FoPSS 2019

3rd Summer School
on Foundations of Programming
and Software Systems

Nominal Techniques

Warsaw, 10-15 September, 2019

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2017: Braga (Portugal)

Probabilistic Programming

2018: Oxford (UK)

Logic and Learning

2019: Warsaw (Poland)

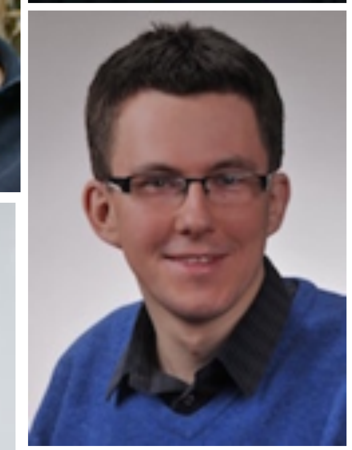
Nominal Techniques

FoPSS 2019

Our lecturers:

- Andrew M. Pitts:
  Nominal sets and functional programming
- Mikołaj Bojańczyk:
  Computation theory with atoms
- Andrzej Murawski:
  Nominal game semantics
- Maribel Fernández:
  Nominal rewriting and unification
- Johannes Borgström:
  Nominal process calculi and modal logics
- Murdoch J. Gabbay:
  Advanced nominal techniques
- Sławomir Lasota:
  Computation theory with atoms II







FoPSS 2019

[A]

Basic Nominal Techniques

Bartek Klin
University of Warsaw

Warsaw, 10-11 September, 2019

What is it all about?

Nominal techniques:

mathematics of, and computation with:

local names
and name dependence

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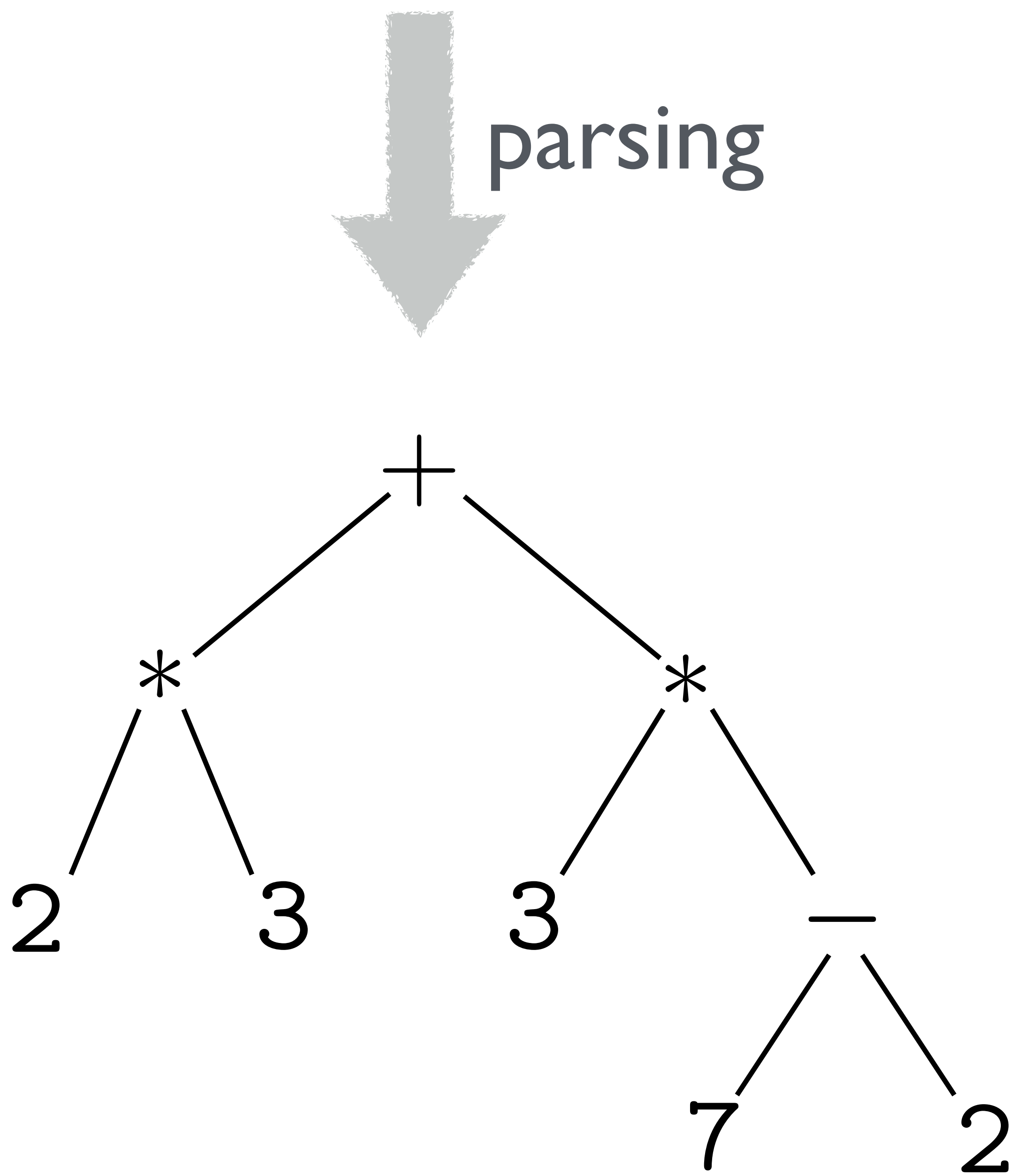
Concrete and abstract syntax

$$2 * 3 + 3 * (7 - 2)$$



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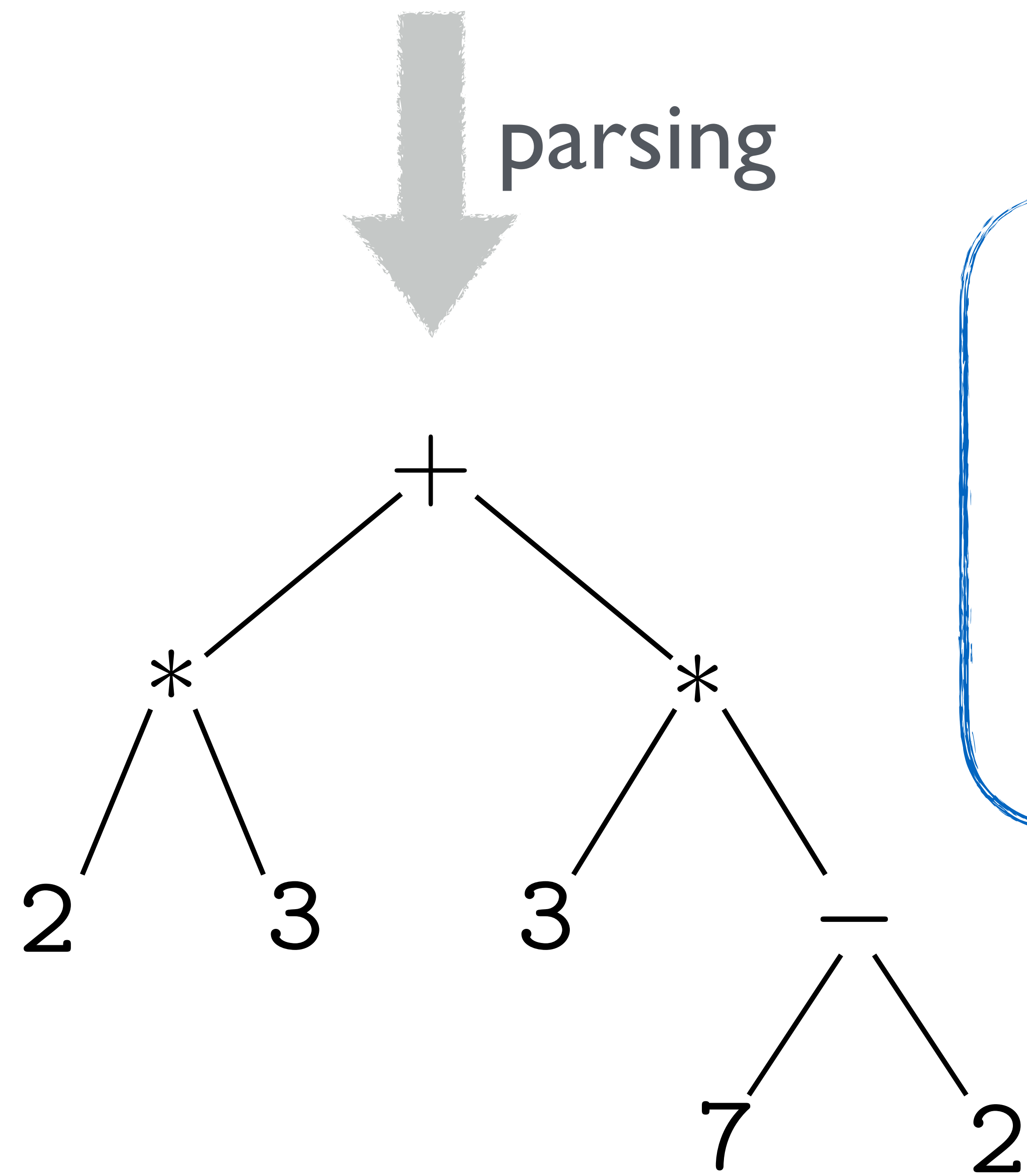
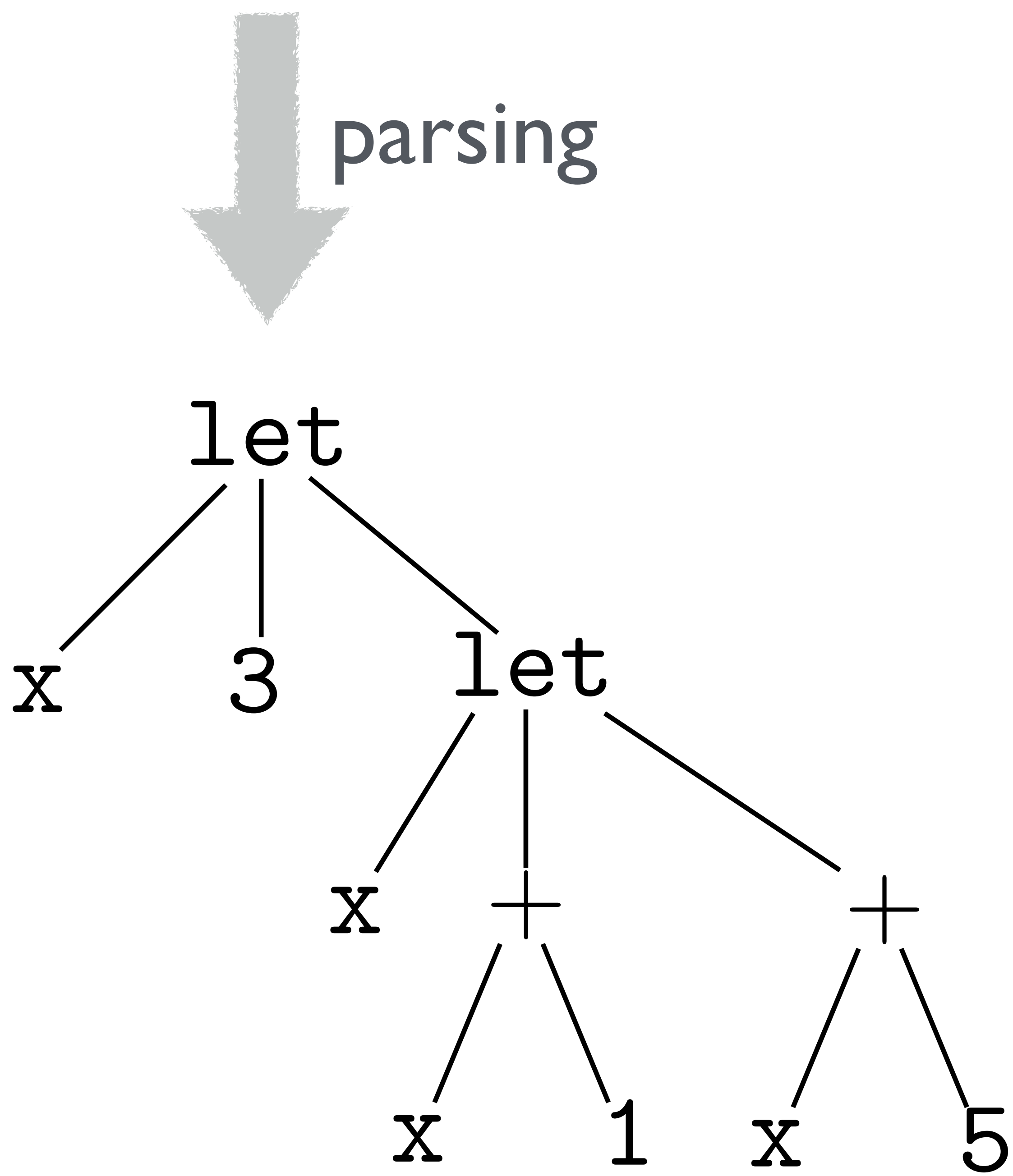


