that

$$\begin{array}{ccc} H^p(S^{(n)}; \mathbb{Q}) \times H^{2n-p}(S^{(n)}; \mathbb{Q}) & \longrightarrow & \mathbb{Q} \\ (\alpha, \beta) & \mapsto & \int_{S^{(n)}} \alpha \cup \beta \end{array} \tag{2.2.6}$$

is a perfect pairing for all p. It follows that $H^p(\gamma): H^p(S^{(n)}; \mathbb{Q}) \to H^p(S^{[n]}; \mathbb{Q})$ is injective for all p. Since γ is an isomorphism outside the irreducible divisor Δ_n we get that $H^p(\gamma)$ is an isomorphism for $p \leq 1$ and that

$$H^2(S^{[n]}; \mathbb{Q}) \cong H^2(S^{(n)}; \mathbb{Q}) \oplus \mathbb{Q} c_1(\mathscr{O}_{S^{[n]}}(\Delta_n)). \tag{2.2.7}$$

Let's pass to integral cohomology. It is not difficult to prove that for $p \leq 2$ every integral Σ_n-invariant p-cohomology class on S^n descends to an integral cohomology class on $S^{(n)}$. More precisely there exists a symmetrization homomorphism

$$t_p: H^p(S; \mathbb{Z}) \longrightarrow H^p(S^{(n)}; \mathbb{Z}), \qquad p \leq 2 \tag{2.2.8}$$

characterized as follows. Let $q: S^n \to S^{(n)}$ be the quotient map and $\pi_i: S^n \to S$ the projection to the i-th factor: then

$$q^* \circ t_p^*(\alpha) = \pi_1^* \alpha + \ldots + \pi_n^* \alpha, \qquad \alpha \in H^p(S; \mathbb{Z}). \tag{2.2.9}$$

For simplicity we will assume from now on that $H^*(S; \mathbb{Z})$ has *no torsion.* It follows by Künneth's decomposition that $H^*(S^n; \mathbb{Z})$ has no torsion and that we have an isomorphism

$$H^p(S^{(n)}; \mathbb{Z}) \cong H^p(S^n; \mathbb{Z})^{\Sigma_n}, \qquad p \leq 2. \tag{2.2.10}$$

Thus we have a series of isomorphisms

$$H^p(S; \mathbb{Z}) \xrightarrow[\sim]{t_p} H^p(S^{(n)}; \mathbb{Z}) \xrightarrow{\sim} H^p(S^{[n]}; \mathbb{Z}), \qquad p \leq 1. \tag{2.2.11}$$

For $p = 1$ this is the same isomorphism which one gets from **Proposition 2.1**. In order to describe integral 2-cohomology we must analyze $c_1(\mathscr{O}_{S^{[n]}}(\Delta_n))$. Let $\widetilde{V}_n$ be as in (2.1.7). The alternating group $A_n < \Sigma_n$ acts on $\widetilde{V}_n$, let $W_n := A_n\backslash\widetilde{V}_n$. Let $U_n \subset S^{n]}$ be the open subset given by (2.1.9). The natural map $\rho\colon W_n \longrightarrow U_n$ is a double cover ramified over $\Delta_n \cap U_n$. The action of $\mathbb{Z}/(2)$ on $\rho_*\mathscr{O}_{W_n}$ gives an eigenspace decomposition

$$\rho_*\mathscr{O}_{W_n} = \mathscr{O}_{U_n} \oplus \mathscr{L}_n^- \tag{2.2.12}$$

where $\mathscr{L}_n^-$ is the (-1)-eigensheaf - an invertible sheaf. By (2.1.4) there is an invertible sheaf $\mathscr{L}_n$ on $S^{[n]}$, unique up to isomorphism, extending $\mathscr{L}_n^-$. Let $\xi_n := c_1(\mathscr{L}_n^{-1})$. By construction

$$2\xi_n = c_1(\mathscr{O}_{S^{[n]}}(\Delta_n)). \tag{2.2.13}$$

Proposition 2.2. *Let S be a smooth complex projective surface. Assume that $H^*(S;\mathbb{Z})$ has no torsion. Then*

$$H^2(S^{[n]};\mathbb{Z}) = \operatorname{Im} t_2 \bigoplus \bigwedge^2 \operatorname{Im} t_1 \bigoplus \mathbb{Z}\xi_n \cong H^2(S;\mathbb{Z}) \bigoplus \bigwedge^2 H^1(S;\mathbb{Z}) \bigoplus \mathbb{Z}. \tag{2.2.14}$$

Moreover t_1 and t_2 are embeddings of Hodge structures.

Proof. By **Proposition 2.1** and the hypothesis that $H^*(S;\mathbb{Z})$ has no torsion we get that $H_1(S^{[n]};\mathbb{Z})$ has no torsion; by the Universal coefficients Theorem it follows that $H^2(S^{[n]};\mathbb{Z})$ has no torsion. By (2.2.7) we know that (2.2.14) holds when we tensor all members by $\mathbb{Q}$. It follows that in order to prove that (2.2.14) holds it suffices to find classes $\beta_1,\ldots,\beta_m \in H_2(S^{[n]};\mathbb{Z})$ (notice: we know $b_2(S^{[n]})$ by (2.2.7)) such that the pairing matrix between $\beta_1,\ldots,\beta_m$ and a basis of $(\operatorname{Im} t_2 \oplus \bigwedge^2 \operatorname{Im} t_1 \oplus \mathbb{Z}\xi_n)$ is unimodular. We leave this as an exercise except for one point. Let $p_1,\ldots,p_{n-1} \in S$ be pairwise distinct and

$$\Gamma_n := \gamma^{-1}(2p_1 + p_2 + \ldots + p_{n-1}) \subset S^{[n]}. \tag{2.2.15}$$

Then Γ_n is isomorphic to $\mathbb{P}^1$. In fact let $f\colon \widetilde{V}_n \to V_n$ be as in (2.1.7) and

$$\widetilde{\Gamma}_n := f^{-1}(2p_1 + p_2 + \ldots + p_{n-1}). \tag{2.2.16}$$

Then $\widetilde{\Gamma}_n$ is isomorphic to $\mathbb{P}^1$ (it is the typical fiber of the map from the exceptional divisor of f to the big diagonal D_n) and the restriction of g (see (2.1.8)) to $\widetilde{\Gamma}_n$ defines an isomorphism $\widetilde{\Gamma}_n \xrightarrow{\sim} \Gamma_n$. Since g is simply ramified along D_n we get that

$$\Delta_n \cdot \Gamma_n = (h^*\Delta_n) \cdot \widetilde{\Gamma}_n = 2D_n \cdot \widetilde{\Gamma}_n = -2. \tag{2.2.17}$$

Thus $\int_{\Gamma_n} \xi_n = -1$. The statement about t_1 and t_2 being embeddings of Hodge structures follows directly from their definition. □

Remark 2.3. The Betti numbers of $S^{[n]}$ have been computed by Göttsche [10], the Hodge numbers have been computed by Göttsche-Soergel [13]. See also the paper of J. Cheah [8].

2.3. **Holomorphic 2-forms on $S^{[n]}$.** Let $\varphi \in H^0(\Omega_S^2)$: we will associate to φ a regular 2-form on $S^{[n]}$. For $1 \le i \le n$ let $\rho_i\colon \widetilde{V}_n \to S$ be the composition of the blow-down map (2.1.7) and projection to the i-th factor. The regular 2-form $\sum_{i=1}^n \rho_i^*\varphi$ is Σ_n-invariant. Let h be quotient map (2.1.9). A local computation shows that $\sum_{i=1}^n \rho_i^*\varphi$ descends to a regular 2-form on U_n. By (2.1.4) we get that the descended 2-form extends to a regular 2-form $\varphi^{[n]}$ on $S^{[n]}$. We have defined a homomorphism

$$\begin{array}{ccc} H^0(\Omega_S^2) & \hookrightarrow & H^0(\Omega_{S^{[n]}}^2) \\ \varphi & \mapsto & \varphi^{[n]} \end{array} \tag{2.3.1}$$

In order to describe $\operatorname{div}(\bigwedge^n \varphi^{[n]})$ we introduce a piece of notation. Let D be an integral curve on S: we let Σ_D be the prime divisor on $S^{(n)}$ given by

$$\Sigma_D^{(n)} := \{A \in S^{(n)} \mid A \cap D \neq \varnothing\}. \tag{2.3.2}$$

Extending by linearity we get a map

$$\begin{array}{ccc} \operatorname{Div}(S) & \longrightarrow & \operatorname{Div}(S^{(n)}) \\ D & \mapsto & \Sigma_D^{(n)} \end{array} \tag{2.3.3}$$

Proposition 2.4. *Keep notation as above and let $0 \neq \varphi \in H^0(\Omega_S^2)$. Let $D = \operatorname{div}(\varphi) \in |K_S|$. Then*

$$\operatorname{div}(\varphi^{[n]}) = \gamma^*(\Sigma_D^{(n)}). \tag{2.3.4}$$

Proof. By (2.1.4) it suffices to prove that Equality (2.3.4) holds on the open subset U_n. This is clear away from $\Delta_n \cap U_n$. Thus $\operatorname{div}(\bigwedge^n \varphi^{[n]}) = a\Delta_n + \gamma^*(\Sigma_D^{(n)})$ for some $a \geq 0$. Let ω_{Δ_n} be the dualizing sheaf of Δ_n. By adjunction we get that $\omega_{\Delta_n} \cong \mathscr{O}_{\Delta_n}((a+1)\Delta_n)$. The restriction of the Hilbert-Chow map to the open dense $\gamma^{-1}(V_n \cap \Delta_n) \subset \Delta_n$ is a $\mathbb{P}^1$-fibration; the fibers are the Γ_n given by (2.2.15). It follows that $\omega_{\Delta_n}|_{\Gamma_n} \cong \mathscr{O}_{\Gamma_n}(-2)$. By (2.2.17) we get that $a = 0$. □

2.4. **Beauville's examples.** Throughout the present subsection $n \geq 2$. Let S be a projective $K3$ surface. Then $S^{[n]}$ is a smooth projective variety of dimension $2n$. We will prove that $S^{[n]}$ is hyperkähler and that

$$b_2(S^{[n]}) = 23. \tag{2.4.1}$$

First $S^{[n]}$ is simply-connected by **Proposition 2.1**. Let $\varphi \in H^0(\Omega_S^2)$ be non-zero. Then φ is symplectic because S is a $K3$ surface and hence $\varphi^{[n]} \in H^0(\Omega^2_{S^{[n]}})$ is symplectic by **Proposition 2.4**. Lastly **Proposition 2.2** gives that $h^{2,0}(S^{[n]}) = 1$ and that (2.4.1) holds (the second Betti number of a $K3$ surface equals 22 by Noether's formula). Now we pass to Beauville's other examples. Let T be an abelian surface. The Hilbert scheme $T^{[n+1]}$ carries a holomorphic symplectic form but it is not HK: in fact the fibration

$$\begin{array}{ccc} T^{[n+1]} & \xrightarrow{s_{n+1}} & T \\ Z & \mapsto & \sum_{p\in T} \ell(\mathscr{O}_{Z,p})p \end{array} \tag{2.4.2}$$

