FoPSS 2019
Andrew Pitts, *Nominal Sets and Functional Programming*
Exercises

Exercise 1. Let Tr, var, $(-) \cdot (-)$ and $=_\alpha$ be as on slides 8,10, 11 and 13.

(i) Prove by induction on the structure of trees $t \in Tr$ that the permutation action $(-) \cdot (-)$ defined on slide 11 satisfies $\mathrm{var}(\pi \cdot t) = \{\pi\, a \mid a \in \mathrm{var}\, t\}$.

(ii) Show that for any $a, a' \in \mathbb{A}$ and $\pi \in \mathrm{Perm}\,\mathbb{A}$ that $\pi \circ (a\ a')$ and $(\pi\, a\ \pi\, a') \circ \pi$ are equal permutations.

(iii) Hence prove by induction on the derivation of $t =_\alpha t'$ from the rules inductively defining $=_\alpha$ on slide 13 that if $t =_\alpha t'$, then $\pi \cdot t =_\alpha \pi \cdot t'$ holds for any $\pi \in \mathrm{Perm}\,\mathbb{A}$.

(iv) Deduce that if $(a\ b) \cdot t =_\alpha (a'\ b) \cdot t'$ holds for some $b \in \mathbb{A} - (\{a, a'\} \cup \mathrm{var}(t\, t'))$, then it holds for any such b. Use this to prove that $=_\alpha$ is an equivalence relation.

Exercise 2. Let Tr and Λ be as on slides 8 and 18. The finite set $\mathrm{fv}\, t$ of free variables of $t \in Tr$ is recursively defined by:

$$\begin{aligned} \mathrm{fv}(\mathtt{V}\, a) &= \{a\} \\ \mathrm{fv}(\mathtt{A}(t, t')) &= (\mathrm{fv}\, t) \cup \mathrm{fv}\, t' \\ \mathrm{fv}(\mathtt{L}(a, t)) &= (\mathrm{fv}\, t) - \{a\}. \end{aligned}$$

(i) Prove that for all $\pi \in \mathrm{Perm}\,\mathbb{A}$ and $t \in Tr$, $\mathrm{fv}(\pi \cdot t) = \{\pi\, a \mid a \in \mathrm{fv}\, t\}$.

(ii) Prove that for all $t \in Tr$, $((\forall a \in \mathrm{fv}\, t)\, \pi\, a = a) \Leftrightarrow \pi \cdot t =_\alpha t$.
[Hint: proceed by induction on the size $|t|$ of abstract syntax trees t, where $|\mathtt{V}\, a| = 0$, $|\mathtt{A}(t, t')| = |t| + |t'| + 1$ and $|\mathtt{L}(a, t)| = |t| + 2$, say. Note that $|(a\ a') \cdot t| = |t|$, so that in the induction step for $\mathtt{L}(a, t)$ one can suitably freshen the bound variable, $\mathtt{L}(a, t) =_\alpha \mathtt{L}(a', (a\ a') \cdot t)$, and apply the induction hypothesis to $(a\ a') \cdot t$.]

(iii) Deduce that the smallest support of an element $c \in \Lambda$ is $\mathrm{fv}\, t$ for any $t \in Tr$ in the α-equivalence class c.

Exercise 3. (i) Show that in the category **Nom** the product of two objects X and Y is given by their cartesian product as sets $X \times Y = \{(x, y) \mid x \in X \wedge y \in Y\}$ with $\mathrm{Perm}\,\mathbb{A}$-action $\pi \cdot (x, y) = (\pi \cdot x, \pi \cdot y)$. (In other words, show that the two product projections $\mathrm{fst}(x, y) = x$ and $\mathrm{snd}(x, y) = y$ are morphisms in **Nom**; and that given any $Z \in$ **Nom** and morphisms $X \xleftarrow{f} Z \xrightarrow{g} Y$, there is a unique morphism $Z \xrightarrow{\langle f, g\rangle} X \times Y$ satisfying $\mathrm{fst} \circ \langle f\,, g\rangle = f$ and $\mathrm{snd} \circ \langle f\,, g\rangle = g$.)

(ii) Prove that for all $(x, y) \in X \times Y$, $\mathrm{supp}(x, y) = \mathrm{supp}\, x \cup \mathrm{supp}\, y$.

Exercise 4. Show that a morphism $X \xrightarrow{f} Y$ in **Nom** is an isomorphism (that is, there is some, necessarily unique, morphism $X \xleftarrow{g} Y$ with $g \circ f = id_X$ and $f \circ g = id_Y$) iff the function f is not only equivariant, but also a bijection.

Exercise 5. Continuing Exercise 3, show that **Nom** is a cartesian closed category. To do this, show that the exponential of two nominal sets X and Y is given by the nominal set $X \to_{\mathrm{fs}} Y$ of finitely supported functions defined on slide 21. You have to define an equivariant function $(X \to_{\mathrm{fs}} Y) \times X \xrightarrow{\mathrm{ap}} Y$ with the property that for any $Z \in \mathbf{Nom}$ and morphism $Z \times X \xrightarrow{f} Y$, there is a unique morphism $Z \xrightarrow{\mathrm{cur}\, f} (X \to_{\mathrm{fs}} Y)$ satisfying $\mathrm{ap} \circ (\mathrm{cur}\, f \times id_X) = f$ (where $Z \times X \xrightarrow{\mathrm{cur}\, f \times id_X} (X \to_{\mathrm{fs}} Y) \times X$ is by definition the morphism $\langle \mathrm{cur}\, f \circ \mathrm{fst}\,, \mathrm{snd}\rangle$).