Algebraic features:

- definitions by recursion
- proofs by induction
- ...

Complications with local names

$$\texttt{let x} = 3 \texttt{ in let x} = \texttt{x} + 1 \texttt{ in x} + 5$$



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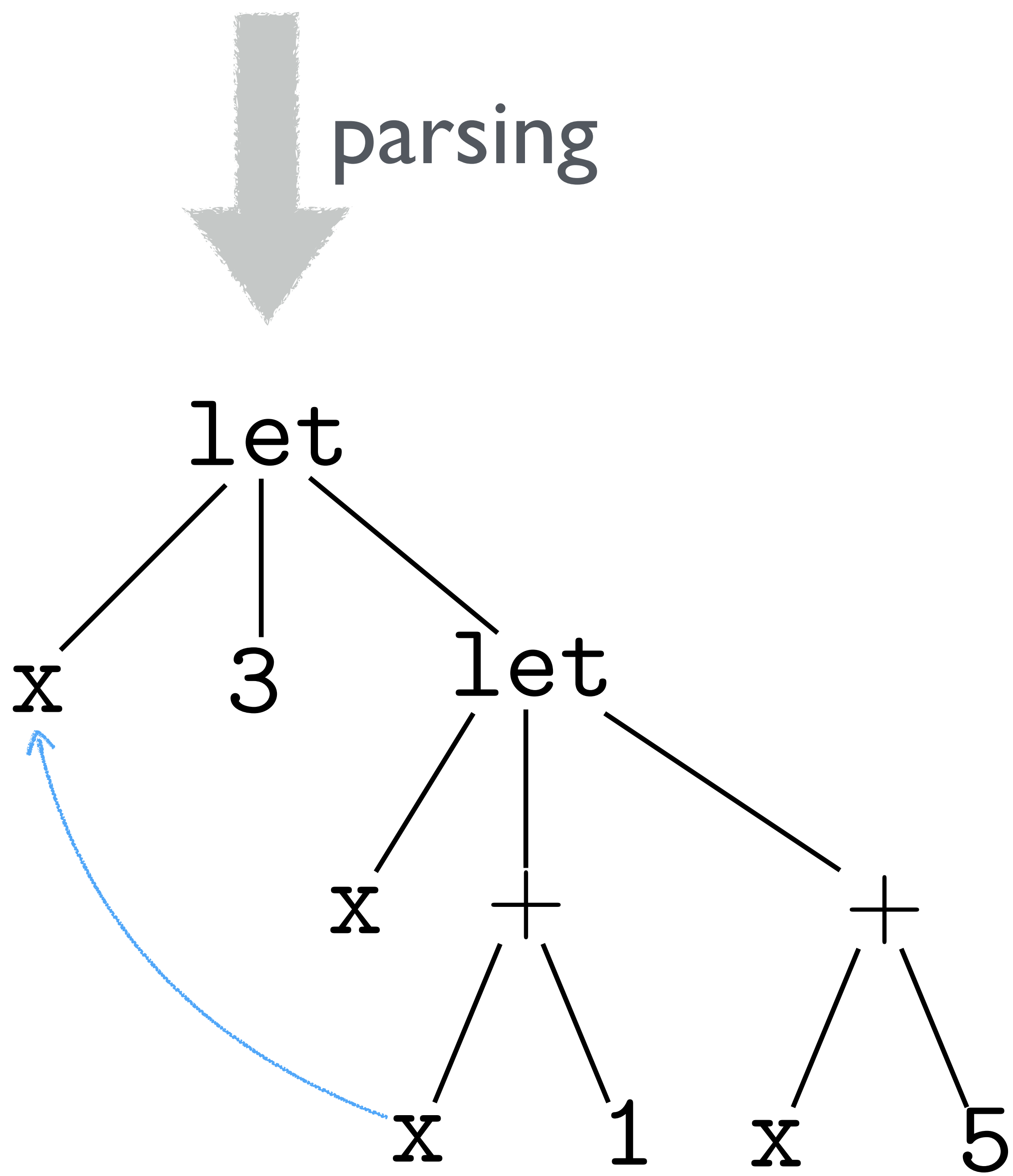

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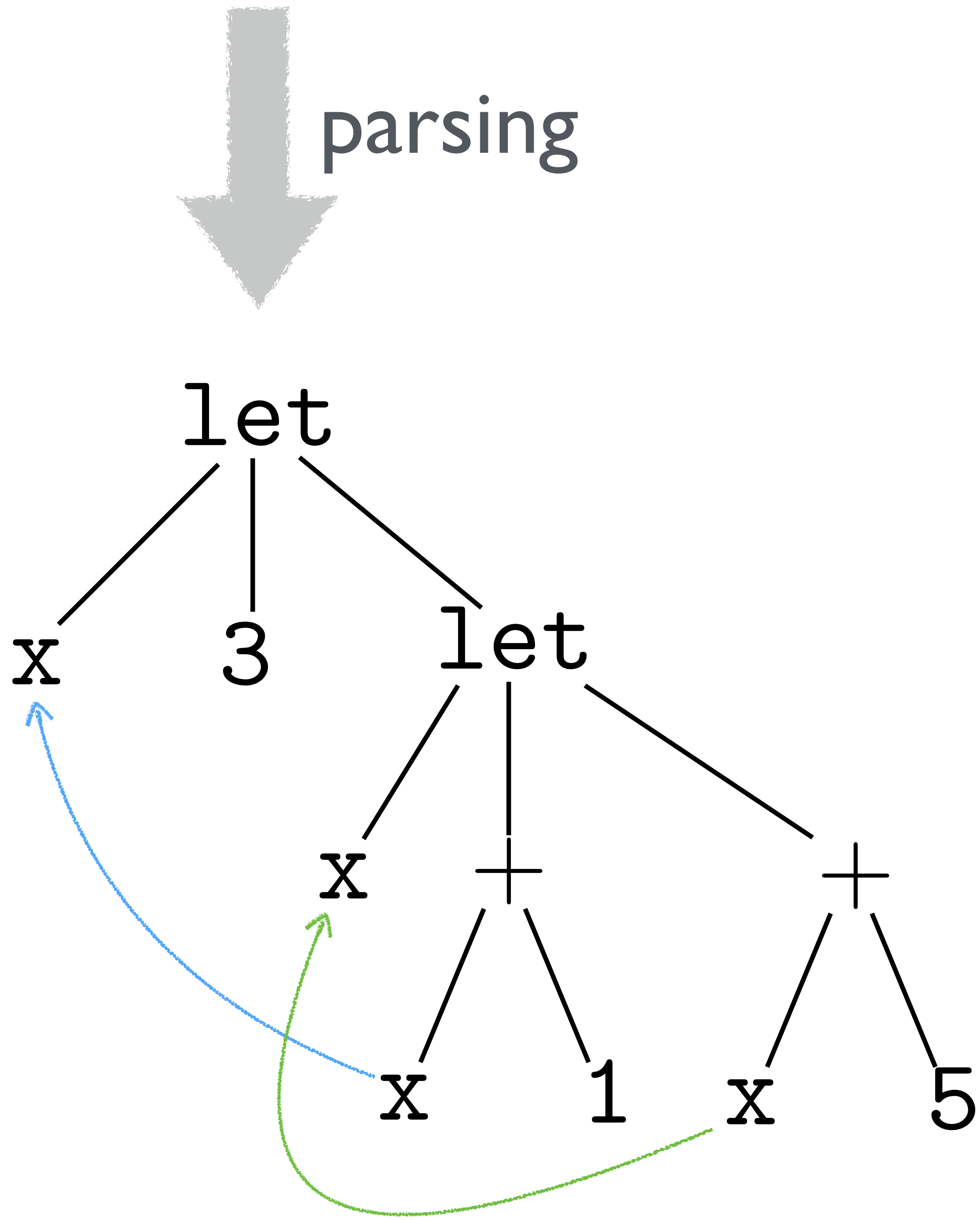