(the sum is in the group T and $\ell(\mathscr{O}_{Z,p})$ is equal to the dimension of $\mathscr{O}_{Z,p}$ as $\mathbb{C}$-vector space) shows that $H_1(T^{[n+1]};\mathbb{Q}) \neq 0$ and also that $T^{[n+1]}$ carries non-zero holomorphic 2-forms which are not symplectic. Let

$$K^{[n]}(T) := s_{n+1}^{-1}(0). \tag{2.4.3}$$

Since $T^{[n+1]}$ has dimension $2(n+1)$ and the summation map s_{n+1} is locally trivial (in the classical or étale topology) the dimension of $K^{[n]}(T)$ is equal to $2n$. The variety $K^{[n]}(T)$ is known as a *generalized Kummer variety* because if $n = 1$ it is isomorphic to the Kummer surface of T (here by Kummer surface we mean the minimal desingularization of the singular Kummer of T given by the quotient $T/\langle(-1)\rangle$ where (-1) is multiplication by (-1)). We will prove that $K^{[n]}(T)$ is a hyperkähler variety and that

$$b_2(K^{[n]}(T)) = 7. \tag{2.4.4}$$

First let's prove that $K^{[n]}(T)$ is simply-connected. The long exact sequence of homotopy groups associated to Fibration (2.4.2) gives an exact sequence

$$\pi_2(T) \longrightarrow \pi_1(K^{[n]}(T)) \longrightarrow \pi_1(T^{[n+1]}) \xrightarrow{s_{n+1,\sharp}} \pi_1(T) \tag{2.4.5}$$

The map $s_{n+1,\sharp}$ is an isomorphism, see the proof of **Proposition 2.1**. Since $\pi_2(T)$ is trivial it follows that $K^{[n]}(T)$ is simply-connected. Next one shows that restriction gives a surjection

$$H^2(T^{[n+1]};\mathbb{Q}) \twoheadrightarrow H^2(K^{[n]}(T);\mathbb{Q}). \tag{2.4.6}$$

The assertion about surjectivity follows from irreducibility of $\Delta_{n+1}|_{K^{[n]}(T)}$ (that is why we need to assume that $n \geq 2$) and from a surjectivity statement involving

$$V_{n+1}^0 := \{(x_1,\ldots,x_{n+1}) \in V_{n+1} \mid x_1 + \ldots + x_{n+1} = 0\}, \tag{2.4.7}$$

namely that restriction gives a surjection $H^2(V_{n+1};\mathbb{Q}) \to H^2(V_{n+1}^0;\mathbb{Q})$. This proves surjectivity of (2.4.6). Now look at Equation (2.2.14) for $S = T$ with n replaced by $(n+1)$. Since $K^{[n]}(T)$ is simply connected we get that restriction defines a surjection

$$H^2(T;\mathbb{Q}) \bigoplus \mathbb{Q}\xi_{n+1} \twoheadrightarrow H^2(K^{[n]}(T);\mathbb{Q}). \tag{2.4.8}$$

In order to prove that the above map is an isomorphism we consider the regular map

$$\begin{array}{ccc} K^{[n]}(T) \times T & \xrightarrow{f} & T^{[n+1]} \\ (Z,a) & \mapsto & \tau_a(Z) \end{array} \tag{2.4.9}$$

where $\tau_a: T \to T$ is translation by a. The map f is Galois with group $T[n+1]$ (the group of $(n+1)$-torsion points of T). Since $K^{[n]}(T)$ is simply connected the Künneth decomposition gives an isomorphism $H^2(K^{[n]}(T) \times T;\mathbb{Q}) \cong H^2(K^{[n]}(T);\mathbb{Q}) \oplus H^2(T;\mathbb{Q})$. Thus $H^2(f)$ defines an injection $H^2(T^{[n+1]};\mathbb{Q}) \hookrightarrow H^2(K^{[n]}(T);\mathbb{Q}) \oplus H^2(T;\mathbb{Q})$; keeping in mind (2.4.8) we get that

$$2b_2(T) + 1 = b_2(T^{[n+1]}) \leq b_2(K^{[n]}(T)) + b_2(T) \leq 2b_2(T) + 1. \tag{2.4.10}$$

Thus the inequalities above are equalities and hence (2.4.8) is an isomorphism. This proves (2.4.4) and hence also that $h^{2,0}(K^{[n]}(T)) = 1$ (see the claim about Hodge structures in the statement of **Proposition 2.2**). It remains to prove that there exists a holomorphic 2-form on $K^{[n]}(T)$. Let $0 \neq \varphi \in H^0(\Omega_T^2)$. By **Proposition 2.4** the holomorphic 2-form $\varphi^{[n+1]}$ on $T^{[n+1]}$ is symplectic. Since f is śtale the pull-back $f^*\varphi^{[n+1]}$ is a symplectic form on $K^{[n]}(T) \times T$. Since $K^{[n]}(T)$ is simply-connected there exist a holomorphic 2-form α on $K^{[n]}(T)$ and a holomorphic 2-form β on T such that $f^*\varphi^{[n+1]} = p^*\alpha + q^*\beta$ where p, q are the projections of $K^{[n]}(T) \times T$ onto the first and second factor respectively. Since $f^*\varphi^{[n+1]}$ is symplectic both α and β are simplectic; thus α is a holomorphic symplectic form on $K^{[n]}(T)$.

Remark 2.5. By **Proposition 2.4** the variety $T^{[n+1]}$ has trivial first Chern class: Map (2.4.9) is its Beauville-Bogomolov decomposition of $T^{[n+1]}$.

Remark 2.6. The Hodge numbers of generalized Kummer varieties have been computed by Göttsche and Soergel [13].

3. Moduli of sheaves on symplectic surfaces

3.1. **Moduli of semistable sheaves.** A general reference for moduli of semistable sheaves is [17]. Let X be a complex projective variety and H an ample Cartier divisor on S. We let $\mathscr{O}_X(1) := \mathscr{O}_X(H)$. Let F be a coherent sheaf on X (unless we state the contrary sheaves are always assumed to be coherent). Let $\mathrm{Ann}(F) \subset \mathscr{O}_X$ be the annihilator of F. Thus $\mathrm{Ann}(F)$ is an ideal sheaf; the *support of* F is the subscheme of X defined by $\mathrm{supp}(F) := V(\mathrm{Ann}(F))$. The *dimension of* F is equal to the dimension of $\mathrm{supp}(F)$; we denote it by $\dim(F)$. The sheaf F is *pure* if any non-zero subsheaf $G \subset F$ has dimension equal to $\dim(F)$.

Example 3.1. If $\dim X = 1$ then a sheaf is pure if and only if it is torsion-free. If $\dim X = 2$ a sheaf is pure of dimension 2 if and only if it is torsion-free. An example of a pure sheaf of dimension 1 on a surface X is given by $F := \iota_* V$ where $\iota: C \hookrightarrow X$ is the inclusion of an irreducible curve and V is a torsion-free sheaf on C.

Given a sheaf F on X we let $F(n) := F \otimes \mathscr{O}_X(n)$. Suppose that F is non-zero and let $d := \dim(F) \geq 0$. The Hilbert polynomial $\chi(F(n))$ is integer-valued, it follows that there exists a unique sequence of integers a_i for $0 \leq i \leq d$ such that

$$\chi(F(n)) = \sum_{i=0}^{d} a_i \binom{n}{i} \qquad \forall n \in \mathbb{Z}. \tag{3.1.1}$$

Furthermore $a_d(F) > 0$; the *multiplicity of* F is equal to $a_d(F)$.

Example 3.2. Suppose that $\dim(F) = \dim X$. Then F is locally-free on an open dense subset $X_0 \subset X$ and the rank of F, denoted by $\mathrm{rk}(F)$, is equal to the rank of the vector-bundle $F|_{X_0}$. Then $a_d(F) = \mathrm{rk}(F) \int_X c_1(H)^d$.

Definition 3.3. Let F be a sheaf on X. Let $d := \dim(F)$. The *reduced Hilbert polynomial of* F, denoted by p_F is defined by

$$p_F(n) := \frac{\chi(F(n))}{a_d(F)}. \tag{3.1.2}$$

The set of isomorphism classes of sheaves on X with fixed Hilbert polynomial does not have a natural structure of quasi-projective variety except in special cases. The largest family of sheaves having a good moduli space is that of pure semistable sheaves.

Definition 3.4. Let X be a smooth irreducible projective variety equipped with an ample divisor H. A non-zero pure sheaf F on X is H-semistable if for every non-zero subsheaf $E \subset F$ we have

$$p_E(n) \leq p_F(n) \qquad \forall n \gg 0. \tag{3.1.3}$$

If strict inequality holds whenever $E \neq F$ then F is H-stable.

Example 3.5. If $\dim F = \dim X$ and F has rank 1 then F is stable for arbitrary H. Suppose that $F = F_1 \oplus F_2$ with $F_i \neq 0$; then F is H-semistable if and only if each F_i is semistable and $p_{F_1} = p_{F_2}$.

We notice that in general (semi)stability does depend on the choice of H.

Claim 3.6. *Let X be a complex projective variety with ample Cartier divisor H and F a pure H-stable sheaf on X. Then F is* simple *i.e.* $\mathrm{Hom}(F,F)=\mathbb{C}\,\mathrm{Id}_F$.

Proof. Let $d := \dim F$. Assume that $\varphi: F \to F$ is a non-zero morphism of sheaves. We claim that φ is an isomorphism. In fact assume that $E := \ker\varphi \neq 0$ and let $G := \mathrm{Im}\,\varphi$. We have an exact sequence of pure d-dimensional sheaves

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0. \tag{3.1.4}$$

In particular $a_d(F) = a_d(E) + a_d(G)$. It follows that

$$p_F(n) = \frac{a_d(E)}{a_d(E)+a_d(G)}p_E(n) + \frac{a_d(G)}{a_d(E)+a_d(G)}p_G(n) \tag{3.1.5}$$

i.e. $p_F(n)$ lies in the segment spanned by $p_E(n)$ and $p_G(n)$. By stability of F we have that $p_E(n) < p_F(n)$ for $n \gg 0$. It follows that $p_F(n) < p_G(n)$ for $n \gg 0$: that is a contradiction because G is a subsheaf of F. We have proved that φ is injective. Thus φ is an injection $F \hookrightarrow F$. By stability we get that $\varphi(F) = F$. This proves that an endomorphism $\varphi: F \to F$ is either zero or an isomorphism. Thus $\mathrm{Hom}(F,F)$ is a finitely generated division $\mathbb{C}$-algebra: since $\mathbb{C}$ is algebraically closed it follows[4] that $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. □

For pure sheaves of dimension equal to $\dim X$ there is the notion of μ(*slope*)-semistability: one replaces the reduced Hilbert polynomial by the slope. The slope of a sheaf F of dimension equal to $\dim X$ is

$$\mu(F) := \frac{1}{\mathrm{rk}(F)}\int_X c_1(F)\cdot c_1(H)^{\dim X-1}. \tag{3.1.6}$$

F is μ-semistable (with respect to H) if for every non-zero subsheaf $E \subset F$ we have

$$\mu(E) \le \mu(F). \tag{3.1.7}$$

If strict inequality holds whenever $E \neq F$ then F is μ-stable. Notice that for a pure sheaf of dimension equal to $\dim X$ we have the following implications:

$$F \text{ is semistable} \Longrightarrow \quad F \text{ is } \mu\text{-semistable} \tag{3.1.8}$$

$$F \text{ is } \mu\text{-stable} \Longrightarrow \quad F \text{ is stable} \tag{3.1.9}$$

In order to obtain a separated moduli space we need to consider an equivalence relation which is weaker than isomorphism. Let F be a pure H-semistable sheaf on X. There exists (see [17]) a *Jordan-Hölder (J-H) filtration of F*

$$0 = F_0 \subset F_1 \subset \cdots \subset F_\ell = F \tag{3.1.10}$$

with the property that each quotient F_i/F_{i-1} is pure, H-stable with reduced Hilbert polynomial equal to P_F. A trivial example: if F is H-stable then a J-H filtration of F is necessarily trivial. Another example: $F = L \otimes_{\mathbb{C}} V$ where L is a line-bundle on X and V is a vector-space of dimension r. In this case the set of J-H filtrations of F is in bijective correspondence with the set of complete flags on V. As we see from the last example a J-H filtration is not unique. One proves that although the J-H filtration is not unique the associated graded sum

$$gr^{JH}(F) := \oplus_{i=1}^{\ell} F_i/F_{i-1} \tag{3.1.11}$$

is unique up to isomorphism.

