FoPSS 2019

Basic Nominal Techniques

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Alternative Formulations

Finite vs. arbitrary atom renamings

Let $\mathrm{Perm}(\mathbb{A})$ be the group of finite bijections on $\mathbb{A}$.

(i.e. such that $\pi(a) = a$ for all but finitely many a)

$\mathrm{Perm}(\mathbb{A})$ canonically acts on the universe $\mathcal{U}$, and the definition of support may be repeated.

Fact: whether we use $\mathrm{Aut}(\mathbb{A})$ or $\mathrm{Perm}(\mathbb{A})$, the same sets are legal and they have the same finite supports.

NB. Not so easy to prove! Essentially a topological argument.

Categories

Legal nominal sets and finitely supported functions form a category.

A category $\mathcal{C}$:

- a collection $|\mathcal{C}|$ of objects
- for each $X, Y \in |\mathcal{C}|$, a set $\mathcal{C}(X, Y)$ of morphisms
- composition operations:

$$\_ \circ \_ : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$$

- identity morphisms: $\mathrm{id}_X \in \mathcal{C}(X, X)$

+ axioms

Another category: equivariant sets and functions. **Nom**

Continuous G-sets

For an equivariant set X, atom renaming acts on X:

$$\_ \cdot \_ : X \times \mathrm{Aut}(\mathbb{A}) \to X$$

Fact: for each $x \in X$ there is a finite $S \subseteq \mathbb{A}$
s.t. for every $\pi, \sigma \in \mathrm{Aut}(\mathbb{A})$
if $\pi|_S = \sigma|_S$ then $x \cdot \pi = x \cdot \sigma$.

we know this!

In other words: $\_ \cdot \_$ is a continuous group action
(X discrete, $\mathrm{Aut}(\mathbb{A})$ with product topology)

$\mathbf{Nom} \approx$ continuous $\mathrm{Aut}(\mathbb{A})$-sets
with equivariant functions between them

Sheaves

Fix an equivariant set X.

For a finite $S \subseteq \mathbb{A}$, define:

$$\hat{X}(S) = \{x \in X \mid \mathrm{supp}(x) \subseteq S\} \subseteq X$$

For an injective function $f : S \to T \subseteq \mathbb{A}$:

- pick any $\pi \in \mathrm{Aut}(\mathbb{A})$ that extends f
- define $\hat{X}(f) : \hat{X}(S) \to \hat{X}(T)$ by:

$$\hat{X}(f)(x) = x \cdot \pi$$

Fact: $\hat{X}(f)(x) \in \hat{X}(T)$

we know this!

Fact: $\hat{X}(f)$ does not depend on the choice of π

Sheaves

We have just shown that $\hat{X}$ is a functor:

$$\hat{X} : \mathbf{I} \to \mathbf{Set}$$

$\mathbf{I}$: the category of finite ~~subsets of $\mathbb{A}$~~ sets
and injective functions

This extends to a correspondence between equivariant functions and natural transformations!

But: not all functors from $\mathbf{I}$ to $\mathbf{Set}$ arise in this way.
Sheaves do.

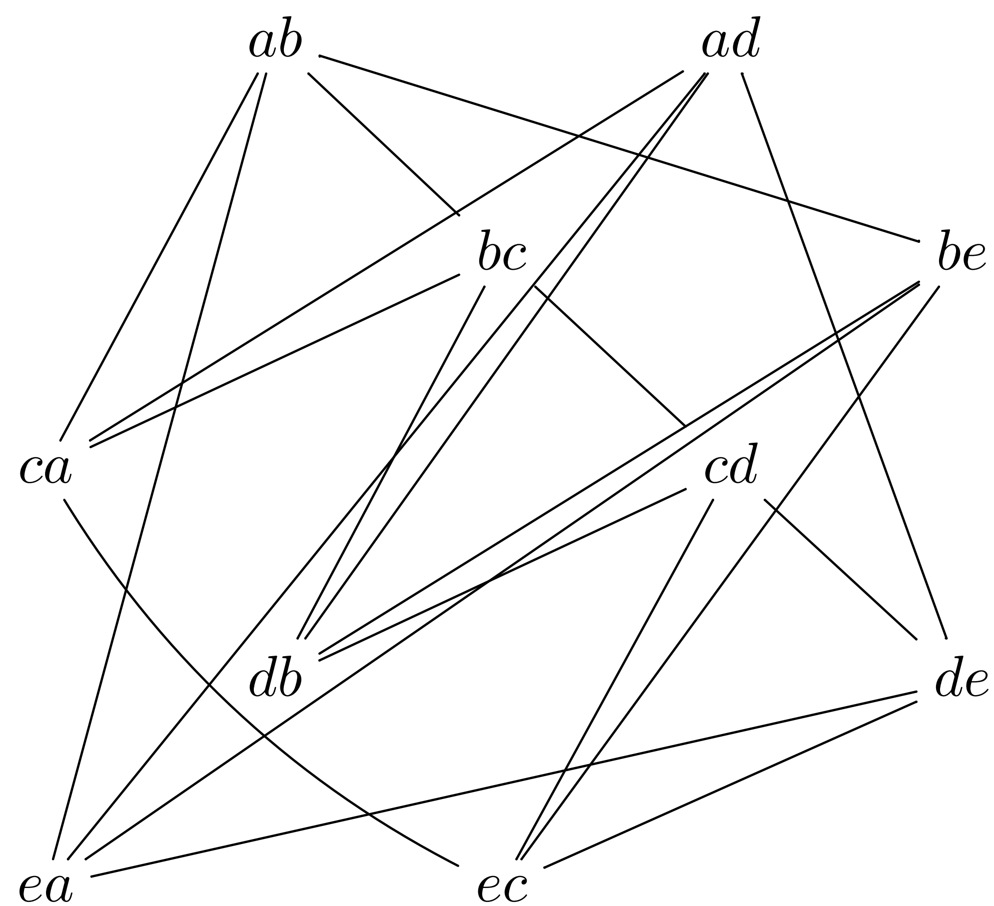
$\mathbf{Nom} \approx$ sheaves on $\mathbf{I}$ and natural transformations