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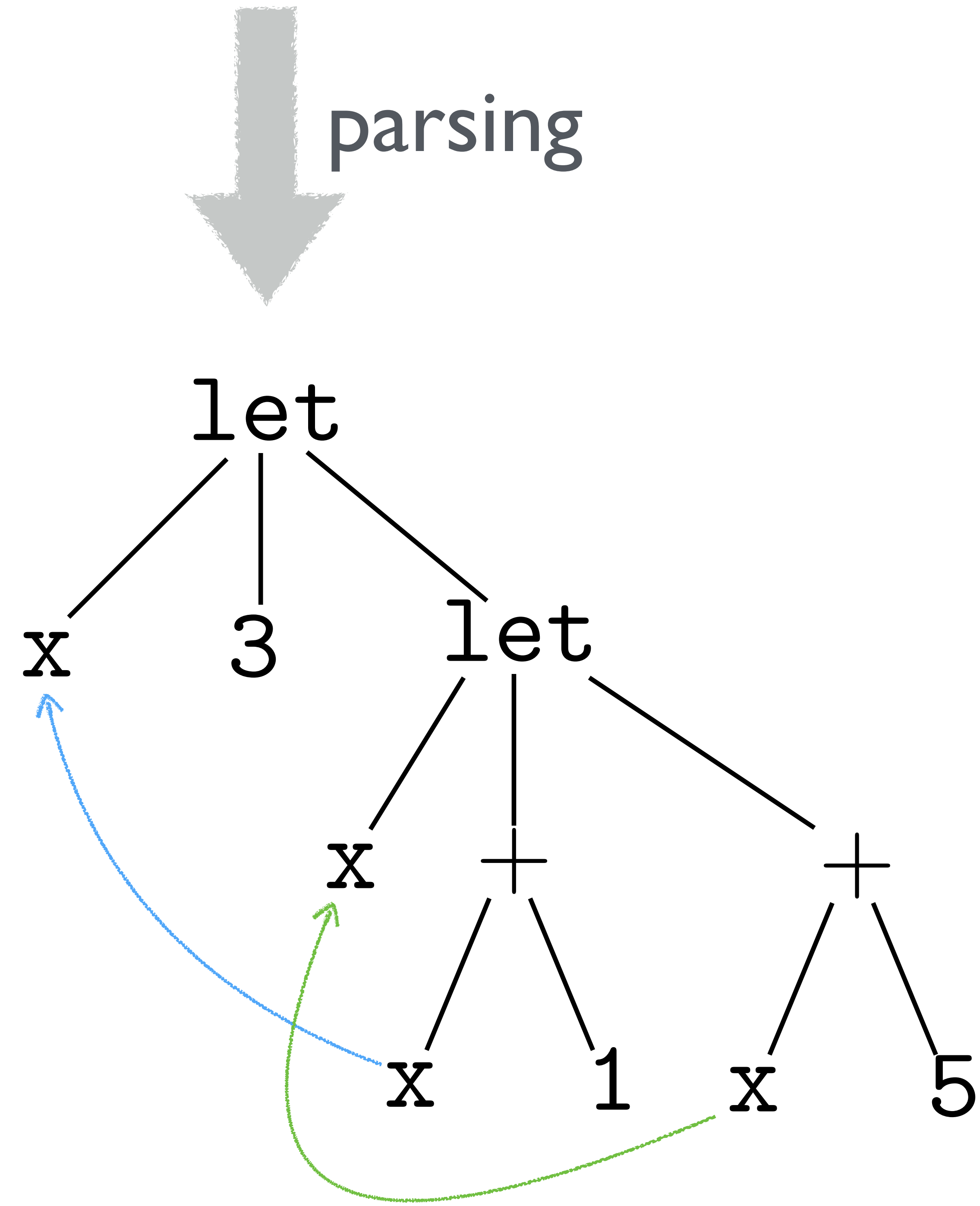

Expressions depend on names!

Name dependence

Idea:

Let every expression come equipped
with an explicit dependence on some names
(or: atoms) that occur in it

Name dependence

Idea: nominal expressions

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More ambitious idea:

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More ambitious idea: nominal sets

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Name dependence

Q:

What does it mean to depend on a name?

Name dependence

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What does it mean to depend on a name?

A:

X depends on a name a
if renaming a to any other name
would alter X

Name dependence

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What does it mean to depend on a name?

A:

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Idea revisited: nominal sets

Let everything come equipped
with information on how renaming names
affects it

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A graph built of atoms

atomic names: $a, b, c, d, e, \ldots$

- nodes: ab $\quad a \neq b$
- edges: ab—bc $\quad a \neq c$

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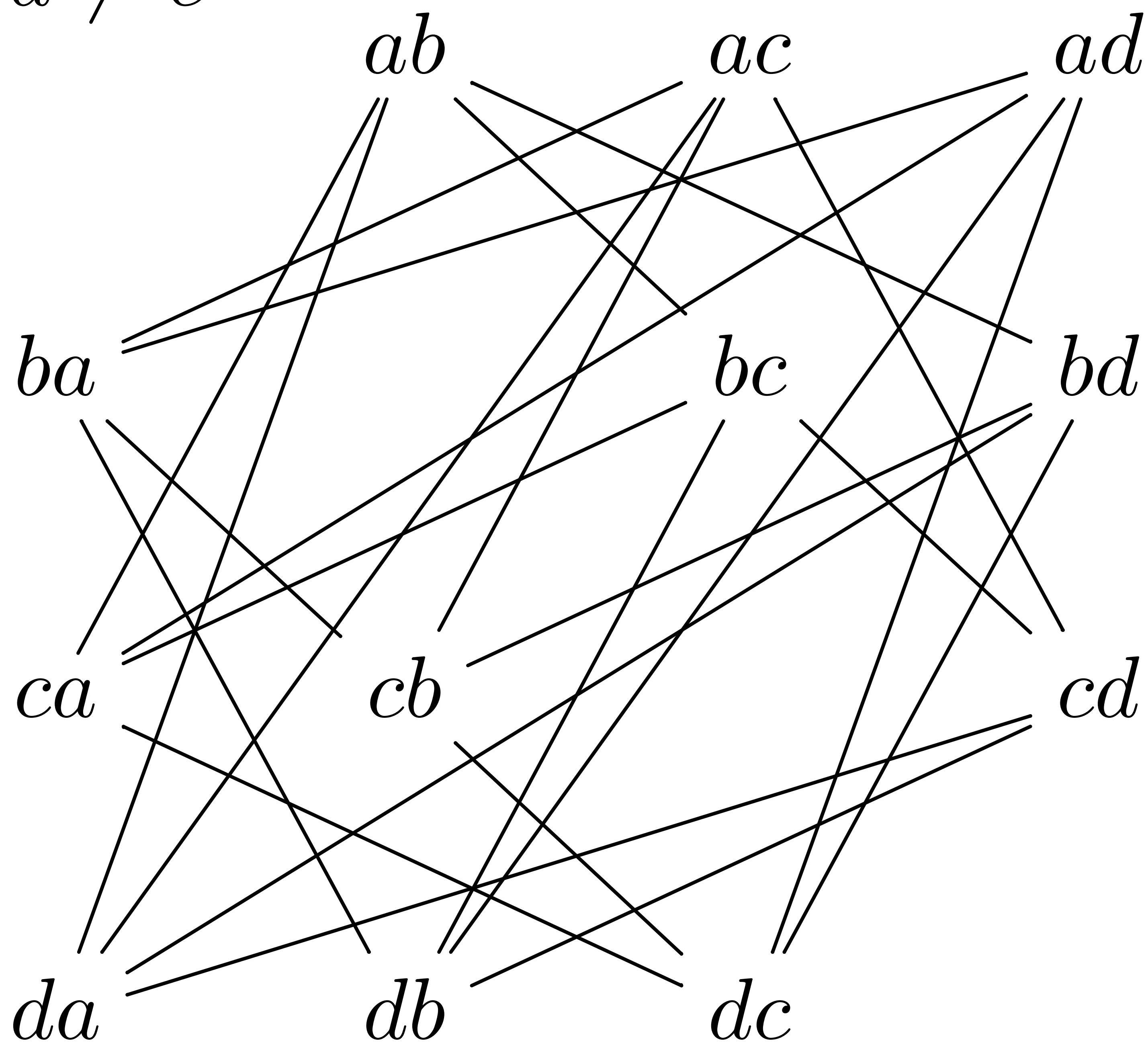

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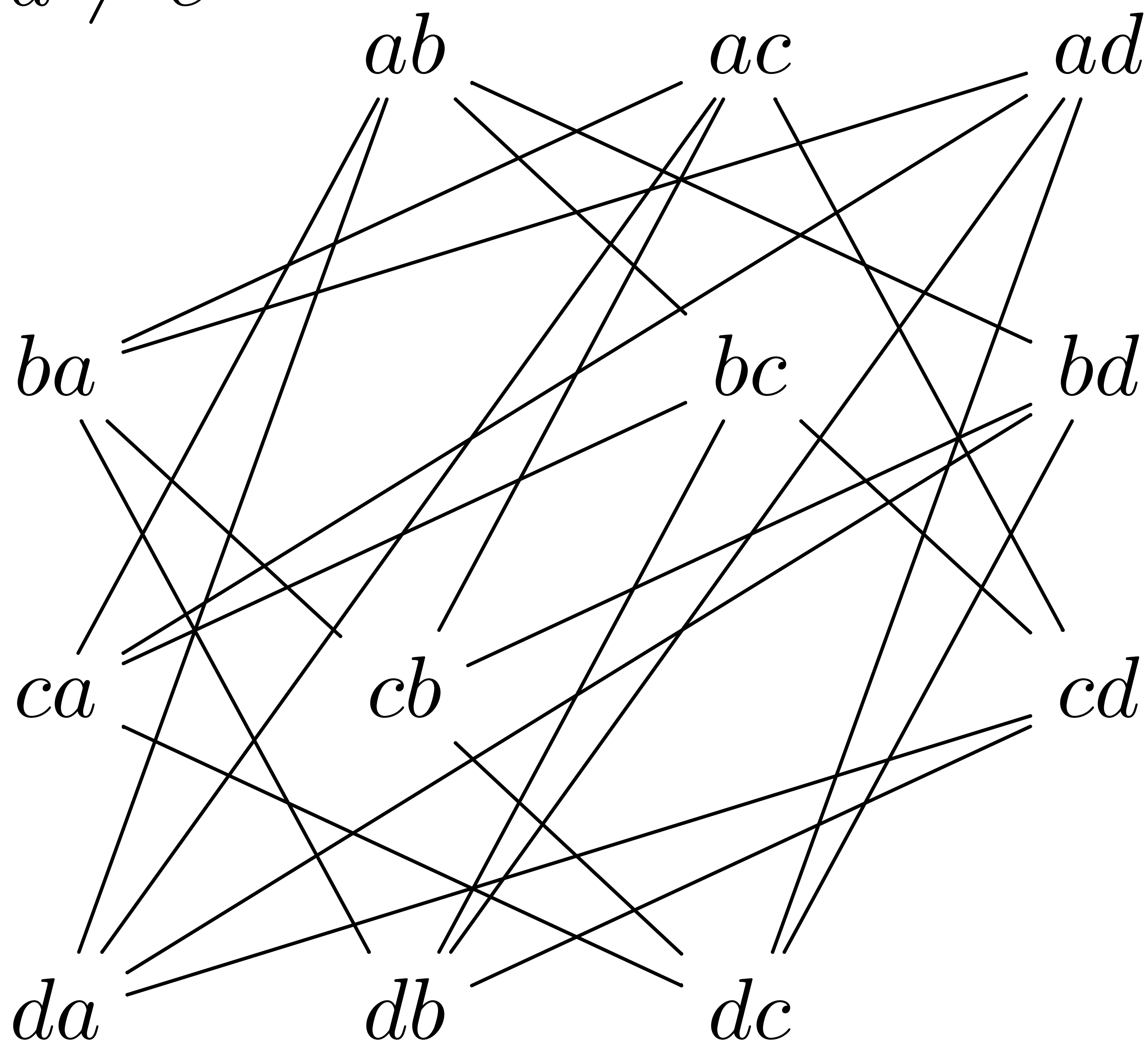