Definition 3.7. Let F and G be pure H-semistable sheaves on X. Then F is S-equivalent to G if $gr^{JH}(F) \cong gr^{JH}(G)$.

If F is H-stable then F is S-equivalent to G if and only if $F \cong G$. On the other hand assume that F fits into the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0 \tag{3.1.12}$$

where E, G are pure with $p_E = p_F = p_G$. Then F is S-equivalent to $E \oplus G$. Let P be an integer-valued polynomial of degree at most $\dim X$; we let

$$\mathscr{M}_X(P) := \{F \text{ pure } H\text{-s.s. sheaf on } X \mid \chi(F(n)) = P(n)\}/S\text{-equivalence} \tag{3.1.13}$$

(We do not include H in the notation although the isomorphism class of the moduli space does depend on H in general.) The main general theorem on moduli of pure sheaves is the result of research that was

[4]Suppose that $\varphi \in (\mathrm{Hom}(F,F) \setminus \mathbb{C}\,\mathrm{Id}_F)$: then φ generates a non-trivial algebraic field extension of $\mathbb{C}$, that is a contradiction.

done during several years. Among the main constributors we quote Mumford, Seshadri, Narasimhan, Gieseker, Maruyama, Simpson. The most general result is due to Simpson [49].

Theorem 3.8. *There exists a projective scheme $\mathscr{M}_X(P)$ with the following properties:*

1. *Let T be a scheme and $\mathscr{F}$ be a sheaf on $X \times T$ which is $\mathscr{O}_T$-flat and such that for all $t \in T$ the restriction $\mathscr{F}|_{X\times\{t\}}$ is pure H-semistable with Hilbert polynomial P. Then there exists a regular map $T \to \mathscr{M}_X(P)$ which associates to closed points their S-equivalence class.*
2. *$\mathscr{M}_X(P)$ "dominates"any other scheme satisfying Item (1).*

Given a pure H-semistable sheaf with Hilbert polynomial P we let $[F] \in \mathscr{M}_X(P)$ be the point corresponding to the S-equivalence class of F. Let $\mathscr{M}_X(P)^{st} \subset \mathscr{M}_X(P)$ be the subset parametrizing stable sheaves: then $\mathscr{M}_X(P)^{st}$ is open. Let $[F] \in \mathscr{M}_X(P)^{st}$: there is a natural identification between the germ of $\mathscr{M}_X(P)$ at $[F]$ and the universal deformation space of F. In particular we get the following result.

Proposition 3.9. *Let $[F] \in \mathscr{M}_X(P)^{st}$. There is a natural isomorphism*

$$\Theta_{[F]}\mathscr{M}_X(P) \cong \operatorname{Ext}^1(F,F). \tag{3.1.14}$$

Let F be a (coherent) sheaf on X; one can define a *trace map*

$$\operatorname{Tr}^i : \operatorname{Ext}^i(F,F) \to H^i(\mathscr{O}_X) \tag{3.1.15}$$

which is the obvious map for $i = 0$ - see [17]. (If F is locally-free then Tr^i is induced by the sheaf map $EndF \xrightarrow{Tr} \mathscr{O}_X$.) We let

$$\operatorname{Ext}^i(F,F)^0 := \ker \operatorname{Tr}^i. \tag{3.1.16}$$

Theorem 3.10 (Mukai [37], Artamkin [3])**.** *Suppose that $[F] \in \mathscr{M}_X(P)^{st}$ and that* $\operatorname{Ext}^2(F,F)^0 = 0$. *Then $\mathscr{M}_X(P)$ is smooth at $[F]$ and its tangent space is canonical identified with* $\operatorname{Ext}^1(F,F)$.

3.2. **Semistable sheaves on symplectic surfaces.** Let S be a symplectic projective surface i.e. a projective $K3$ or an abelian surface. We let

$$\widetilde{H}(S) := H^0(S) \oplus H^2(S) \oplus H^4(S) \tag{3.2.1}$$

We denote elements of $\widetilde{H}(S)$ by (r,ℓ,s) where $\ell \in H^2(S)$ and $r,s \in \mathbb{C}$ (we identify $H^4(S)$ with $\mathbb{C}$ via the orientation class η of S). Given $\alpha = (r,\ell,s) \in \widetilde{H}(S)$ we let

$$\alpha^\vee := (r,-\ell,s). \tag{3.2.2}$$

One gives $\widetilde{H}(S)$ a Hodge structure of weight 2 as follows:

$$\widetilde{H}(S)^{2,0} = H^{2,0}(S), \quad \widetilde{H}(S)^{0,2} = H^{0,2}(S), \quad \widetilde{H}(S)^{1,1} = H^0(S) \oplus H^{1,1}(S) \oplus H^4(S). \tag{3.2.3}$$

Thus $\widetilde{H}(S)$ has an integral Hodge structure - the integral structure coming from $\widetilde{H}(S;\mathbb{Z})$. The *Mukai lattice* [36] of S is the group $\widetilde{H}(S;\mathbb{Z})$ equipped with the symmetric bilinear form

$$\langle (r,\ell,s),\ (r',\ell',s')\rangle := \deg(\ell \cup \ell') - rs' - r's = \deg((r+\ell+s\eta) \cup (r'+\ell'+s'\eta)^\vee). \tag{3.2.4}$$

(Here $\eta \in H^4(S)$ is the orientation class of S.) Notice that $\langle,\rangle$ is even, unimodular, of signature $(4, b_2(S)-2)$ i.e. $(4,20)$ if S is a $K3$ and $(4,4)$ if S is an abelian surface. Let F be a coherent sheaf on S; following Mukai [36] one sets

$$v(F) := \operatorname{ch}(F)\sqrt{\operatorname{Td}(S)} = ch(F)(1+\epsilon\eta), \tag{3.2.5}$$

where $\eta \in H^4(S;\mathbb{Z})$ is the orientation class and ϵ is equal to 1 if S is a $K3$ surface and is equal to 0 if S is an abelian surface. Notice that $v(F) \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S)$. By Hirzebruch-Riemann-Roch we have

$$\langle v(E), v(F)\rangle = -\chi(E,F) := -\sum_{i=0}^{2}(-1)^i \dim \operatorname{Ext}^i(E,F). \tag{3.2.6}$$

By Serre duality we have $\operatorname{Ext}^2(F,F) \cong \operatorname{Hom}(F,F)^\vee$ and hence (3.2.6) gives

$$\dim \operatorname{Ext}^1(F,F) = 2\dim \operatorname{Hom}(F,F) + \langle v(F), v(F)\rangle. \tag{3.2.7}$$

Definition 3.11. A *Mukai vector* is a

$$\mathbf{v} = (r,\ell,s) \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S) \tag{3.2.8}$$

such that $r \geq 0$ and such that ℓ is effective if $r = 0$.

Notice that if F is a pure sheaf of dimension 2 or 1 then $v(F)$ is a Mukai vector. Now let H be an ample divisor on S and let $\mathscr{O}_S(1) \cong \mathscr{O}_S(H)$. One indicizes moduli spaces of H-semistable pure sheaves on S by Mukai vectors. Let $\mathbf{v} \in \widetilde{H}^{1,1}$ be a Mukai vector: the Hilbert polynomial $\chi(F(n))$ of a sheaf F such that $v(F) = \mathbf{v}$ is independent of F, call it $P_{\mathbf{v}}$. Let

$$\mathscr{M}_S(\mathbf{v}) := \{[F] \in \mathscr{M}_S(P_{\mathbf{v}}) | \; v(F) = \mathbf{v}\}. \tag{3.2.9}$$

The above subset of $\mathscr{M}_S(P_{\mathbf{v}})$ is open hence it inherits a natural structure of scheme. Of course $\mathscr{M}_S(\mathbf{v})$ depends on H: whenever there is the potential for confusion we will denote $\mathscr{M}_S(\mathbf{v})$ by $\mathscr{M}_{S,H}(\mathbf{v})$.

Proposition 3.12. *Let S be a projective symplectic surface and H an ample divisor on S. Let $\mathbf{v}$ be a Mukai vector. Then*

1. *$\mathscr{M}_S(\mathbf{v})$ is a projective scheme.*
2. *Suppose that $[F] \in \mathscr{M}_S(\mathbf{v})^{st}$. Then $\mathscr{M}_S(\mathbf{v})$ is smooth at $[F]$ and*

$$\dim_{[F]} \mathscr{M}_S(\mathbf{v}) = 2 + \langle \mathbf{v}, \mathbf{v}\rangle. \tag{3.2.10}$$

In particular if $\mathbf{v}^2 < (-2)$ then $\mathscr{M}_S^{st}(\mathbf{v})$ is empty.

Proof. (1): This is because $\mathscr{M}_S(\mathbf{v})$ is a union of connected components of $\mathscr{M}_S(P_{\mathbf{v}})$ (it is both open and closed). (2): Since F is stable **Claim 3.6** gives that $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. By Serre duality it follows that $\mathrm{Ext}^2(F,F)^0 = 0$. By **Theorem 3.10** we get that $\mathscr{M}_S(\mathbf{v})$ is smooth at $[F]$ with tangent space canonically identified with $\mathrm{Ext}^1(F,F)$. Thus Equation (3.2.10) follows from (3.2.7) and $\mathrm{Hom}(F,F) = \mathbb{C}\,\mathrm{Id}_F$. □

Example 3.13. Let S be a $K3$ surface. We have an isomorphism

$$\begin{array}{ccc} S^{[n]} & \overset{\sim}{\longrightarrow} & \mathscr{M}_S(1,0,(n-1)) \\ [Z] & \mapsto & [I_Z]. \end{array} \tag{3.2.11}$$

Moduli spaces of semistable sheaves on a symplectic surface S are relevant for the theory of HK manifolds because one can (following Mukai) associate to a homolorphic symplectic form on S a homolorphic symplectic form on $\mathscr{M}_S(\mathbf{v})^{st}$. In fact let $\varphi \in \Gamma(\Omega_S^2)$: one defines a 2-form $\tau(\varphi)$ on $\mathscr{M}_S(\mathbf{v})^{st}$ by setting

$$\tau(\varphi)(\alpha,\beta) := \int_S \varphi \wedge \mathrm{Tr}^2(\alpha \cup \beta), \tag{3.2.12}$$

where $\cup$ denotes Yoneda product. If F is locally-free $\cup$ is induced by the map of sheaves

$$\begin{array}{ccc} End F \otimes End F & \longrightarrow & End F \\ (\phi,\psi) & \mapsto & \phi \circ \psi \end{array} \tag{3.2.13}$$

Proposition 3.14 (Mukai [37])**.** *Keep notation and hypothese as above. Then $\tau(\varphi)$ is holomorphic and closed. If φ is non-zero then $\tau(\varphi)$ is symplectic at each point of $\mathscr{M}_S(\mathbf{v})^{st}$.*

(Actually closedness of $\tau(\varphi)$ without the assumption that $\mathscr{M}_S(\mathbf{v})^{st}$ is closed is proved elsewhere - see for example [39].) Notice that non-degeneracy of $\tau(\varphi)$ follows immediately from Serre-duality.