$[\mathbb{A}]$

Orbit Finite Sets

An example problem revisited

- nodes: ab $\quad a \neq b$
- edges: ab—bc $\quad a \neq c$



Is 3-colorability decidable?

Orbits

The orbit of x is the set $\{x \cdot \pi \mid \pi \in \mathrm{Aut}(\mathbb{A})\}$

Every equivariant set is a disjoint union of orbits.

Orbit-finite set if the union is finite.

More generally: the S-orbit of x is

$$\{x \cdot \pi \mid \pi \in \mathrm{Aut}_S(\mathbb{A})\}$$

Fact: An orbit-finite set is S-orbit-finite for every finite S.

Examples

Orbit-finite sets:

$$\mathbb{A} \qquad \mathbb{A}^n \qquad \binom{\mathbb{A}}{n}$$

$$\mathbb{A}^{\triangleleft} = \{\{(a,b,c),(b,c,a),(c,a,b)\} \mid a,b,c \in \mathbb{A}\}$$

- closed under finite union, intersection difference, finite Cartesian product
- but not under (even finite) powerset!

Not orbit-finite:

$$\mathbb{A}^* \qquad \mathcal{P}_{\text{fin}}(\mathbb{A})$$

Group representation

Some single-orbit sets: $\mathbb{A}^{(n)}$ $\binom{\mathbb{A}}{n}$ $\mathbb{A}^{\triangleleft}$

More generally, for $n \in \mathbb{N}$ and $G \leq \mathrm{Sym}(n)$

define an equiv. relation $\sim_G$ on $\mathbb{A}^{(n)}$:

$$(a_1, \ldots, a_n) \sim_G (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$$

for $\sigma \in G$.

Fact: $\mathbb{A}^{(n)}/_{\sim_G}$ is an equivariant, single-orbit set.

Theorem: Every equivariant, single-orbit set is in equivariant bijection with one of this form.

Example

Remember the graph?

- nodes: $\{\ ab \ \mid a \neq b \in \mathbb{A}\}$

$$n = 2 \qquad G = 1$$

- edges: $\{\ ab — bc \ \mid a \neq c \in \mathbb{A}\}$

$$n = 3 \qquad G = 1$$

Problems:

- not well suited for modular representation
- inefficient: $\mathbb{A}^n$ has exponentially many orbits

Example ctd.

Still the same puzzle:

- nodes: $\{\ ab \mid a \neq b \in \mathbb{A}\}$
- edges: $\{\ ab — bc \mid a \neq c \in \mathbb{A}\}$

This is a reasonable finite presentation already!

We keep writing down finite descriptions of infinite sets all the time.

Let's make that formal.

Logical presentation

A set-builder expression:

$$\{e \mid a_1, \ldots, a_n \in \mathbb{A},\; \phi[a_1, \ldots, a_n, b_1, \ldots, b_m]\}$$

- expression: e
- bound variables: $a_1, \ldots, a_n$
- FO(=)-formula: ϕ
- free variables: $b_1, \ldots, b_m$

Add also $\emptyset$ and $\cup$.

Fact: s.-b. e. + interpretation of free vars. as atoms
= a hereditarily orbit-finite set with atoms

Fact: Every h. o.-f. set is of this form.

Examples

The graph:

$$G = (V, E)$$
$$V = \{(a,b) \mid a, b \in \mathbb{A}, a \neq b\}$$
$$E = \{\{(a,b),(b,c)\} \mid a, b, c \in \mathbb{A}, a \neq b \neq c\}$$

(encode pairs with standard set-theoretic trickery)

Descriptions like this can be input to algorithms, for example:

Is 3-colorability of orbit-finite graphs decidable?

Exercises. Prove that:

1. An equivariant orbit-finite set has only finitely many equivariant subsets.

2. For equivariant, orbit-finite sets X and Y, there are finitely many equivariant functions from X to Y.

3. If X is orbit-finite then every $S \subseteq_{\mathrm{fin}} \mathbb{A}$ supports only finitely many elements of X.

4. The converse implication to 3. does not hold.

Set theory with atoms

Nominal sets form a topos

A lot of mathematics can be done with atoms

set → nominal set

finite → orbit-finite

function → equivariant function

EXCEPT:

- axiom of choice fails, even orbit-finite choice
- powerset does not preserve orbit-finiteness

Slogans

X = set, function, relation, automaton, Turing machine, grammar, graph, system of equations...

Nominal X

Infinite but with lots of symmetries

orbit-finite

Infinite but symbolically finitely presentable

We can compute on them