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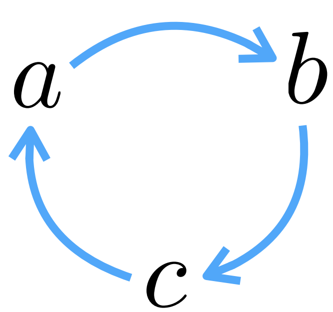

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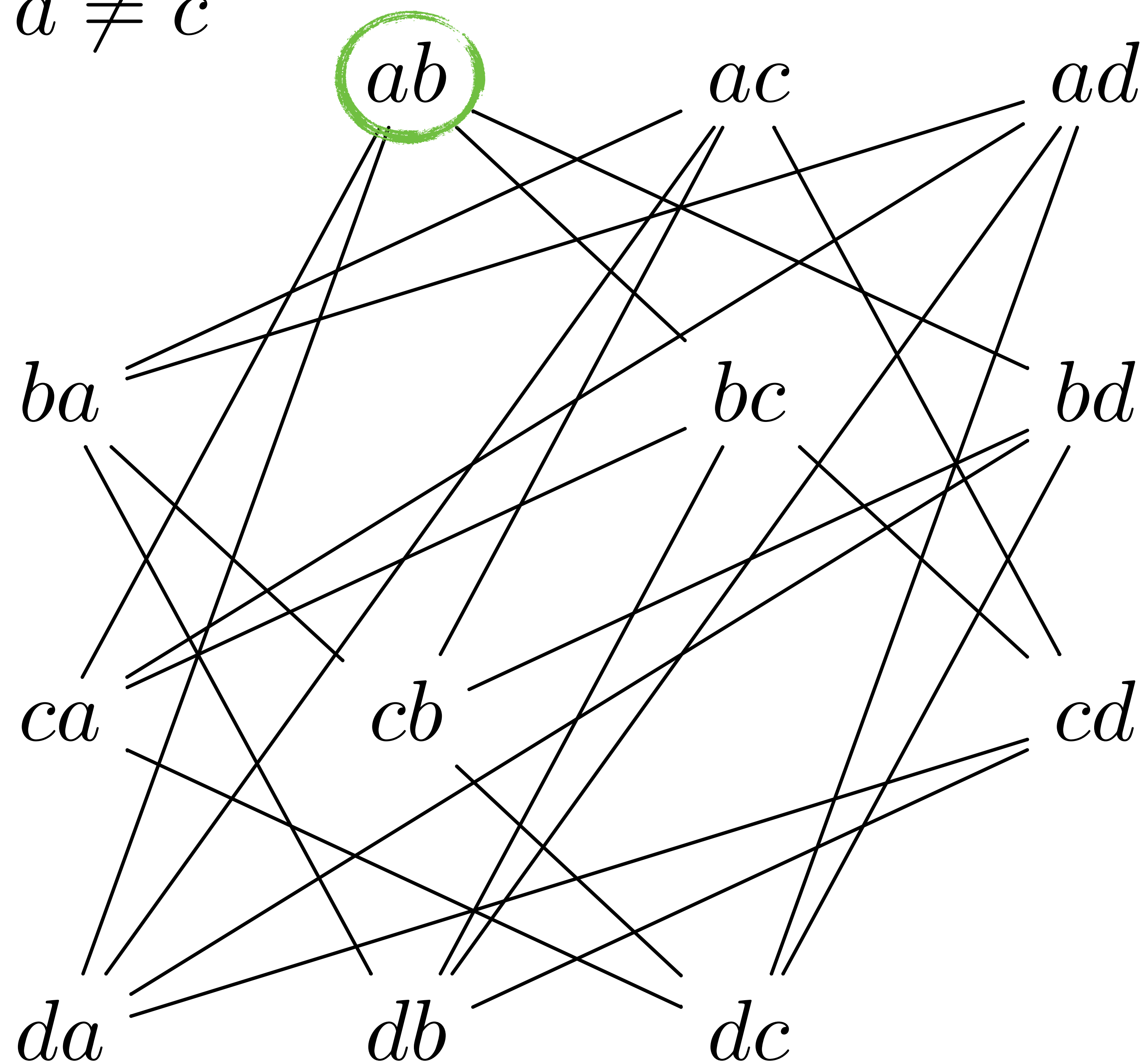

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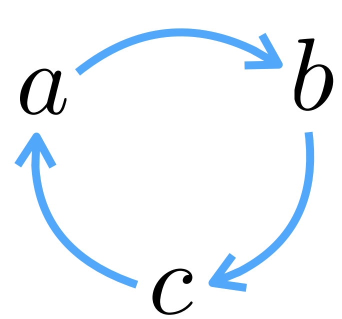

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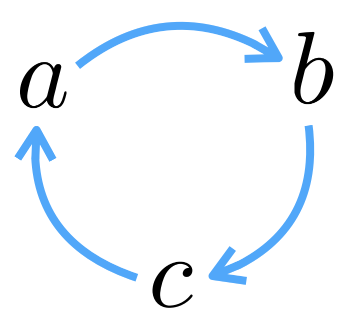