Remark 3.15. Let S be a $K3$ surface and $\varphi \in H^0(\Omega_S^2)$. The Hilbert scheme $S^{[n]}$ is identified with the moduli space $\mathscr{M}_S(1,0,-(n-1))$, see **Example 3.13**, and hence we have the holomorphic 2-forms $\varphi^{[n]}$ and $\tau(\varphi)$. The relation between the forms is the following:

$$\tau(\varphi) = -4\pi^2 \varphi^{[n]}. \tag{3.2.14}$$

In order to study moduli of semistable sheaves on $K3$ and abelian surfaces one must introduce the notion of generic polarization. Let S be a smooth projective surface. Let $\mathrm{NS}(S)$ be the Néron-Severi group of S i.e. $H^{1,1}_{\mathbb{Z}}(S)$ and $\mathrm{NS}(S)_{\mathbb{R}} := \mathrm{NS}(S) \otimes_{\mathbb{Z}} \mathbb{R}(S)$. Let $A(S) \subset \mathrm{NS}(S)$ be the ample cone and $A(S)_{\mathbb{R}} \subset \mathrm{NS}(S)_{\mathbb{R}}$ be its tensor product with $\mathbb{R}$. A *wall* consist of $W_D := D^{\perp} \cap A(S)_{\mathbb{R}}$ where D is a divisor on S with *strictly negative* self-intersection.

Proposition 3.16. *Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S. There exists a union of walls $\mathscr{W} = \bigcup_{D \in \mathscr{C}} W_D$ with the following properties:*

1. *$\mathscr{W}$ is locally finite and hence the complement in $A(S)_{\mathbb{R}}$ is a dense open subset.*
2. *Let $H \in (A(S) \setminus \mathscr{W})$ and F be a strictly H-semistable (i.e. semistable but* not *stable) sheaf with $v(F) = \mathbf{v}$. Then $gr^{JH}(F) = \bigoplus_i E_i$ where for each i we have $v(E_i) = a_i \mathbf{v}$ with $a_i \in \mathbb{Q}$.*

(3) *Let $H_1, H_2 \in (A(S) \setminus \mathscr{W})$ belong to the same connected component of $(A(S) \setminus \mathscr{W})$. A sheaf F with $v(F) = \mathbf{v}$ is H_1-semistable if and only if it is H_2-semistable.*

Definition 3.17. Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S. An ample divisor H on S is $\mathbf{v}$-*generic* if it lies outside the minimal union of walls $\mathscr{W}$ for which the conclusions of **Proposition 3.16** hold.

The following result is an immediate consequence of **Proposition 3.16**.

Corollary 3.18. *Let S be a projective symplectic surface and $\mathbf{v}$ be an* indivisible *Mukai vector for S. Let H be an ample $\mathbf{v}$-generic divisor. Then $\mathscr{M}_S(\mathbf{v})^{st} = \mathscr{M}_S(\mathbf{v})$.*

Suppose that $\mathbf{v}$ is an indivisible Mukai vector for S and H is a $\mathbf{v}$-generic ample divisor on S. If $\mathscr{M}_S(\mathbf{v})$ is non-empty then by **Corollary 3.18** and **Proposition 3.14** the moduli space $\mathscr{M}_S(\mathbf{v})$ is a smooth projective variety of dimension $(2+\mathbf{v}^2)$ carrying a holomorphic symplectic form. In **Subsection 3.3** we will indicate how one proves that $\mathscr{M}_S(\mathbf{v})$ is a deformation of $K3^{[n]}$ if S is a $K3$ (if S is an abelian surface then the Beauville-Bogomolov decomposition of $\mathscr{M}_S(\mathbf{v})$ is the product of S, its dual variety and a deformation of the generalized Kummer $K^{[n]}(S)$).

Remark 3.19. Let S be a projective symplectic surface and $\mathbf{v} = (r, \ell, s)$ a Mukai vector for S with $-2 \le \mathbf{v}^2$ (see Item (2) of **Proposition 3.12**) and $r > 0$. Let

$$k := \begin{cases} \frac{r^2}{4}\mathbf{v}^2 + \frac{r^4}{2} & \text{if } S \text{ is a } K3, \\ \frac{r^2}{4}\mathbf{v}^2 & \text{if } S \text{ is an abelian surface.} \end{cases} \tag{3.2.15}$$

Let $\mathscr{W} \subset A(S)_{\mathbb{R}}$ be the union of the walls W_D where D runs through the set of divisors such that $-k \le D \cdot D < 0$. Then the conclusions of **Proposition 3.16** hold for the above $\mathscr{W}$.

3.3. **Moduli of semistable sheaves on symplectic surfaces: indivisible Mukai vector.** Let (S, H) be a polarized symplectic surface and $\mathbf{v} \in \widetilde{H}(S)$ a Mukai vector. Suppose that $\mathscr{M}_S(\mathbf{v}) = \mathscr{M}_S(\mathbf{v})^{st}$ - this will be the case if $\mathbf{v}$ is indivisible and H is $\mathbf{v}$-generic by **Corollary 3.18**. Following Mukai [36] one define a map $\widetilde{H}(S) \to H^2(\mathscr{M}_S(\mathbf{v}))$ as follows. Let us assume for the moment that there exists a tautological sheaf $\mathscr{F}$ on $S \times \mathscr{M}_S(\mathbf{v})$ i.e. a sheaf which is flat over $\mathscr{M}_S(\mathbf{v})$ and such that for all $[F] \in \mathscr{M}_S(\mathbf{v})$ the restriction $\mathscr{F}|_{S \times \{[F]\}}$ is isomorphic to F. Let $\pi: S \times \mathscr{M}_S(\mathbf{v}) \to S$ and $\rho: S \times \mathscr{M}_S(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ be the projections. We let

$$\begin{array}{ccc} \widetilde{H}(S) & \longrightarrow & H^2(\mathscr{M}_S(\mathbf{v})) \\ \alpha & \mapsto & \rho_*\left(\left[\mathrm{ch}(\mathscr{F}) \cup \pi^* \sqrt{\mathrm{Td}\, S} \cup \pi^* \alpha^\vee\right]_3\right) \end{array} \tag{3.3.1}$$

where $\alpha^\vee$ is as in (3.2.2) and $[\ldots]_3$ means the component lying in $H^6(S \times \mathscr{M}_S(\mathbf{v}))$. A tautological sheaf does not necessarily exist, but a quasi-tautological sheaf exists and one may define a map as above with $\mathscr{F}$ replaced by a quasi-tautological sheaf of similarity m and then divide by m, see [36] for details. The above map depends on the (quasi)tautological sheaf, but its restriction to $\mathbf{v}^\perp$ is independent of the (quasi)tautological sheaf: one denotes it by

$$\theta_{\mathbf{v}}: \mathbf{v}^\perp \longrightarrow H^2(\mathscr{M}_S(\mathbf{v})). \tag{3.3.2}$$

Clearly $\theta_{\mathbf{v}}$ is a morphism of Hodge structures - the definition of Hodge structure on $\widetilde{H}(S)$ is motivated by this observation.

Theorem 3.20 (Mukai, Göttsche - Huybrechts, O'Grady, Yoshioka)**.** *Let S be a projective $K3$ surface. Let $\mathbf{v}$ be Mukai vector as in* (3.2.8) *and suppose that*

(1) $\mathbf{v}$ *is indivisible,*
(2) $-2 \le \mathbf{v}^2$,
(3) $(r, s) \ne (0, 0)$.

Let H be a $\mathbf{v}$-generic ample divisor on S. Then $\mathscr{M}_S(\mathbf{v})$ is an irreducible symplectic variety deformation equivalent to $S^{[n]}$ where $2n = (2+\mathbf{v}^2)$. Now suppose that $0 \le \mathbf{v}^2$. Then the map $\theta_{\mathbf{v}}$ of (3.3.2) *is integral, the restriction to integral cohomology surjects onto $H^2(\mathscr{M}_S(\mathbf{v}); \mathbb{Z})$, and*

(1) $\ker(\theta_{\mathbf{v}}) = \mathbb{C}\mathbf{v}$ *if* $\mathbf{v}^2 = 0$,
(2) $\theta_{\mathbf{v}}$ *is injective if* $2 \le \mathbf{v}^2$.

Mukai [36] proved **Theorem 3.20** when $\dim \mathscr{M}_S(\mathbf{v}) = 2$, Göttsche and Huybrechts [12] proved it for rank 2 and c_1 is indivisible, O'Grady [40] assumed that the rank is non-zero and c_1 is indivisible, Yoshioka [50, 51] proved it in general. Moreover the statements in [12, 40] are that $\mathscr{M}_S(\mathbf{v})$ is an irreducible symplectic variety which deforms to a variety birational to $(K3)^{[n]}$; the stronger statement follows by applying a general theorem of Huybrechts [14] which states that birational HK manifolds are deformation equivalent.

Remark 3.21. Let (S,H), $\mathbf{v}$ be as in **Theorem 3.20**. In general the moduli space $\mathscr{M}_S(\mathbf{v})$ is not isomorphic to a Hilbert scheme $F^{[n]}$, not even birational. Thus **Theorem 3.20** provides explicit examples of HK deformations of $K3^{[n]}$ which are not isomorphic to a Hilbert scheme of a $K3$ surface.

There is a result valid for moduli spaces of semistable sheaves on a polarized abelian surface (T,Θ) which is analogous to **Theorem 3.20**. Given a 0-cycle $Z = \sum_i n_i(p_i)$ on T we let $\sigma(Z) \in T$ be given by $\sigma(Z) := \sum_i n_i p_i$ (we take the sum in the group T); if Z' is rationally equivalent to Z then $\sigma(Z') = \sigma(Z)$ and hence we have a well-defined homomorphism $\sigma\colon \mathrm{CH}_0(T) \to T$. Let $\mathbf{v}$ be a Mukai vector on T and

$$\begin{array}{ccc} \mathscr{M}_T(\mathbf{v}) & \xrightarrow{A_{\mathbf{v}}} & T \times Pic(T) \\ [F] & \mapsto & (\sum c_2^{CH}(F), [c_1^{CH}(F)]) \end{array} \tag{3.3.3}$$

where $c_i^{CH}(F)$ denotes the i-th Chern class in the Chow group of T. The map $A_{\mathbf{v}}$ is regular. Choose $[F_0] \in \mathscr{M}_S(\mathbf{v})$ (assuming $\mathscr{M}_S(\mathbf{v})$ is non-empty) and let $\alpha_0 := c_1^{CH}(F_0)$. We let

$$\mathscr{M}_T(\mathbf{v})^0 := A_{\mathbf{v}}^{-1}(0,\alpha_0). \tag{3.3.4}$$

The isomorphism class of $\mathscr{M}_T(\mathbf{v})^0$ is independent of the choice of F_0 as soon as $4 \le \langle \mathbf{v},\mathbf{v}\rangle$.

Theorem 3.22 (Yoshioka [51])**.** *Let T be an abelian surface. Let $\mathbf{v}$ be a Mukai vector as in* (3.2.8) *and assume that*

1. $\mathbf{v}$ *is indivisible,*
2. $4 \le \mathbf{v}^2$,
3. $(r,s) \ne (0,0)$.