Exercise 6. Show that name abstraction nominal sets $[\mathbb{A}]X$ defined on slide 28 give a right adjoint to the functor $(-) * \mathbb{A} : \mathbf{Nom} \longrightarrow \mathbf{Nom}$ which sends each $X \in \mathbf{Nom}$ to

$$X * \mathbb{A} \triangleq \{(x, a) \in X \times \mathbb{A} \mid a \mathbin{\#} x\}$$

(with $\mathrm{Perm}\,\mathbb{A}$-action inherited from the product $X \times \mathbb{A}$) and each morphism $X \xrightarrow{f} Y$ to the morphism $X * \mathbb{A} \xrightarrow{f * \mathbb{A}} Y * \mathbb{A}$ defined by $(f * \mathbb{A})(x, a) = (f\, x, a)$.

To do this, first show that there is a well-defined equivariant function $(-) \mathbin{@} (-) : ([\mathbb{A}]X) * \mathbb{A} \longrightarrow X$ satisfying $(\langle a \rangle x) \mathbin{@} b = (a\ b) \cdot x$. This is called *concretion* and is the counit of the adjunction: show that if $Y * \mathbb{A} \xrightarrow{f} X$, then there is a unique morphism $Y \xrightarrow{\hat{f}} [\mathbb{A}]X$ satisfying $f(y, a) = (\hat{f}\, y) \mathbin{@} a$, for all $(y, a) \in Y * \mathbb{A}$.

Exercise 7. Coproducts in **Nom** are given by disjoint union, $X + Y \triangleq \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$ with $\mathrm{Perm}\,\mathbb{A}$-action given by $\begin{cases} \pi \cdot (0, x) = (0, \pi \cdot x) \\ \pi \cdot (1, y) = (1, \pi \cdot y). \end{cases}$
Show that $[\mathbb{A}](X + Y)$ is isomorphic to $([\mathbb{A}]X) + ([\mathbb{A}]Y)$.

Exercise 8. Show that $[\mathbb{A}]\mathbb{A}$ is isomorphic in the category **Nom** to the coproduct $\mathbb{A} + 1$.

Exercise 9. Every set S determines a nominal set by endowing it with the trivial permutation action $\pi \cdot s = s$. (What is the support of s in this case?) We call such nominal sets *discrete*. For any discrete nominal set S, show that $[\mathbb{A}]S$ is isomorphic to S in **Nom**.

Exercise 10. Show that for any $X, Y \in \mathbf{Nom}$, $[\mathbb{A}](X \times Y)$ is isomorphic to $([\mathbb{A}]X) \times ([\mathbb{A}]Y)$.

Exercise 11. Show that for any $X, Y \in \mathbf{Nom}$, $[\mathbb{A}](X \to_{\mathrm{fs}} Y)$ is isomorphic to $([\mathbb{A}]X) \to_{\mathrm{fs}} ([\mathbb{A}]Y)$.

Exercise 12. Show that a subset S of the nominal set $\mathbb{A}$ is finitely supported iff it is either finite or cofinite (that is, its complement $\mathbb{A} - S$ is finite).

Exercise 13. Suppose $\varphi(a)$ and $\varphi'(a)$ are properties of atomic names $a \in \mathbb{A}$ whose extensions $\{a \mid \varphi(a)\}$ and $\{a \mid \varphi'(a)\}$ give finitely supported subsets of $\mathbb{A}$ (see slide 21). Writing $(\text{И}a)\, \varphi(a)$ to indicate that $\{a \mid \varphi(a)\}$ is a cofinite set of atoms (that is, its complement in $\mathbb{A}$ is finite), show that this 'freshness quantifier' has the following properties:

(i) $\neg(\text{И}a)\,\varphi(a) \;\Leftrightarrow\; (\text{И}a)\,\neg\varphi(a)$.

(ii) $((\text{И}a)\,\varphi(a) \wedge (\text{И}a)\,\varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\,(\varphi(a) \wedge \varphi'(a))$.

(iii) $((\text{И}a)\,\varphi(a) \vee (\text{И}a)\,\varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\,(\varphi(a) \vee \varphi'(a))$.

(iv) $((\text{И}a)\,\varphi(a) \Rightarrow (\text{И}a)\,\varphi'(a)) \;\Leftrightarrow\; (\text{И}a)\,(\varphi(a) \Rightarrow \varphi'(a))$.

If $X \in \mathbf{Nom}$ and $\varphi(a, x)$ determines a finitely supported subset of $\mathbb{A} \times X$, what in general is the relationship between $(\exists x \in X)(\text{И}a)\,\varphi(a, x)$ and $(\text{И}a)(\exists x \in X)\,\varphi(a, x)$? And between $(\forall x \in X)(\text{И}a)\,\varphi(a, x)$ and $(\text{И}a)(\forall x \in X)\,\varphi(a, x)$?

Exercise 14. Use the α-structural recursion theorem for λ-terms from slide 38 to prove the following *α-structural induction principle* for the nominal set Λ of λ-terms modulo α-equivalence: if $P \in \mathrm{P}_{\mathrm{fs}}\Lambda$ satisfies

$$
\begin{aligned}
&(\forall a \in \mathbb{A})\ a \in P \\
&\wedge\ (\forall e_1, e_2 \in \Lambda)\ e_1 \in P \wedge e_2 \in P \Rightarrow e_1\, e_2 \in P \\
&\wedge\ (\text{И}a)(\forall e \in \Lambda)\ e \in P \Rightarrow \lambda a.\, e \in P
\end{aligned}
$$

then $(\forall e \in \Lambda)\ e \in P$. [Hint: for any nominal set X, $\mathrm{P}_{\mathrm{fs}}X$ is isomorphic to $X \to_{\mathrm{fs}} 2$; so we can apply the recursion principle to functions from Λ to 2.]