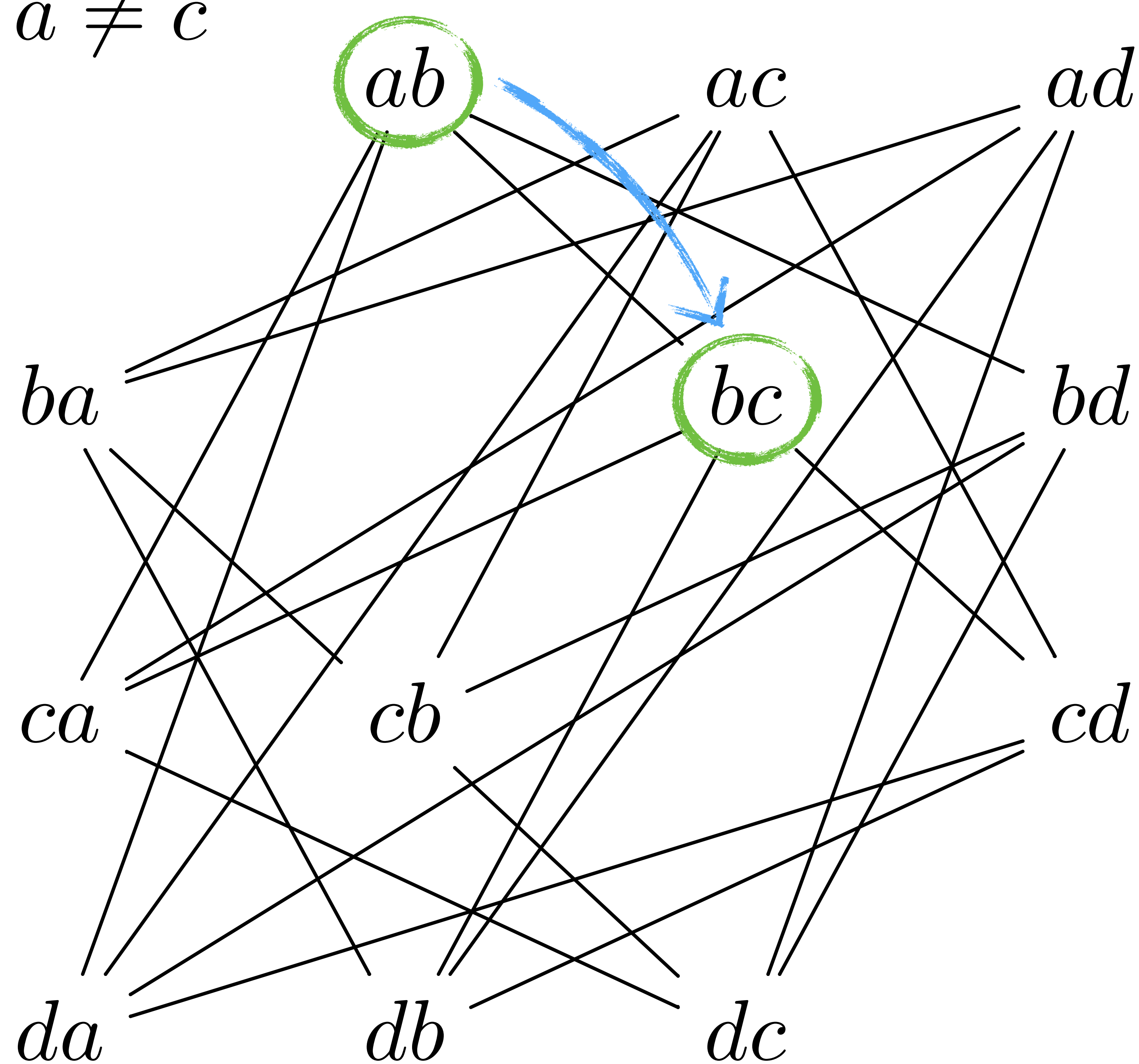

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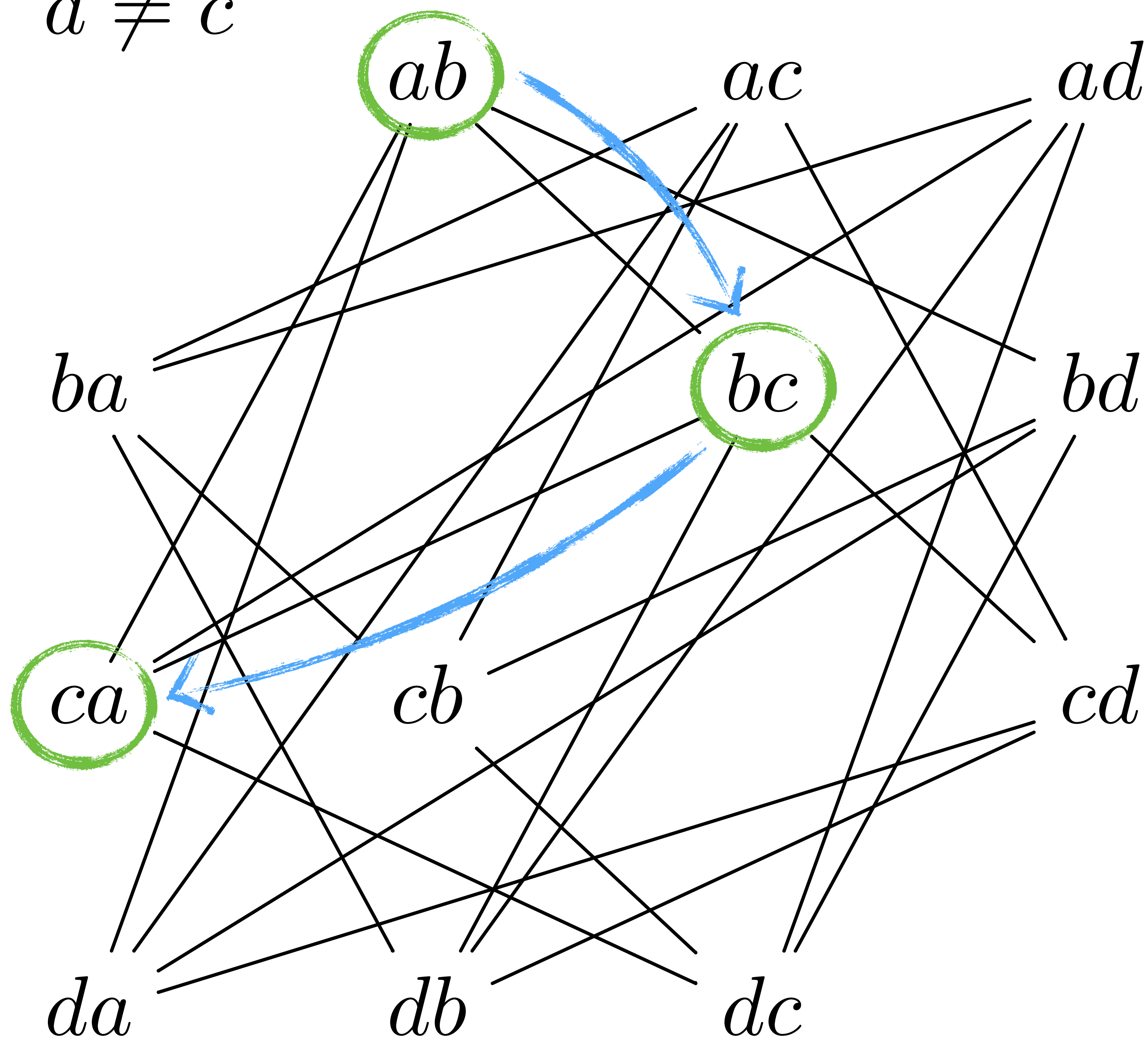

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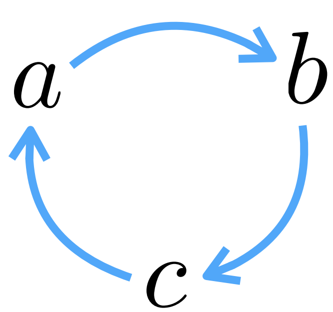

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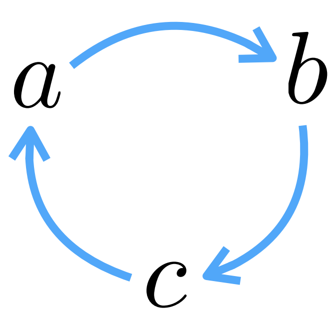

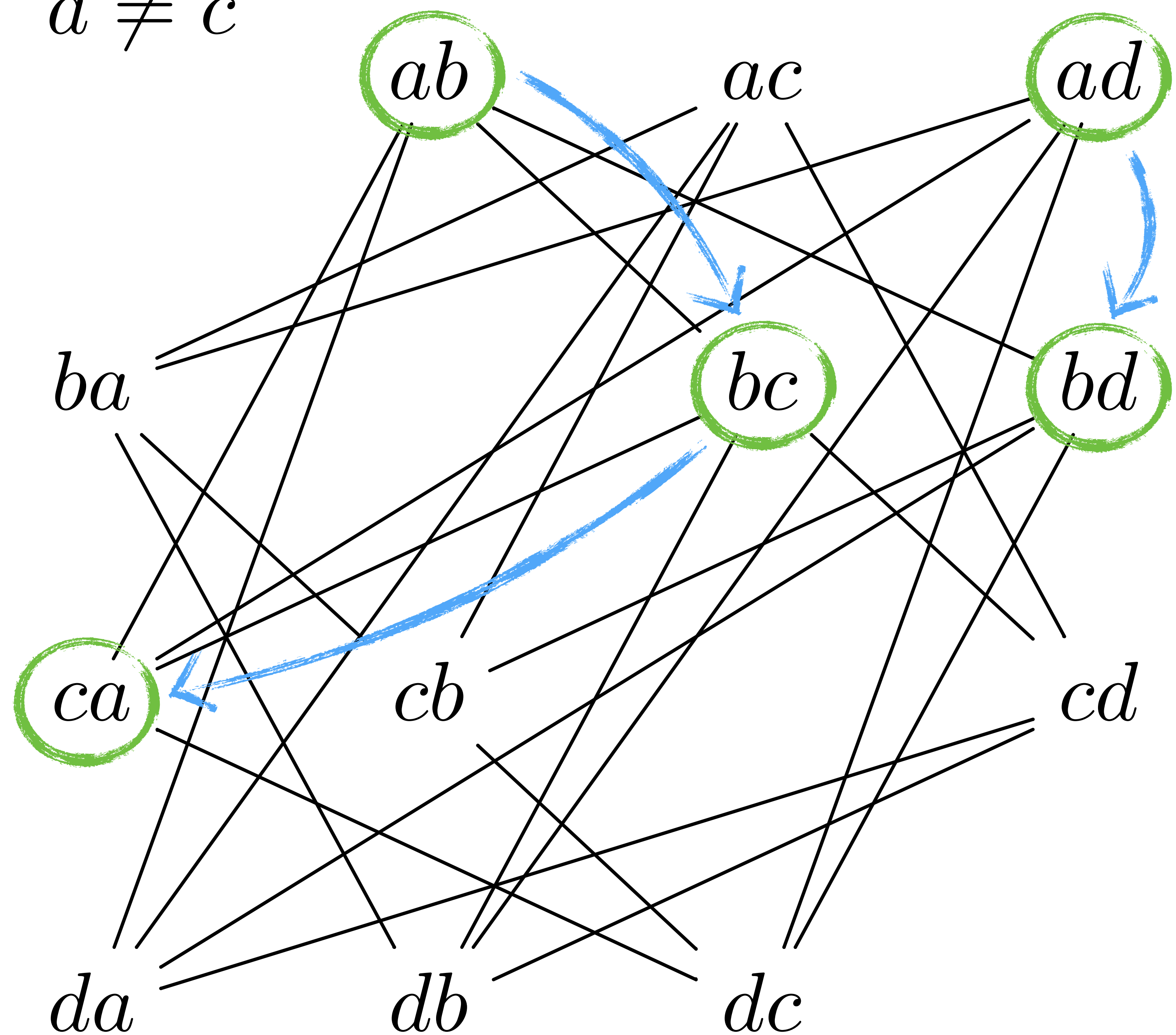

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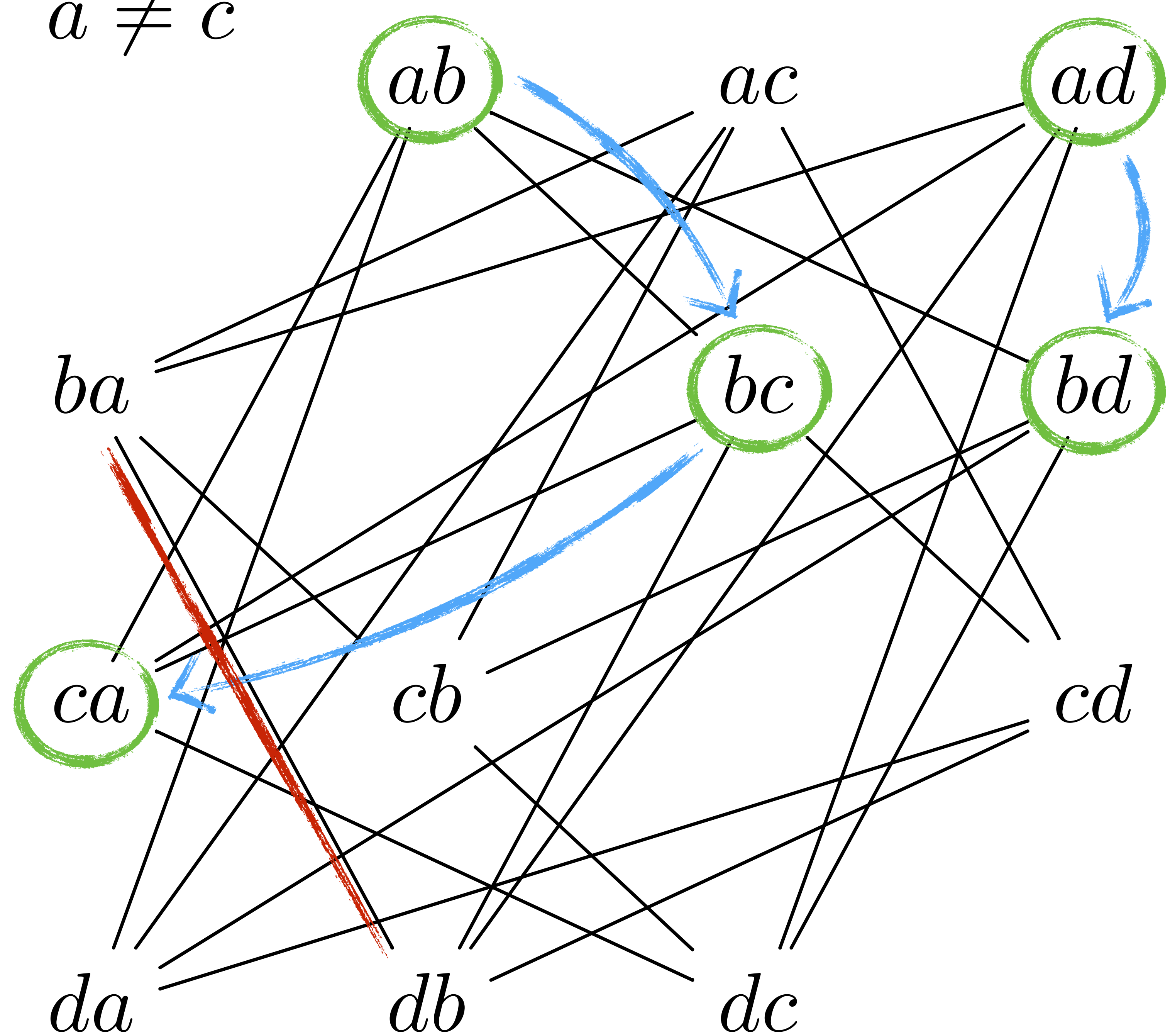

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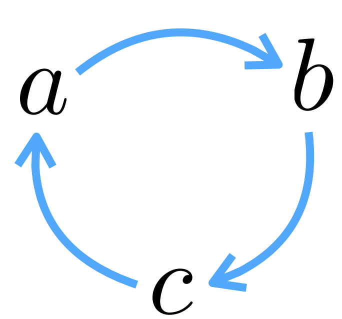