Let Θ be a $\mathbf{v}$-generic ample divisor on T. Then $\mathscr{M}_T(\mathbf{v})^0$ is an irreducible symplectic variety deformation equivalent to $K^{[n]}(T)$ where $2n = (\mathbf{v}^2 - 2)$. Now suppose that $6 \le \mathbf{v}^2$. Then the map $\theta_{\mathbf{v}}$ of (3.3.2) *is integral and the restriction to integral cohomology is an isomorphism $(\mathbf{v}^{\perp})_{\mathbb{Z}} \xrightarrow{\sim} H^2(\mathscr{M}_T(\mathbf{v})^0;\mathbb{Z})$.*

3.4. **Moduli spaces of sheaves on symplectic surfaces: modifications.** We will describe "operations" on moduli spaces of sheaves on symplectic surfaces; they are the main ingredients of the proofs of **Theorem 3.20** and **Theorem 3.22** for Mukai vectors of strictly positive square. (The analysis of $\mathscr{M}_S(\mathbf{v})$ for indivisible $\mathbf{v}$ with $\mathbf{v}^2 \le 0$ was done by Mukai [35, 36], and is more direct.) Lastly we will sketch the proof of **Theorem 3.20** in the case when r is coprime to the divisibility of ℓ.

3.4.1. *Deformations of the symplectic surface.* Let S be a projective symplectic surface and H an indivisible ample divisor on S. Let $\deg(H\cdot H) = 2d$. Let $\pi\colon \mathscr{S}_{2d} \to T_{2d}$ be a complete family of deformations of (S,H) i.e. the following hold:

1. $\pi\colon \mathscr{S}_{2d} \to T_{2d}$ is projective with a relativley ample divisor $\mathscr{H}$.
2. Given $t \in T_{2d}$ and letting $S_t := \pi^{-1}(t)$, $H_t := \mathscr{H}|_{S_t}$, the couple (S_t,H_t) is a deformation of (S,H).
3. If (S',H') is a deformation of (S,H) there exists $t \in T_{2d}$ such that $(S',H') \cong (S_t,H_t)$.

Since deformations of symplectic surfaces are unobstructed we may assume that T_{2d} is smooth. We may also assume that T_{2d} is irreducible (for $K3$ surfaces this is a highly non-trivial result). Now let $r \in \mathbb{N}$ and $a,s \in \mathbb{Z}$: we assume that

$$\mathbf{v}_t := (r, ac_1(H_t), s) \in \widetilde{H}(S_t)$$

is a Mukai vector for all $t \in T_{2d}$ (i.e. $a \ge 0$ if $r = 0$). There exists a relative moduli space $\mathscr{M}_{\mathscr{S}} \to T_{2d}$, i.e. a projective map such that the fiber over $t \in T_{2d}$ is isomorphic $\mathscr{M}_{(S_t,H_t)}(\mathbf{v}_t)$. Let $T_{2d}(\mathbf{v}) \subset T_{2d}$ (here $\mathbf{v} = (r,a,s)$) be the subset of t such that H_t is $\mathbf{v}_t$-generic. Then $T_{2d}(\mathbf{v})$ is open dense in T_{2d}. We let

$$\varphi\colon \mathscr{M}_{\mathscr{S}}(\mathbf{v}) \to T_{2d}(\mathbf{v}) \tag{3.4.1}$$

be the restriction of the relative moduli space. Now let's make the following assumption

$$\gcd(r,a,s) = 1. \tag{3.4.2}$$

Let $t \in T_{2d}(\mathbf{v})$: by **Corollary 3.18** every sheaf F parametrized by $\mathscr{M}_{(S_t,H_t)}(\mathbf{v}_t)$ is stable. It follows that the trace-less part $\mathrm{Ext}^2(F,F)^0$ vanishes and hence "sideways" deformations of F are unobstructed, i.e. the differential of φ at $[F]$ is surjective. Thus φ is submersive and hence any two moduli spaces $\mathscr{M}_{(S_t,H_t)}(\mathbf{v}_t)$, $\mathscr{M}_{(S_u,H_u)}(\mathbf{v}_u)$ are equivalent by deformation. One also gets that the maps

$$\theta_{\mathbf{v}_t}\colon \mathbf{v}_t^{\perp} \longrightarrow H^2(\mathscr{M}_{(S_t,H_t)}(\mathbf{v}_t)) \tag{3.4.3}$$

are locally constant.

3.4.2. *Tensorization by a line-bundle.* Let S be a projective symplectic surface and H an indivisible ample divisor on S. Let F be a sheaf on S which is H-slope-stable: if L is an invertible sheaf on S then $F\otimes L$ is H-slope-stable[5]. The following slight modification of **Proposition 3.16** is useful in the present context.

Proposition 3.23. *Let S be a projective symplectic surface and $\mathbf{v}=(r,a\ell_0,s)$ a Mukai vector for S with ℓ_0* indivisible *and r,a* coprime. *There exists a union of walls $\mathscr{V}=\bigcup_{D\in\mathscr{C}}V_D$ with the following properties:*

1. *$\mathscr{V}$ is locally finite and hence the complement in $A(S)_{\mathbb{R}}$ is a dense open subset.*
2. *Let $H\in(A(S)\smallsetminus\mathscr{V})$ and F be H-sope-semistable sheaf with $v(F)=\mathbf{v}$. Then F is H-sope-stable.*
3. *Let $H_1,H_2\in(A(S)\smallsetminus\mathscr{V})$ belong to the same connected component of $(A(S)\smallsetminus\mathscr{V})$. A sheaf F with $v(F)=\mathbf{v}$ is H_1-slope-stable if and only if it is H_2-slope-stable.*

Thus if $\mathbf{v}$ is a Mukai vector for S as in **Proposition 3.23** and $H\in(A(S)\smallsetminus\mathscr{V})$ then all sheaves parametrized by $\mathscr{M}_S(\mathbf{v})$ are slope-stable: it follows that tensorization by an invertible sheaf L on S defines an isomorphism

$$\begin{array}{ccc} \mathscr{M}_S(\mathbf{v}) & \xrightarrow{\sim} & \mathscr{M}_S(\mathbf{v}\cup\mathrm{ch}(L)) \\ [F] & \mapsto & [F\otimes L] \end{array} \tag{3.4.4}$$

3.4.3. *Fourier-Mukai transform.* Let S be a projective symplectic surface and $\mathbf{u}$ an indivisible Mukai vector for S with $\mathbf{u}^2=0$. Let H be a $\mathbf{u}$-generic ample divisor on S. Mukai [36, 35] proved that $\mathscr{M}_S(\mathbf{u})$ is a $K3$ surface if S is, and an abelian surface if S is. We make the assumption that there exists a tautological sheaf $\mathscr{P}$ on $S\times\mathscr{M}_S(\mathbf{u})$.

Example 3.24.
1. S is a $K3$ and $\mathbf{u}=(1,0,0)$. Then $\mathscr{M}_S(\mathbf{u})\cong S$ and the tautological sheaf is $\mathscr{I}_\Delta$ where $\Delta\subset S\times S$ is the diagonal.
2. S is an elliptic $K3$ with a section of the elliptic fibration. Let C an elliptic fiber and $\mathbf{u}=(0,c_1(\mathscr{O}_S(C)),s)$. The generic sheaf parametrized by $\mathscr{M}_S(\mathbf{u})$ is isomorphic to $j_{t,*}(L)$ where $j_t\colon C_t\hookrightarrow S$ is the inclusion of an elliptic fiber and $L\in\mathrm{Pic}^s(C_t)$. Notice that $\mathscr{M}_S(\mathbf{u})\cong S$ because S has a section of the elliptic fibration.
3. S is an abelian surface and $\mathbf{u}=(1,0,0)$. Then $\mathscr{M}_S(\mathbf{u})\cong S^\vee=\mathrm{Pic}^0(S)$ and the tautological sheaf of choice is the *normalized* Poincaré line-bundle.

Given a projective scheme X we let $D^b(X)$ be the category of boundede complexes of sheaves on X. Let S, $T:=\mathscr{M}_S(\mathbf{u})$ and $\mathscr{P}$ be as above. The *Fourier-Mukai functor* $\Phi^{\mathscr{P}}_{S\to T}\colon D^b(S)\longrightarrow D^b(T)$ is defined as

$$\begin{array}{ccc} D^b(S) & \xrightarrow{\Phi^{\mathscr{P}}_{S\to T}} & D^b(T) \\ F & \mapsto & \pi_{T,*}(\mathscr{P}\overset{L}{\otimes}\pi_S^*F) \end{array} \tag{3.4.5}$$

where π_S,π_T are the projections of $S\times T$ to S and T respectively.

Theorem 3.25 (Mukai [35], Bridgeland [7])**.** *Keep hypotheses and notation as above. Then $\Phi^{\mathscr{P}}_{S\to T}$ is an equivalence of triangulated categories.*

We will be interested in geometric versions of the Fourier-Mukai transform. Under suitable hypotheses the following will hold.

Assumption 3.26. Keep notation and assumptions as above. Let $\mathbf{v}$ be a Mukai vector of S and H an ample divisor on S such that $\mathscr{M}_{S,H}(\mathbf{v})=\mathscr{M}_{S,H}(\mathbf{v})^{st}$. Let L be an ample divisor on $T=\mathscr{M}_S(\mathbf{u})$. If F is the *generic* sheaf parametrized by $\mathscr{M}_{S,H}(\mathbf{v})$ then $\Phi^{\mathscr{P}}_{S\to T}(F)$ is represented by an L-stable sheaf G with Mukai vector $\mathbf{w}\in\widetilde{H}(T)$ (i.e. by $G[i]$ for some i independent of the generic sheaf F).

[5]Gieseker-stability might not be preserved after tensorization with L.

Suppose that **Assumption 3.26** holds. Then we get a birational map

$$\begin{array}{ccc} \mathscr{M}_{S,H}(\mathbf{v}) & \dashrightarrow & \mathscr{M}_{T,H}(\mathbf{w}) \\ [F] & \mapsto & [G] \end{array} \tag{3.4.6}$$

where G is as in **Assumption 3.26**.

Example 3.27. Let S and $\mathbf{u} = (1,0,0)$ be as in Item (1) of **Example 3.24**. Let $\mathbf{v} = (r, \ell, s)$ be a Mukai vector of S and H an ample divisor on S such that the generic F parametrized by $\mathscr{M}_{S,H}(\mathbf{v})$ is globally generated (this implies that $s > 0$). Then **Assumption 3.26** holds under suitable extra assumptions on ℓ and H, for example $\mathrm{Pic}(S) = \mathbb{Z}[H]$ and $\ell = c_1(\mathscr{O}_S(H))$ are sufficient (we let $L = H$). With these hypotheses the rational map (3.4.6) is *Mukai's reflection* wich associates to the generic F parametrized by $\mathscr{M}_{S,H}(\mathbf{v})$ (and hence globally generated) the sheaf E fitting into the exact sequence

$$0 \longrightarrow E \longrightarrow H^0(F) \otimes \mathscr{O}_S \longrightarrow F \longrightarrow 0. \tag{3.4.7}$$

Let $\mathbf{w} := (s, -\ell, r)$: Mukai's reflection is a birational map $\mathscr{M}_S(\mathbf{v}) \dashrightarrow \mathscr{M}_S(\mathbf{w})$.

3.4.4. **Theorem 3.20***: sketch of proof.* Let $\mathbf{v} = (r, a\ell_0, s)$ where ℓ_0 is indivisible and $\gcd(r,a,s) = 1$. We will sketch the proof that $\mathscr{M}_S(\mathbf{v})$ is a HK variety which deforms to a HK variety birational to $K3^{[n]}$, where $2n = (2 + \mathbf{v}^2)$.

Step 1: By deforming (S,H) to a $K3$ with a suitable Picard group of rank 2 and tensorizing sheaves by a line-bundle we may assume that $a = \gcd(r,a)$ and that Mukai's reflection (see **Example 3.27**) is a birational map $\mathscr{M}_S(\mathbf{v}) \dashrightarrow \mathscr{M}_S(s, -a\ell_0, r)$.