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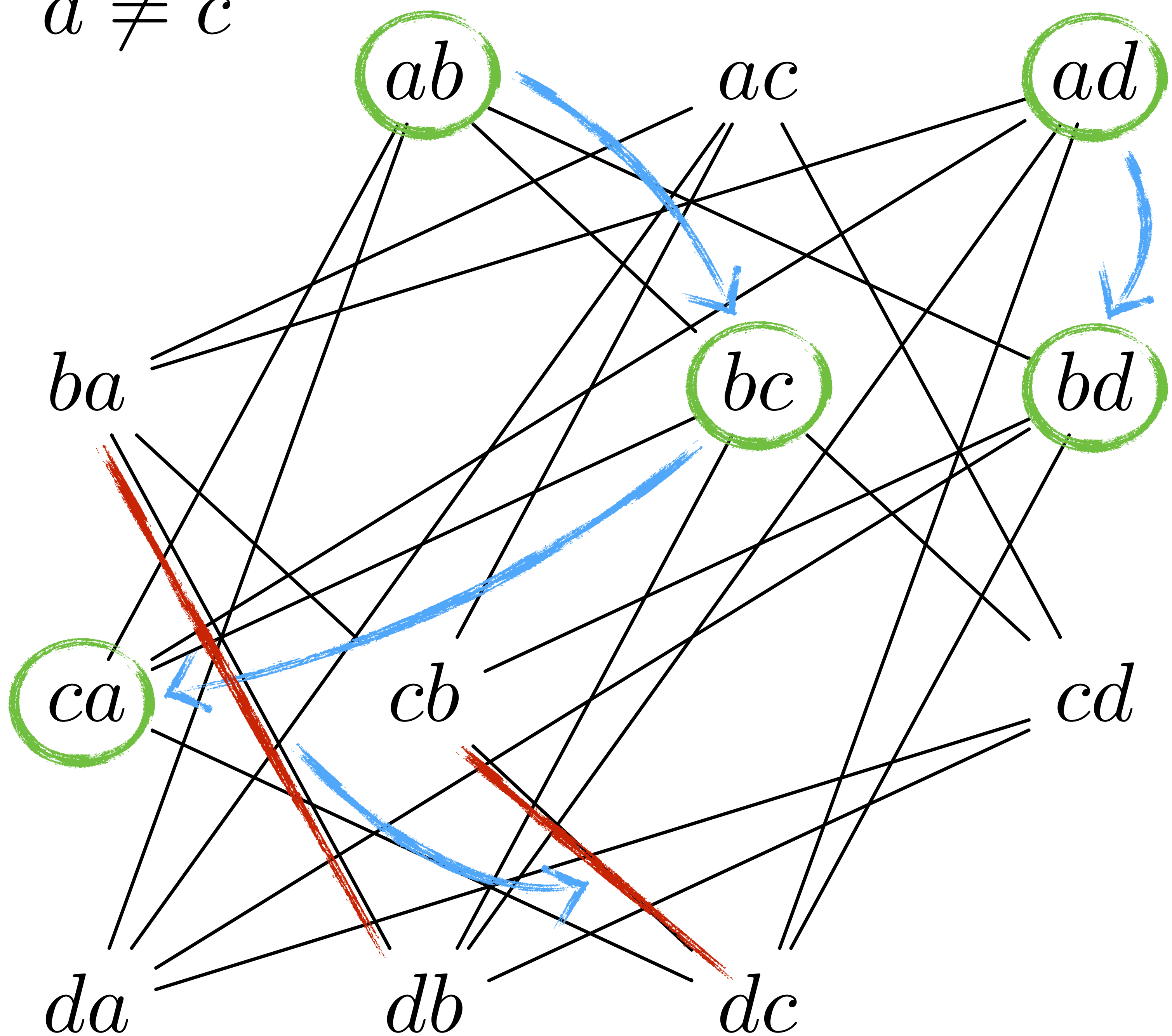

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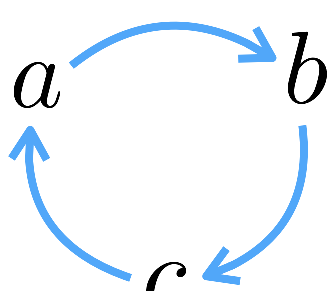

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Slightly infinite

The same graph:

- nodes: ab $\qquad a \neq b$
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Slightly infinite

The same graph:

- nodes: ab $a \neq b$
- edges: ab—bc $a \neq c$

- nodes: $\{(a,b) : a,b \in \mathbb{A} : a \neq b\}$
- edges: $\big\{\{(a,b),(b,c)\} : a,b,c \in \mathbb{A}$

  $: a \neq b \wedge b \neq c \wedge a \neq c\big\}$

Slightly infinite

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Infinite, but presented by finite means

An example problem

- nodes: ab $a \neq b$
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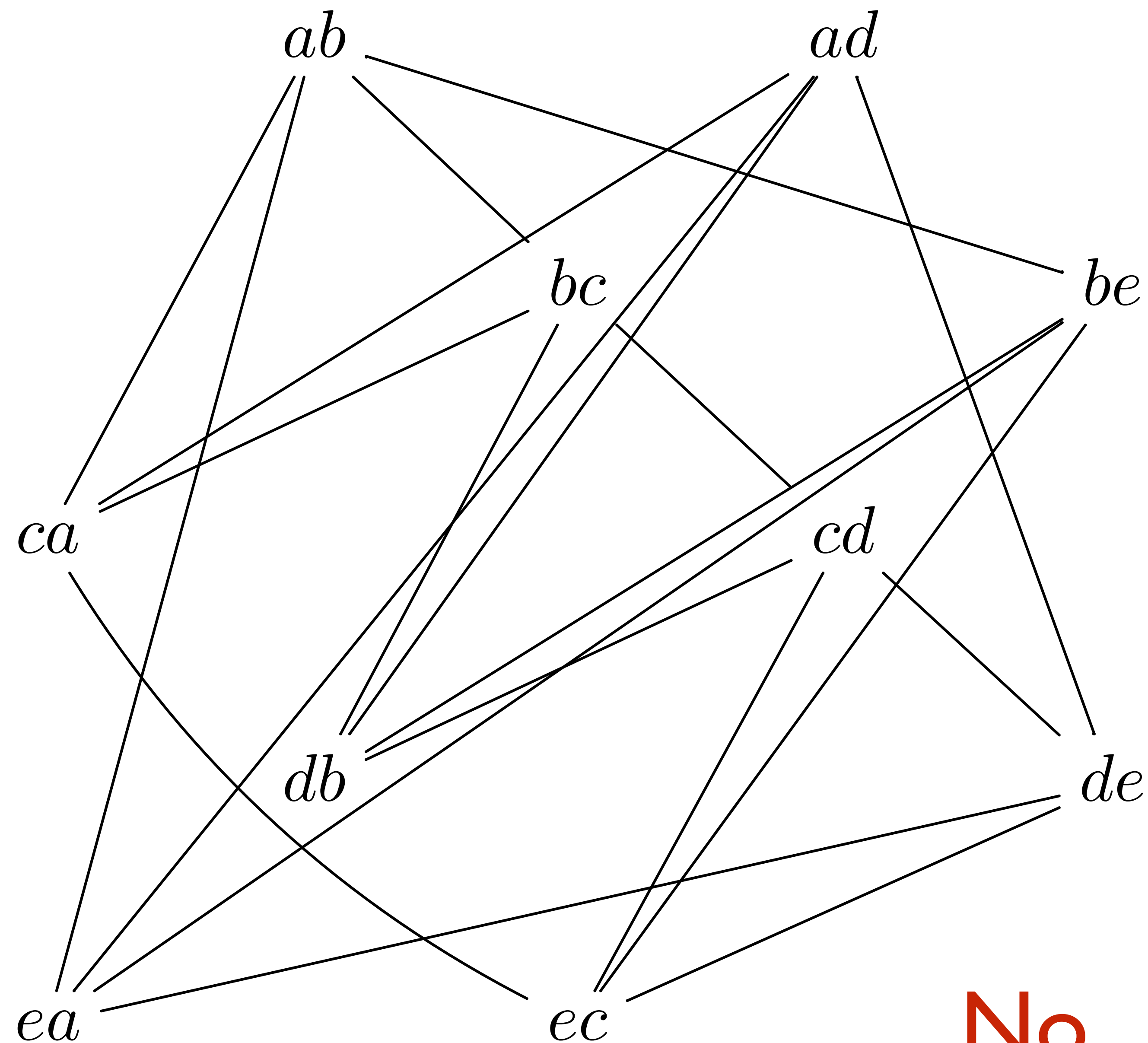

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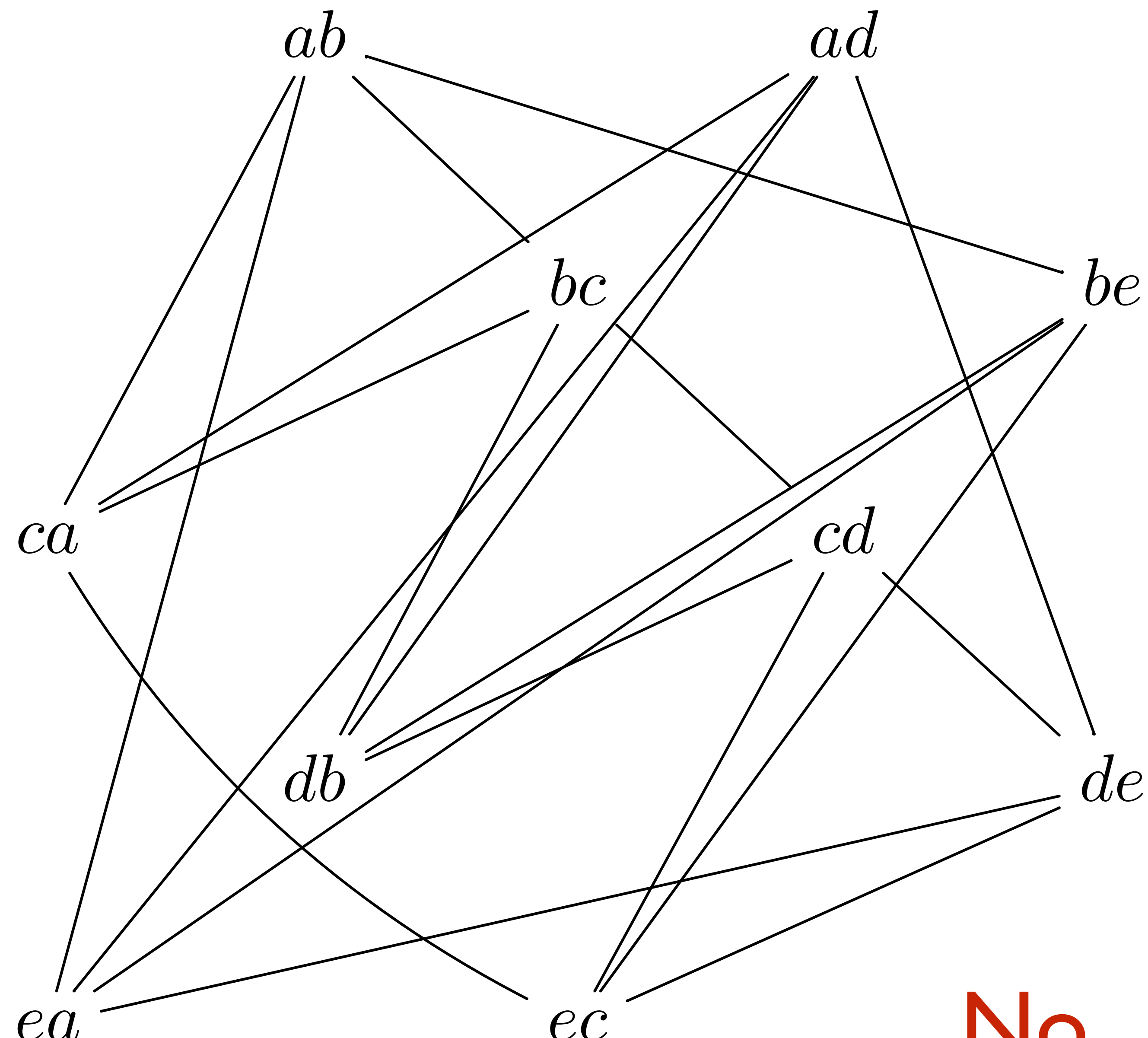

No.

Is 3-colorability decidable?

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Computer Science 101

Theorem:

Every algorithm to sort n numbers must work in time $\Omega(n \log n)$.

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This amounts to restricting the class of legal atom renamings.

Nominal Sets:
Basic Defnitions

$[\mathbb{A}]$

or: Sets with Atoms

Nominal Sets:
Basic Defnitions

Atoms

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$$(\pi \cdot \sigma) \cdot \rho = \pi \cdot (\sigma \cdot \rho)$$

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$(a\ b) \in \mathrm{Aut}(\mathbb{A})$ - the swap of a and b

For example: $(a\ b)(b\ c)(c\ a) = (b\ c)$

$$(a\ b)^{-1} = (a\ b)$$

Von Neumann hierarchy

A hierarchy of universes:

$$\mathcal{U}_0 = \emptyset$$

$$\mathcal{U}_{\alpha+1} = \mathcal{P}\mathcal{U}_\alpha$$

$$\mathcal{U}_\beta = \bigcup_{\alpha<\beta} \mathcal{U}_\alpha$$

defined for every ordinal number.

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Elements of sets are other sets,
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Every set sits somewhere in this hierarchy.

Sets with atoms

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Elements of sets with atoms are atoms or other sets with atoms, in a well founded way

Renaming atoms

A canonical renaming action:

$$\_ \cdot \_ : \mathcal{U} \times \mathrm{Aut}(\mathbb{A}) \to \mathcal{U}$$

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This is a group action of $\mathrm{Aut}(\mathbb{A})$:

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Fact: For every π , the function $\_ \cdot \pi$ is a bijection on $\mathcal{U}$.

Finite support

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A legal set with atoms, or nominal set:

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A set is equivariant if it has empty support.

Examples

$a \in \mathbb{A}$ is supported by $\{a\}$

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Basic Properties

Closure properties

Fact: if X and Y are legal sets then

$X \cup Y,\ \ X \cap Y,\ X + Y,\ X \setminus Y,\ X \times Y$ are legal.

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S supports X and T supports Y

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(But: $S \cap T$ does not support $X \cap Y$!)

Fact: if X is legal and $Y \subseteq X$ is finitely supported then Y is legal.

Powersets

Fact: $\mathcal{P}\mathbb{A}$ is not legal (though it is equivariant).

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NB. these proofs are "easy".

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A (binary) relation is a set of pairs.

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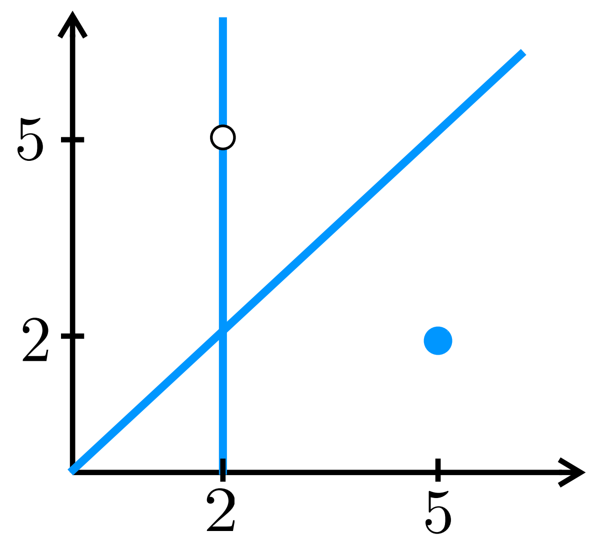

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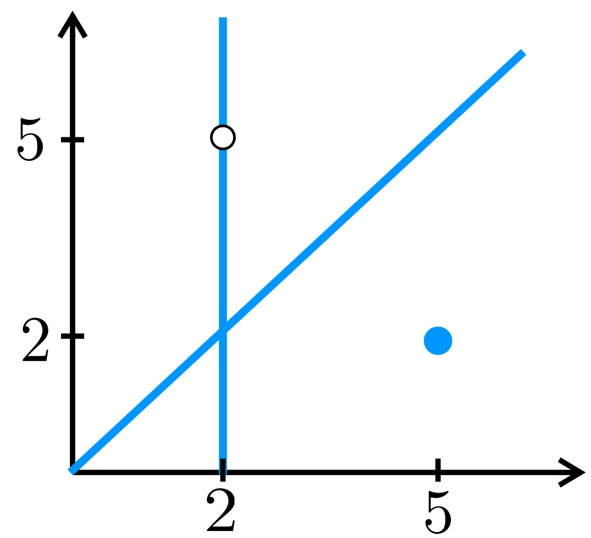

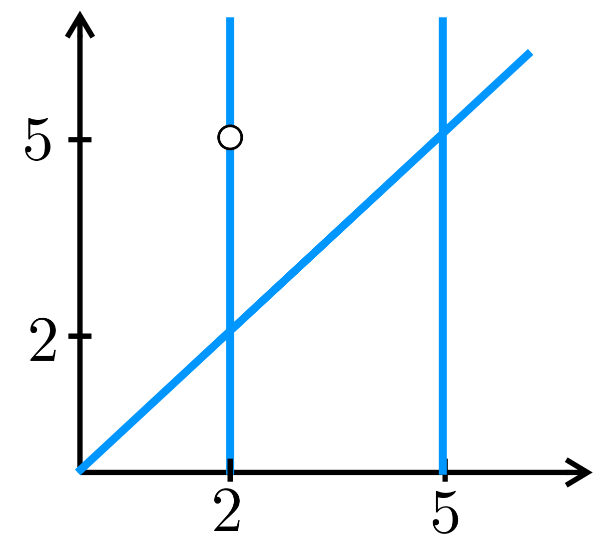

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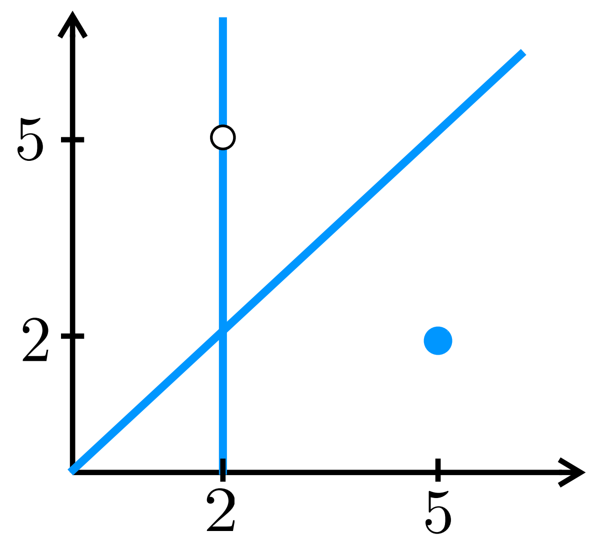

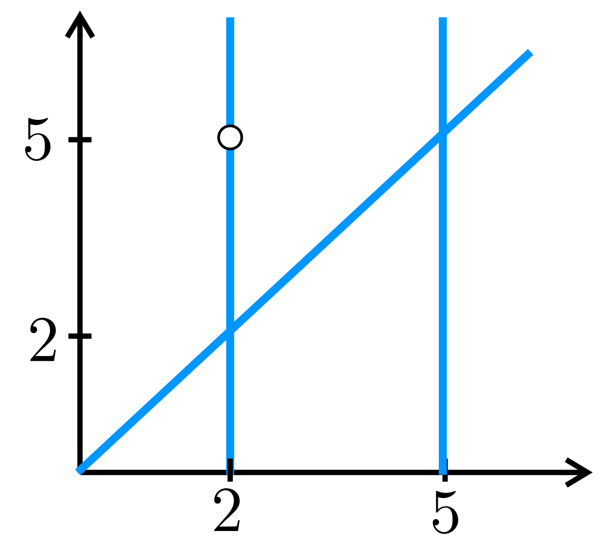

R^*

R , R^* are supported by $\{2, 5\}$

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Only equiv. function from $\mathbb{A}$ to $\mathbb{A}^2$ is the diagonal

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A relation/function/... is equivariant
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A relation/function/... supported by S ,
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NB. another "easy" proof.

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Fact: for finite S and T,

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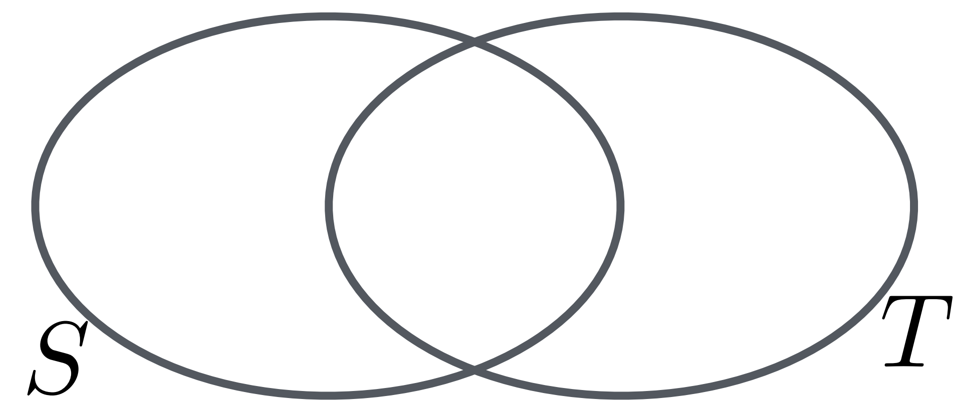
NB. This is harder to prove!

One way: induction on $|S \triangle T|$.

Proof

Assume S and T support X.