Step 2: Since $\gcd(r,a,s) = 1$ and $a = \gcd(r,a)$ we have $\gcd(a,s) = 1$. Iterating the procedure of Step 1 we may assume that $a = 1$, i.e. $\mathbf{v} = (r, -\ell_0, s)$.

Step 3: Via another iteration of Step 1 we are reduced to proving the Theorem for S an elliptic $K3$ surface with a section and $\mathbf{v} = (r, \ell, s)$ where ℓ is a numerical section i.e. $\deg(\ell \cdot C) = 1$ for C an elliptic fiber. Moreover we may assume that the ample divisor H is very close to C (how close depends on $\mathbf{v}^2$) so that a sheaf with $v(F) = \mathbf{v}$ is H-slope-semistable if and only if its restriction to the generic elliptic fiber C_t is stable.

Step 4: There exists an integer k such that if $v(F) = \mathbf{v}$ then $\chi(F \otimes \mathscr{O}_S(kC)) = 1$. Replace $\mathbf{v}$ by $\mathbf{v} \cdot \mathrm{ch}(\mathscr{O}_S(kC))$. One proves that if $[F] \in \mathscr{M}_S(\mathbf{v})$ is generic then $\chi(F) = 1$. Let $[F] \in \mathscr{M}_S(\mathbf{v})$ be generic: then we have an exact sequence

$$0 \longrightarrow \mathscr{O}_S \longrightarrow F \longrightarrow E \longrightarrow 0 \tag{3.4.8}$$

where E is an H-semistable sheaf. Let $\mathbf{w} := v(E)$. The map

$$\begin{array}{ccc} \mathscr{M}_{S,H}(\mathbf{v}) & \dashrightarrow & \mathscr{M}_{S,H}(\mathbf{w}) \\ [F] & \mapsto & [E] \end{array} \tag{3.4.9}$$

is birational. This can be interpreted as the birational map induced by the Fourier-Mukai transform given in Item (2) of **Example 3.24**, see [30].

Step 5: The moduli space $\mathscr{M}_{S,H}(\mathbf{w})$ is a moduli space of sheaves on the elliptic surface similar to the one above, in particular

$$\mathbf{w} = (r-1, \ell', s'), \qquad \deg(\ell' \cdot C) = 1.$$

Iterating the above procedure we get down to a moduli space $\mathscr{M}_{S,H}(1, \ell, s)$ i.e. $S^{[n]}$ for a suitable n.

4. O'Grady's examples

4.1. **Main results.** Let S be a projective symplectic surface and $\mathbf{v}$ a Mukai vector for S which is *divisible*. Thus

$$\mathbf{v} = m\mathbf{v}_0, \qquad \mathbf{v}_0 \in \widetilde{H}^{1,1}_{\mathbb{Z}}(S) \text{ indivisible}, \; m \in \mathbb{N}, \; m \geq 2. \tag{4.1.1}$$

Let H be a $\mathbf{v}$-generic ample divisor on S. Suppose that $\mathscr{M}_S(\mathbf{v})^{st}$ and $\mathscr{M}_S(\mathbf{v}_0)^{st}$ are non-empty. Let $F := \bigoplus_{i=1}^m E_i$ where E_i is a stable sheaf such that $v(E_i) = \mathbf{v}_0$. Then F is a strictly semistable sheaf parametrized by a point of $\mathscr{M}_S(\mathbf{v})$. We expect that $\mathscr{M}_S(\mathbf{v})$ is singular at these points. One may ask the following question: does there exist a desingularization $\widetilde{\mathscr{M}}_S(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ such that the holomorphic symplectic form on $\mathscr{M}_S(\mathbf{v})^{st}$ (viewed as a subset of $\widetilde{\mathscr{M}}_S(\mathbf{v})$) extends to a holomorphic symplectic form on $\widetilde{\mathscr{M}}_S(\mathbf{v})$? Such a desingularization is a *symplectic desingularization.* The following result, summarizing the work of many mathematicians, answers the above question.

Theorem 4.1 (O'Grady, Kiem, Rapagnetta, Kaledin, Lehn, Sorger, Perego)**.** *Let S be a symplectic projective surface. Let $\mathbf{v}$ be a divisible Mukai vector as in* (4.1.1)*. Suppose that $\mathbf{v}_0^2 \ge 2$ and that $(r,s) \neq (0,0)$. Let H be a $\mathbf{v}$-generic ample divisor on S. Then $\mathscr{M}_S(\mathbf{v})$ is non-empty, irreducible of dimension $(2+\mathbf{v}^2)$ and its smooth locus is equal to $\mathscr{M}_S(\mathbf{v})^{st}$. There exits a symplectic desingularization $\widetilde{\pi}\colon\widetilde{\mathscr{M}}_S(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ if and only if $m=2$ and $\mathbf{v}_0^2=2$. Now suppose that $m=2=\mathbf{v}_0^2$.*

1. *If S is a $K3$ surface then $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ is a 10-dimensional HK variety and $b_2(\widetilde{\mathscr{M}}_S(\mathbf{v}_0)) = 24$.*
2. *If S is an abelian surface let $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0 := f^{-1}(\mathscr{M}_S(2\mathbf{v}_0)^0)$. Then $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ is a 6-dimensional HK variety and $b_2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0) = 8$.*
3. *Let S and S' be $K3$ surfaces, $\mathbf{v}_0$ and $\mathbf{v}_0'$ Mukai vectors for S and S' with $2=\mathbf{v}_0^2=(\mathbf{v}_0')^2$ and H, H' ample divisors on S and S' respectively which are $2\mathbf{v}_0$ and $2\mathbf{v}_0'$ generic respectively. Then $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ is deformation equivalent to $\widetilde{\mathscr{M}}_{S'}(2\mathbf{v}_0')$. A similar statement holds for abelian surfaces.*

Let $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ and $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ be as in Items (1) and (2) of **Theorem 4.1**. Since $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ has second Betti number different from that of $(K3)^{[n]}$ and of a generalized Kummer it is not not a deformation of the Beauville examples. A similar statement holds for $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$. Thus we get two new deformation classes of HK manifolds. A word about the contributions of the mathematicians quoted in the statement of **Theorem 4.1**. Suppose that $\mathbf{v}_0=(1,0,-1)$: a sheaf F parametrized by $\mathscr{M}_S(2\mathbf{v}_0)$ has rank 2, $c_1=0$ and c_2 equal to 4 if S is a $K3$ and 2 if S is an abelian surface. O'Grady [41, 42] proved that $\mathscr{M}_S(2(1,0,-))$ has a symplectic desingularization $\widetilde{\mathscr{M}}_S(2(1,0,-1))$ and that if S is a $K3$ then $\widetilde{\mathscr{M}}_S(2(1,0,-1))$ is a 10-dimensional HK variety with $b_2 \ge 24$ (and hence not a deformation of Beauville's examples), and that if S is an abelian surface then $\widetilde{\mathscr{M}}_S(2(1,0,-1))^0$ is a 6-dimensional HK variety with $b_2=8$ (and hence not a deformation of Beauville's examples). Kiem [25] proved non-existence of a symplectic desingularization of $\mathscr{M}_S(2\mathbf{v}_0)$ for some choices of $\mathbf{v}_0$. Rapagnetta [46] proved that if S is a $K3$ surface then $b_2(\widetilde{\mathscr{M}}_S(2(1,0,-1)))=24$. Suppose that $\mathbf{v}_0^2=2$: Lehn and Sorger [26] showed that the symplectic desingularization $f\colon\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)\to\mathscr{M}_S(2\mathbf{v}_0)$ can be obtained by a single blow-up, namely the blow-up of the singular locus of $\mathscr{M}_S(2\mathbf{v}_0)$. Kaledin, Lehn and Sorger [24] proved non-existence of a symplectic desingularization for all $\mathbf{v}$ with $m>2$ or $\mathbf{v}_0^2>2$. Perego and Rapagnetta [44] proved Item (3) of **Theorem 4.1**.

Remark 4.2. We do not know all of the Betti numbers of the 6 and 10 dimensional HK varieties appearing in **Theorem 4.1** - that is in contrast with the case of Hilbert schemes of $K3$ surfaces (or any surface) or of generalized Kummers. Rapagnetta [45] proved that the topological Euler characterstic of the 6-dimensional variety is equal to 1920.

4.2. **Singularities and desingularizations.** Let S be a projective symplectic surface and $\mathbf{v}_0$ an *indivisible* Mukai vector such that $2\le\mathbf{v}_0^2$. We will go through some results which suggest that there should exist a symplectic desingularization of $\mathscr{M}_S(2\mathbf{v}_0)$ when $\mathbf{v}_0^2=2$ and that there should *not* exist such a desingularization if $2<\mathbf{v}_0^2$. (The polarization of S is $(2\mathbf{v}_0)$-generic.) Lastly we will mention the main point of the proof by Kaledin - Lehn - Sorger [24] that $\mathscr{M}_S(m\mathbf{v}_0)$ does not have a symplectic desingularization if $m\ge 3$ and $2\le\mathbf{v}_0^2$ or $m=2$ and $2<\mathbf{v}_0^2$.

4.2.1. $\mathbf{v}=2\mathbf{v}_0$. In the present subsubsection we let $\mathbf{v}_0$ be an *indivisible* Mukai vector such that $2\le\mathbf{v}_0^2$ and we let $\mathbf{v}:=2\mathbf{v}_0$. We let H be a $\mathbf{v}$-generic polarization of S. Since $\mathscr{M}_S(\mathbf{v})^{st}$ is smooth we must examine $\mathscr{M}_S(\mathbf{v})$ in a neighborhood of a point x parametrizing a strictly semistable sheaf. The GIT construction of the moduli space gives that one must examine the deformation space of the poly-stable representative F of x i.e. such that $gr^{JH}(F)\cong F$.

Claim 4.3. *Keep notation and assumptions as above. If F is a non-stable H-polystable sheaf with $v(F)=\mathbf{v}$ then one of the following holds:*

1. *$F\cong G_1\oplus G_2$ where G_1, G_2 are H-stable* non-isomorphic *sheaves with $v(G_1)=v(G_2)=\mathbf{v}_0$.*
2. *$F\cong G\oplus G$ where G is an H-stable sheaf with $v(G)=\mathbf{v}_0$.*

Proof. This follows at once from the hypothesis that the polarization is $\mathbf{v}$-generic and $\mathbf{v}_0$ is indivisible. □

Let $\Omega(\mathbf{v})\subset\mathscr{M}_S(\mathbf{v})$ be the subset of points represented by a polystable sheaf as in Item (2) of **Claim 4.3**: this a closed subset isomorphic to $\mathscr{M}(\mathbf{v}_0)$, and hence of dimension $(2+\mathbf{v}_0^2)$. Let $\Sigma(\mathbf{v})\subset\mathscr{M}_S(\mathbf{v})$ be the subset of points represented by a polystable sheaf as in Item (1) of **Claim 4.3**: this is a closed subset isomorphic to the *symmetric square* of $\mathscr{M}(\mathbf{v}_0)$, and hence of dimension $2(2+\mathbf{v}_0^2)$. Notice that $\Omega(\mathbf{v})$ is contained in $\Sigma(\mathbf{v})$ and is equal to the singular locus of $\Sigma(\mathbf{v})$. Suppose that $\mathbf{v}_0^2 = 2$: in [41] we proved that

$$\operatorname{sing}\mathscr{M}_S(\mathbf{v}) = \Sigma(\mathbf{v}) \tag{4.2.1}$$

and we obtained a symplectic desingularization of $\mathscr{M}_S(\mathbf{v})$ by going through the following steps:

Step 1.: Let $\overline{\mathscr{M}}_S(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ be the blow-up of $\Omega(\mathbf{v})$.