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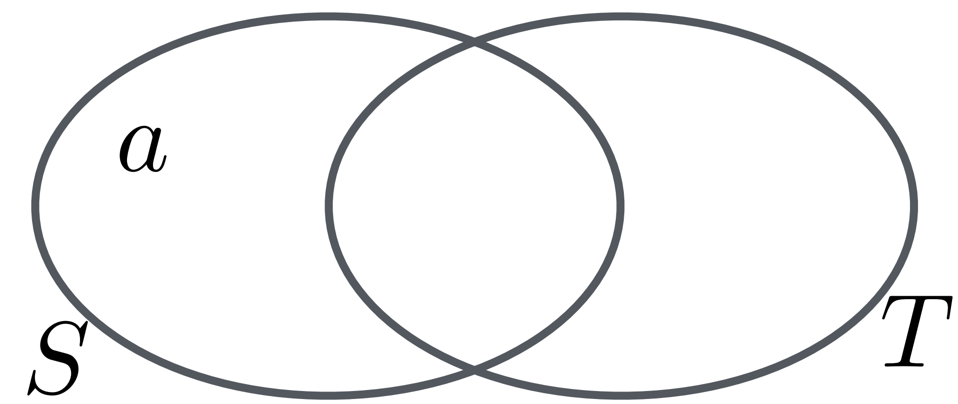

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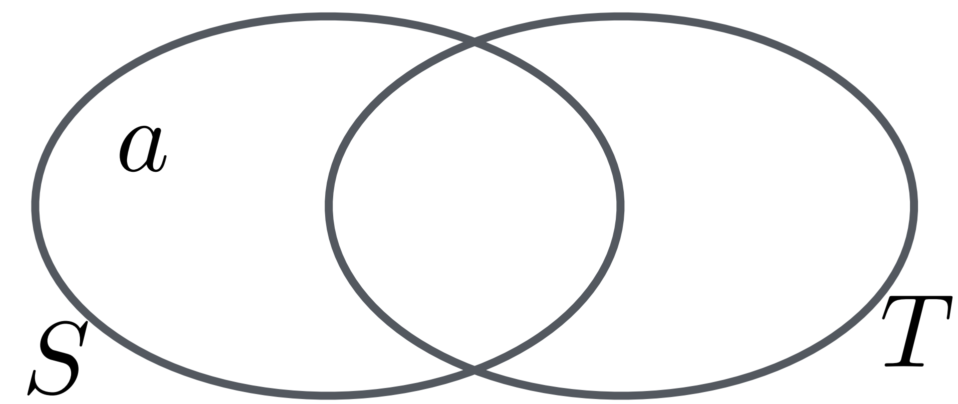

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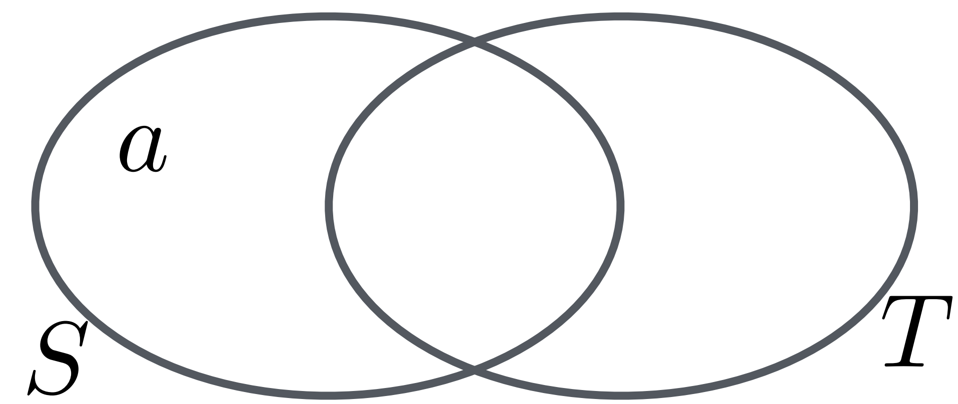

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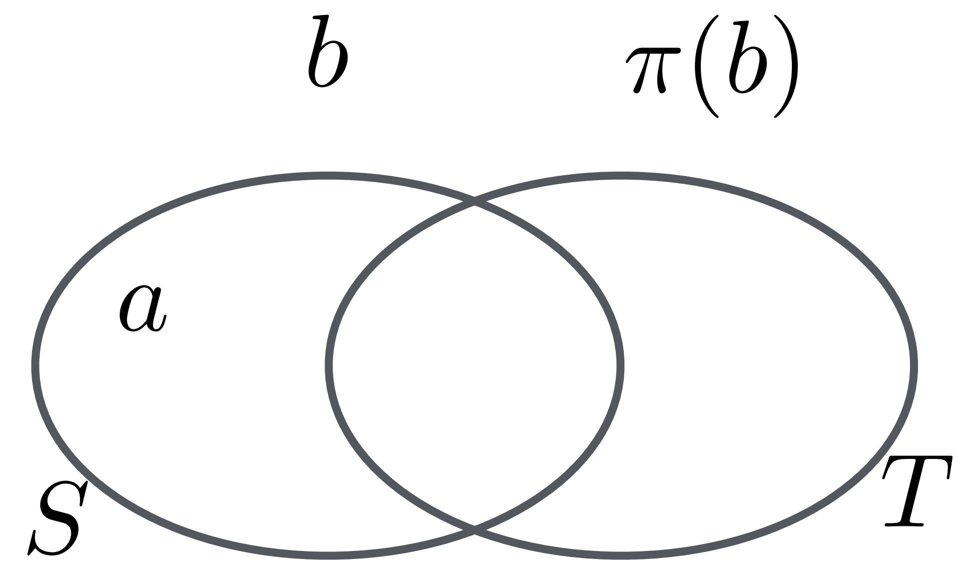

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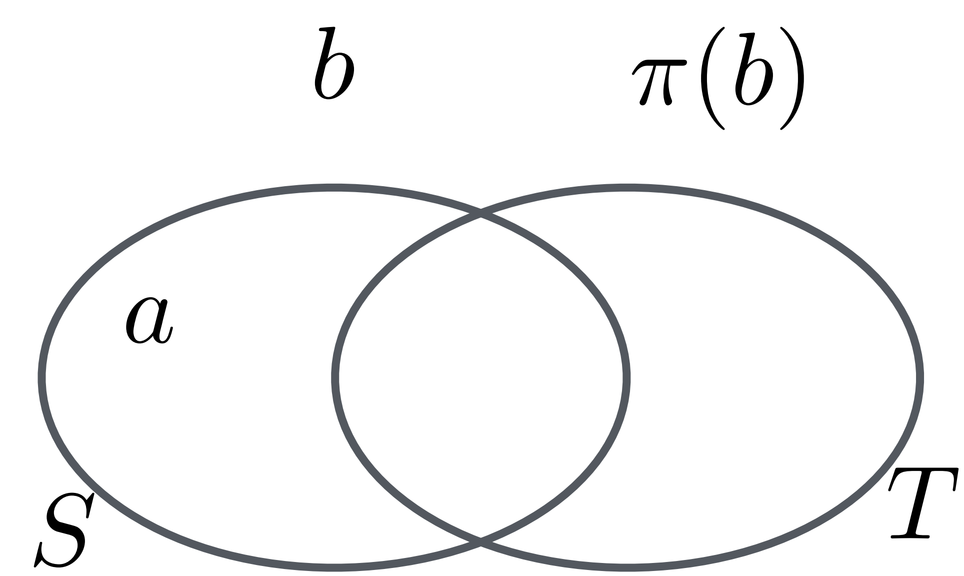

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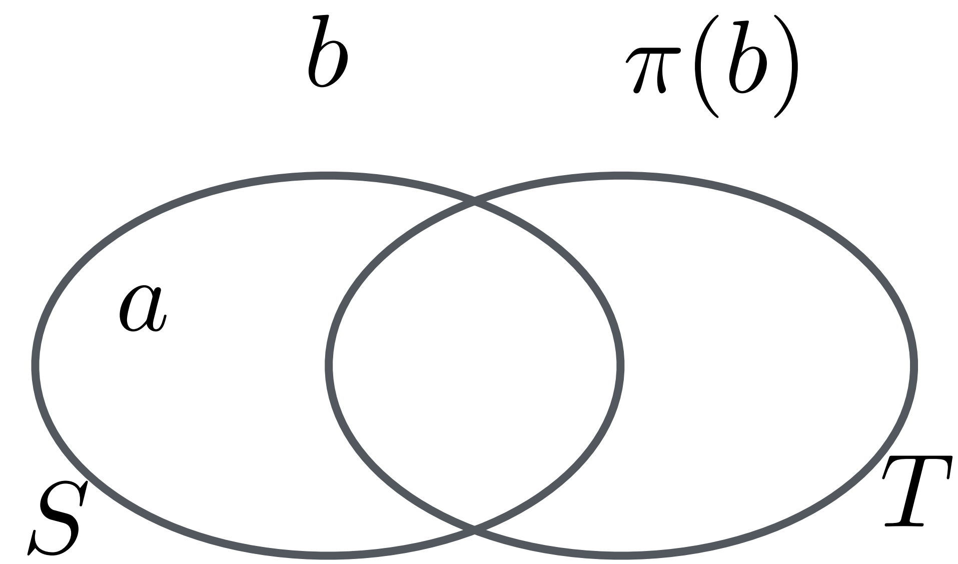

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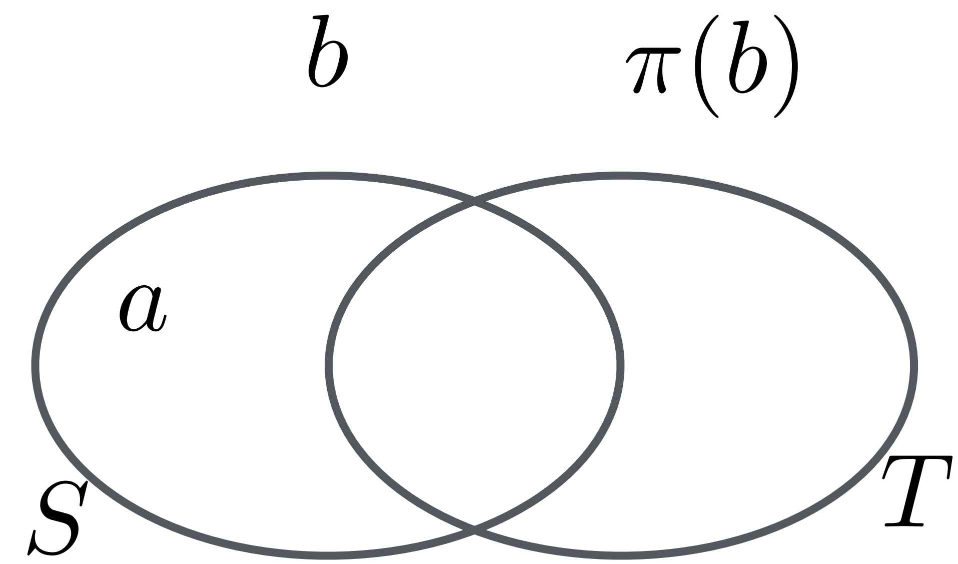

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For an (equivariant) set X,
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