Step 2.: Let $\widehat{\mathscr{M}_S}(\mathbf{v}) \to \overline{\mathscr{M}}_S(\mathbf{v})$ be the blow-up of the strict transform of $\Sigma(\mathbf{v})$: the space $\widehat{\mathscr{M}_S}(\mathbf{v})$ is Kirwan's partial desingularization of $\mathscr{M}_S(\mathbf{v})$. Let $\widehat{\pi}: \widehat{\mathscr{M}_S}(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ be the natural map.

Step 3.: The symplectic form on $\mathscr{M}_S(\mathbf{v})^{st}$ (viewed as a subset of $\widehat{\mathscr{M}_S}(\mathbf{v})$, this makes sense by (4.2.1)) extends to a symplectic form on $\widehat{\mathscr{M}_S}(\mathbf{v})$ which degeneretas on the (irreducible) inverse image $\widehat{\Omega}(\mathbf{v}) := \widehat{\pi}^{-1}(\Omega(\mathbf{v}))$. There exists a K-negative ray of $\widehat{\mathscr{M}_S}(\mathbf{v})$ whose associated contraction $\theta: \widehat{\mathscr{M}_S}(\mathbf{v}) \to \widetilde{\mathscr{M}_S}(\mathbf{v})$ contracts exactly $\widehat{\Omega}(\mathbf{v})$ and such that $\widetilde{\mathscr{M}_S}(\mathbf{v})$ is *smooth*. It follows that $\widetilde{\mathscr{M}_S}(\mathbf{v})$ is a HK variety.

Step 4.: One shows that the natural (bi)rational map $\widetilde{\mathscr{M}_S}(\mathbf{v}) \to \mathscr{M}_S(\mathbf{v})$ is regular: this is a symplectic desingularization of $\mathscr{M}_S(\mathbf{v})$.

Lehn and Sorger [26] have proved that one obtains the same desingularization by blowing up $\Sigma(\mathbf{v})$. Let us explain why this migth be true and what fails if $2 < \mathbf{v}_0^2$. First we need to recall a few results from deformation theory. Let F be a sheaf on S. Small deformations of F are governed by the *Kuranishi map*

$$(\operatorname{Ext}^1(F,F),0) \xrightarrow{\Psi} (\operatorname{Ext}^2(F,F)^0,0),$$

a regular map of analytic germs. In fact the deformation space $\operatorname{Def}(F)$ is the analytic germ given by

$$\operatorname{Def}(F) = \Psi^{-1}(0). \tag{4.2.2}$$

The linear term Ψ_1 of Ψ vanishes (this is consistent with (3.1.14)) and the quadratic term Ψ_2 is given by

$$\Psi_2(\alpha) = \frac{1}{2}\alpha \cup \alpha \tag{4.2.3}$$

where $\cup$ is Yoneda product. The group $\operatorname{Aut}(F)$ acts naturally on $\operatorname{Ext}^1(F,F)$, $\operatorname{Ext}^2(F,F)^0$ and $\operatorname{Def}(F)$, and the actions are compatible with (4.2.2). The subgroup of $\operatorname{Aut}(F)$ given by homotheties acts trivially and hence we have an induced action of

$$\operatorname{Aut}_0(F) := \operatorname{Aut}(F)/\mathbb{C}^\times \tag{4.2.4}$$

on $\operatorname{Ext}^1(F,F)$, $\operatorname{Ext}^2(F,F)^0$ and $\operatorname{Def}(F)$. Let us identify $\operatorname{Ext}^1(F,F)$, $\operatorname{Ext}^2(F,F)^0$, Ψ_2 and $\operatorname{Aut}_0(F)$ for the non-stable polystable sheaves F parametrized by $\mathscr{M}_S(\mathbf{v})$ where $\mathbf{v} = 2\mathbf{v}_0$ with $\mathbf{v}_0$ indivisible and $2 \le \mathbf{v}_0^2$.

***F* as in Item (1) of Claim 4.3:** Then

$$\operatorname{Ext}^1(F,F) = \bigoplus_{1\le i,j\le 2} \operatorname{Ext}^1(G_i,G_j) \tag{4.2.5}$$

and

$$\operatorname{Ext}^2(F,F)^0 = \{(\alpha,\beta) \in \operatorname{Ext}^2(G_1,G_1) \oplus \operatorname{Ext}^2(G_2,G_2) \mid \operatorname{Tr}(\alpha) + \operatorname{Tr}(\beta) = 0\}.$$

The quadratic term Ψ_2 is given by

$$\Psi_2(\gamma_{11},\ldots,\gamma_{22}) = \frac{1}{2}(\gamma_{12} \cup \gamma_{21},\ \gamma_{21} \cup \gamma_{12})$$

where for $\gamma \in \operatorname{Ext}^1(F,F)$ we let γ_{ij} be its components according to Decomposition (4.2.5). Moreover $\operatorname{Aut}_0(F) \cong \mathbb{C}^\times$, its action on $\operatorname{Ext}^2(F,F)^0$ is trivial, while on $\operatorname{Ext}^1(F,F)$ it is given by

$$t(\gamma_{11},\ldots,\gamma_{22}) = (\gamma_{11}, t\gamma_{12}, t^{-1}\gamma_{21}, \gamma_{22}). \tag{4.2.6}$$

***F* as in Item (2) of Claim 4.3:** We write F as $F = G \otimes_{\mathbb{C}} \mathbb{C}^2$ and hence

$$\operatorname{Ext}^1(F,F) = \operatorname{Ext}^1(G,G) \otimes_{\mathbb{C}} \mathfrak{gl}_2(\mathbb{C}), \qquad \operatorname{Ext}^2(F,F)^0 = \operatorname{Ext}^2(G,G) \otimes_{\mathbb{C}} \mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}). \tag{4.2.7}$$

Moreover $\operatorname{Aut}_0(F) \cong \operatorname{PGL}_2(\mathbb{C})$ and the action on $\operatorname{Ext}^1(F,F)$, $\operatorname{Ext}^2(F,F)^0$ is induced by the natural action on $\mathfrak{gl}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$. Now let us proceed to describe Ψ_2. First the kernel of

Ψ_2 is the summand $\mathrm{Ext}^1(G,G)\otimes_{\mathbb{C}}\mathbb{C}\,\mathrm{Id}$, hence Ψ_2 may be identified with a quadratic form $\overline{\Psi}_2$ on $\mathrm{Ext}^1(G,G)\otimes_{\mathbb{C}}\mathfrak{sl}_2(\mathbb{C})$. The Killing form gives identifications

$$\mathrm{Ext}^1(G,G)\otimes_{\mathbb{C}}\mathfrak{sl}_2(\mathbb{C})\cong \mathrm{Hom}(\mathfrak{sl}_2(\mathbb{C}),\mathrm{Ext}^1(G,G)),\qquad \mathfrak{sl}_2(\mathbb{C})\cong \bigwedge^2 \mathfrak{sl}_2(\mathbb{C})^\vee. \tag{4.2.8}$$

The choice of a symplectic form on S determines Mukai's symplectic form on $\mathrm{Ext}^1(G,G)$, call it ω. Now let $f\in \mathrm{Hom}(\mathfrak{sl}_2(\mathbb{C}),\mathrm{Ext}^1(G,G))$: then (up to a non-zero multiplicative factor) we have

$$\overline{\Psi}_2(f)=f^*\omega. \tag{4.2.9}$$

Now let us make the following hypothesis.

Assumption 4.4. For the sheaves F above we may identify the Kuranishi map Ψ with its second-order term Ψ_2, compatibly with the action of $\mathrm{Aut}(F)$, and hence the germ of $\mathscr{M}_S(\mathbf{v})$ at $[F]$ is isomorphic to $\Psi_2^{-1}(0)//\mathrm{Aut}_0(F)$.

We will analyze the germ of $\mathscr{M}_S(\mathbf{v})$ at properly semistable points distinguishing between two cases, according to **Claim 4.3**. Let

$$\widetilde{\mathscr{M}}_S(\mathbf{v})\longrightarrow \mathscr{M}_S(\mathbf{v}) \tag{4.2.10}$$

be the blow-up of $\Sigma(\mathbf{v})$.

$[F]\in(\Sigma(\mathbf{v})\smallsetminus\Omega(\mathbf{v}))$: Thus $F=G_1\oplus G_2$ where $v(G_1)=v(G_2)=\mathbf{v}_0$, G_1, G_2 are stable and $G_1\ncong G_2$. The description above together with **Assumption 4.4** give easily that the normal cone to $\Sigma(\mathbf{v})$ in $\mathscr{M}_S(\mathbf{v})$ is isomorphic (in the analytic topology) to the germ at 0 of the affine cone over the incidence subset

$$I(\mathbf{v}):=\{([x],[f])\in\mathbb{P}(\mathrm{Ext}^1(G_1,G_2))\times\mathbb{P}(\mathrm{Ext}^1(G_2,G_1))\mid f(x)=0\}. \tag{4.2.11}$$

Here $\mathrm{Ext}^1(G_2,G_1)=\mathrm{Ext}^1(G_1,G_2)^\vee$ via Serre duality. Notice that $\dim\mathrm{Ext}^1(G_1,G_2)=\mathbf{v}_0^2$. It follows that if $\mathbf{v}_0^2=2$ the normal cone to $\Sigma(\mathbf{v})$ in $\mathscr{M}_S(\mathbf{v})$ at $[F]$ is an ordinary 2-dimensional quadratic point and hence the blow-up of $\Sigma(\mathbf{v})$ will desingularize such a point, and since it is a crepant singularity the symplectic form extends across the exceptional divisor to a symplectic form. Next suppose that $2<\mathbf{v}_0^2$. Then $\widetilde{\mathscr{M}}_S(\mathbf{v})$ is smooth over $[F]$ but is *not* crepant. Locally around $[F]$ (analytic topology) the exceptional divisor of $\widetilde{\mathscr{M}}_S(\mathbf{v})\to\mathscr{M}_S(\mathbf{v})$ has two $\mathbb{P}^{(\mathbf{v}_0^2-2)}$-fibrations and the normal bundle of the exceptional divisor restricts to $\mathscr{O}(-1)$ on the fibers of these fibrations: one gets local crepant desingularizations of $\mathscr{M}_S(\mathbf{v})$ near $[F]$ by blowing down the exceptional divisor along either one of these fibrations. Since the two fibrations get exchanged by monodromy there is no way of making global sense of the above blow-down.

$[F]\in\Omega(\mathbf{v})$: Identifying the $\mathrm{PGL}_2(\mathbb{C})$-module $\mathfrak{sl}_2(\mathbb{C})$ with $\mathfrak{so}_3(\mathbb{C})$ we get that the normal slice to $\Omega(\mathbf{v})$ in $\mathscr{M}_S(\mathbf{v})$ at $[F]$ is isomorphic to

$$\mathscr{N}_G:=\{q\in \mathrm{S}^2\,\mathrm{Ext}^1(G,G)\mid \mathrm{rk}\,q\le 3,\quad \ker q \text{ contains an } \omega\text{-lagrangian subspace}\}. \tag{4.2.12}$$

Moreover abusing notation we have

$$\Sigma(\mathbf{v})\cap\mathscr{N}_G=\mathscr{W}_G:=\{q\in\mathrm{S}^2\,\mathrm{Ext}^1(G,G)\mid \mathrm{rk}\,q\le 1\}. \tag{4.2.13}$$

Now notice the following difference between the case $\mathbf{v}_0^2=2$ and $2<\mathbf{v}_0^2$. In the former case the rank of elements of $\mathscr{V}_G$ is at most 2 because $\dim\mathrm{Ext}^1(G,G)=4$: it follows that the blow-up of $\Sigma(\mathbf{v})$ desingularizes $\mathscr{M}_S(\mathbf{v})$, and that the symplectic form extends to a symplectic form on the blow-up. In the latter case the rank of a generic element of $\mathscr{V}_G$ is 3: it follows that $\widetilde{\mathscr{M}}_S(\mathbf{v})$ is *not* smooth over points $[F]\in\Omega(\mathbf{v})$.

Remark 4.5. **Assumption 4.4** actually holds. In fact a series of works by Kaledin [22], Kaledin - Lehn [23], Lunts [28], Zhang [53], Manetti [29] prove *formality* of the DG Lie algebra controlling deformations of F, and this proves that the Kuranishi map may be identified with its second-order term. In addition to that Arbarello - Saccà [2] show that the identification can be assumed to be compatible with the action of $\mathrm{Aut}_0(F)$.

4.2.2. $\mathbf{v}=m\mathbf{v}_0$, *$m>2$ or $m=2$ and $2<\mathbf{v}_0^2$*. Kaledin, Lehn and Sorger [24] have proved that the singularities of $\mathscr{M}_S(\mathbf{v})$ are factorial. It follows that there is no symplectic desingularization of $\mathscr{M}_S(\mathbf{v})$ because a hypothetical symplectic desingularization $\widetilde{\mathscr{M}_S}(\mathbf{v})\to\mathscr{M}_S(\mathbf{v})$ would be semi-small by a result of Kaledin [21] (Prop. 4.4 and Remk. 4.5) and Namikawa [38] (Prop. 1.4) and that is not compatible with the fact (proved in [24]) that the singular locus of $\mathscr{M}_S(\mathbf{v})$ has codimension at least 4.

4.3. **Topology of the symplectic desingularizations.** Let S be a projective symplectic surface and $\mathbf{v}_0$ an indivisible Mukai vector of S with $\mathbf{v}_0^2 = 2$. In order to prove that if S is a $K3$ the symplectic desingularization $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ is HK, and that if S is a an abelian surface then $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ is a HK one must show that $h^{2,0}(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)) = 1$ and $\pi_1(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)) = 1$ if S is a $K3$ and that $h^{2,0}(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0) = 1$ and $\pi_1(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0) = 1$ if S is an abelian surface. Then we must compute the second Betti number of $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ and $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ respectively. The details are too involved to be explained here, see [41, 42, 46]. What can be done fairly easily is show that $b_2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)) \geq 24$ if S is a $K3$ and $b_2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0) \geq 8$ if S is an abelian surface. In fact one can define a linear map $\theta_{\mathbf{v}}\colon \mathbf{v}^\perp \to H^2(\mathscr{M}_S(2\mathbf{v}_0))$ as in the case of $\mathbf{v}$ primitive (see [44]). Composing with the pull-back $H^2(\mathscr{M}_S(2\mathbf{v}_0)) \to H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0))$ (followed by restriction to $H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0)$ if S is an abelian surface) we get an injective map $\mathbf{v}^\perp \to H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0))$ if S is a $K3$ and an injective map $\mathbf{v}^\perp \to H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0)$ if S is an abelian surface. Thus we get a rank-23 subspace $V_S \subset H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0))$ if S is a $K3$ and a rank-7 subspace $V_S \subset H^2(\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0)$ if S is an abelian surface. The Poincarè dual of the exceptional divisor of the symplectic desingularization gives an element of H^2 which is not contained in V_S and hence $b_2 \geq (b_2(S) + 2)$.

Remark 4.6. Little is known regarding the topology of $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ for a S a $K3$ and $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ for S an abelian surface beyond the rank of H^2 (and the Beauville-Bogomolov quadratic form on H^2, see [46]). Let S be a $K3$ surface: Mozgovoy [34] has proved that the topological Euler characteristic of $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)$ is equal to 176904. Let S be an abelian surface: Rapaganetta [45] has proved that the topological Euler characteristic of $\widetilde{\mathscr{M}}_S(2\mathbf{v}_0)^0$ is equal to 1920. The fact is that the cohomology of Beauville's examples is related to the cohomology of the symmetric product of a surface, and the latter is easily described, while we do not have (yet) models of our 10-dimensional example or our 6-dimensional example which are related to easily described varieties.

References

1. N. Addington - M. Lehn, *On the symplectic eightfold associated to a Pfaffian cubic fourfold*, arXiv:1404.5657 [math.AG]
2. E. Arbarello - G. Saccà, *Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quivers varieties*, in preparation.
3. I. V. Artamkin, *On the deformation of sheaves*, Math. USSR-Izv. 32 (1989), pp. 663-668.
4. A. Bayer - B. Hassett - Y. Tschinkel *Mori cones of holomorphic symplectic varieties of* $K3$ *type*, arXiv:1307.2291 [math.AG]
5. A. Beauville, *Variétes Kähleriennes dont la premiére classe de Chern est nulle*, J. Differential geometry 18 (1983), pp. 755-782.
6. A. Beauville, R. Donagi, *La variétés des droites d'une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), pp. 703-706.
7. T. Bridgeland, *Equivalences of triangulated categories and Fourier-Mukai transforms*, Bull. London Math. Soc. 31 (1999), pp. 25-34.
8. J. Cheah, *On the cohomology of Hilbert schemes of points*, J. Algebraic Geom. 5 (1996), pp. 479-511.
9. O. Debarre - C. Voisin, *Hyper-Kähler fourfolds and Grassmann geometry*, J. Reine Angew. Math. 649 (2010), pp. 63-87.
10. L. Göttsche, *The Betti numbers of the Hilbert scheme of points on a smooth projective surface*, Math. Ann. 286 (1990), pp. 193-207.
11. J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math 90 (1968), pp. 511-521.
12. L. Göttsche, D. Huybrechts, *Hodge numbers of moduli spaces of stable bundles on* $K3$ *surfaces*, Internat. J. Math. 7, 1996, pp. 359-372.
13. L. Göttsche, W. Soergel *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces*, Math. Ann. 296 (1993), pp. 235-245.
14. D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. 326 (2003), pp. 499-513.
15. D. Huybrechts, *Finiteness results for compact hyperkähler manifolds*, J. Reine Angew. Math. 558 (2003), pp. 15-22.
16. D. Huybrechts, *A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]*, Séminaire Bourbaki. Astérisque 348 (2012), Exp. No. 1040, pp. 375-403.
17. D. Huybrechts, M. Lehn, *The geometry of moduli spaces of shaves*, Aspects of Mathematics E 31, Vieweg (1997).
18. A. Iarrobino, *Punctual Hilbert schemes*, Bull. Amer. Math. Soc. 78 (1972), pp. 819-823.
19. A. Iliev - K. Ranestad, *K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds*, Trans. Amer. Math. Soc. 353, 2001, pp. 1455-1468.

20. A. Iliev - K. Ranestad, *Addendum to "K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds"*, C. R. Acad. Bulgare Sci. 60, 2007, pp. 1265-1270.
21. D. Kaledin, *On crepant resolutions of symplectic quotient singularities*, Selecta Math. 9 (2003), pp. 529-555.
22. D. Kaledin, *Some remarks on formality in families*, Mosc. Math. J. 7 (2007), pp. 643-652.
23. D. Kaledin - M. Lehn, *Local structure of hyperkähler singularities in O'Grady's examples*, Mosc. Math. J. 7 (2007), pp. 653-672.
24. D. Kaledin - M. Lehn - Ch. Sorger, *Singular symplectic moduli spaces*, Invent. Math. 164, 2006, pp. 591-614.
25. Y-H. Kiem, *On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface*, Compos. Math. 141 (2005), pp. 902-906.
26. M. Lehn - Ch. Sorger, *La singularité de O'Grady*, J. Algebraic Geom. 15, 2006, pp. 753-770.
27. C. Lehn - M. Lehn - C. Sorger - D. van Straten, *Twisted cubics on cubic fourfolds*, arXiv:1305.0178 [math.AG]
28. V. Lunts, *Formality of DG algebras (after Kaledin)*, J. Algebra 323 (2010), pp. 878-898.
29. M. Manetti, *On some formality criteria for DG Lie algebras*, arXiv1310.3048 [math.QA]
30. A. Marian - D. Oprea, *Generic strange duality for K3 surfaces*, (With an appendix by Kota Yoshioka) Duke Math. J. 162 (2013), pp. 1463-1501.
31. E. Markman. *Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface*, Internat. J. Math. 21 (2010), pp. 169-223.
32. E. Markman. *A survey of Torelli and monodromy results for holomorphic-symplectic varieties*, Complex and differential geometry, Springer Proc. Math. 8 (2011), pp. 257-322.
33. G. Mongardi. *A note on the Kähler and Mori cones of hyperkähler manifolds*, arXiv:1307.0393 [math.AG]
34. S. Mozgovoy. *The Euler number of O'Grady's 10-dimensional symplectic manifold*, Ph.D. Thesis, J. Gutenberg Universität, Mainz (Deutschland), (2006).
35. S. Mukai, *Duality between $D(X)$ and $D(X^\vee)$ with its application to Picard sheaves*, Nagoya Math. J. 81 (1981), pp. 153-175.
36. S. Mukai, *On the moduli space of bundles on K3 surfaces, I*, Vector Bundles on Algebraic Varieties, TIFR, Bombay, O.U.P. (1987), pp. 341-413.
37. S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*, Invent. math 77 (1984), pp. 101-116.
38. Y. Namikawa, *Deformation theory of singular symplectic n-folds*, Math. Ann. 319 (2001), pp. 597-623.
39. K. G. O'Grady, *Moduli of vector-bundles on surfaces*, Proc. Sympos. Pure Math., 62, AMS (1997), pp. 101-126.
40. K. O'Grady, *The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface*, J. of Alg. Geom. 6, 1997, pp. 599-644.
41. K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. fur die reine und angew. Math. 512, 1999, pp. 49-117.
42. K. G. O'Grady, *A new six-dimensional irreducible symplectic variety*, J. Algebraic Geom. 12 (2003), pp. 435-505.
43. K. O'Grady, *Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics*, Duke Math. J. 34, 2006, pp. 99-137.
44. A. Perego - A. Rapagnetta, *Deformation of the O'Grady moduli spaces*, J. Reine Angew. Math. 678 (2013), pp. 1-34.
45. A. Rapagnetta, *Topological invariants of O'Grady's six dimensional irreducible symplectic variety*, Math. Z. 256 (2007), pp. 1-34.
46. A. Rapagnetta, *On the Beauville form of the known irreducible symplectic varieties*, Math. Ann. 340 (2008), pp. 77-95.
47. S. M. Salamon, *On the cohomology of Kähler and hyper-Kähler manifolds*, Topology 35, 1996, pp. 137-155.
48. J. Sawon, *On the discriminant locus of a Lagrangian fibrations*, Math. Ann. 341 (2008), pp. 201-221.
49. C. T. Simpson *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. No. 79 (1994), pp. 47-129.
50. K. Yoshioka, *Some examples of Mukai's reflections on K3 surfaces*, J. Reine Angew. Math. 515 (1999), pp. 97-123.
51. K. Yoshioka, *Moduli spaces of stable sheaves on abelian surfaces*, Math. Ann. 321 (2001), pp. 817-884.
52. M. Verbitsky, *Mapping class group and a global Torelli theorem for hyperkähler manifolds*, Duke Math. J. 162 (2013), pp. 2929-2986.
53. Z. Zhang, *A note on formality and singularities of moduli spaces*, Mosc. Math. J. 12 (2012), pp. 863-879.