# Emergence of Scale-free Spike Flow Graphs in Recurrent Neural Networks

Filip Piekniewski and Tomasz Schreiber
Nicolaus Copernicus University, Poland

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  - Are biological neural networks scale-free?
- Motivations to unify scale-free concept and NN
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# Introduction

- Random graph theory has gathered a lot of attention due to development of scale-free network concept
- A.-L. Barabasi and R.Albert "Emergence of scaling in random networks", Science, no 393, pp. 509-512, October 1999
- R. Albert and A.-L. Barabasi, "Statistical Mechanics of complex networks", Reviews of modern physics, vol 74, p 47, 2002.
- Many other papers ...

# Scale-free graphs

- For a graph *G=(V, E)* we define the degree of vertex v as the number of vertices adjacent to *v* with respect to edge set *E*
- We define in-degree and out-degree in a directed graph similarly as the number of incoming and outgoing edges
- For any graph we may analyze its degree distribution which is a histogram of vertex degrees
- In a truly random graphs (Erdos-Renyi model) the degree distribution follows binomial distribution which approaches Poisson distribution for large graphs

# Scale-free graphs

- For a broad class of real-world network the degree distribution does not resemble Poisson distribution
- The distribution follows a power law $k^{-\gamma}$, thus the system has the same statistical properties independent of the scale
- These examples include
  - the World Wide Web and the Internet
  - science collaboration network and citation network
  - cellular metabolic network
  - many others ...

# Scale-free graphs

- The discovery had put in question the validity of models of these networks based on Erdos-Renyi random graphs
- For long it was not obvious what organising principles and mechanisms are responsible for such a phenomenon
- A.-L. Barabasi and R. Albert in 1999 had proposed a model that led to a construction of a scale-free network, based on two principles:
  - **Preferential attachment**
  - **Model growth**
- Both these principles are to some extent present in nervous systems...

# Previous results in context of Neural Networks

- It was natural to ask whether NN also have scale-free property?

- Some of the known neural topologies have been studied in context of scale-free property - the result was negative

  C. Koch and G.Laurent "Complexity of Nervous System", Science, vol 284, no. 5411, pp. 96-98, 1999;

  L.A. Amaral, A.Scala, M. Barthelemy and H.E. Stanley "Classes of small world networks", Proc Natl Acad Sci USA, vol 97, no 21,2000

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Why?

# Motivation

- The motivation of this research is to link recent results in random graph theory to neuroscience
- Perhaps finding certain structures (organizing principles) will help us find the complexity level at which the essential computation is done
- With relatively good models of single neurons, maybe its time to pay more attention to connectivity?

# The spike flow concept

- Weighted fully connected graph G
- Weights are drawn independently from normal distribution *N(0,1)* and remain fixed in course of the dynamics
- Each vertex v incorporates a non negative integer which refers to a number of potential available in v
- The network is equipped with a Hamiltonian:

$$\mathcal{H}(\bar{\sigma}) := \frac{1}{2} \sum_{i \neq j} w_{ij} |\sigma_i - \sigma_j|$$

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Positive $w_{ij}$ favours agreement of i and j, whereas negative $w_{ij}$ favours disagreement

$$\mathcal{H}(\bar{\sigma}) := \frac{1}{2} \sum_{i \neq j} w_{ij} |\sigma_i - \sigma_j|$$

# Dynamics

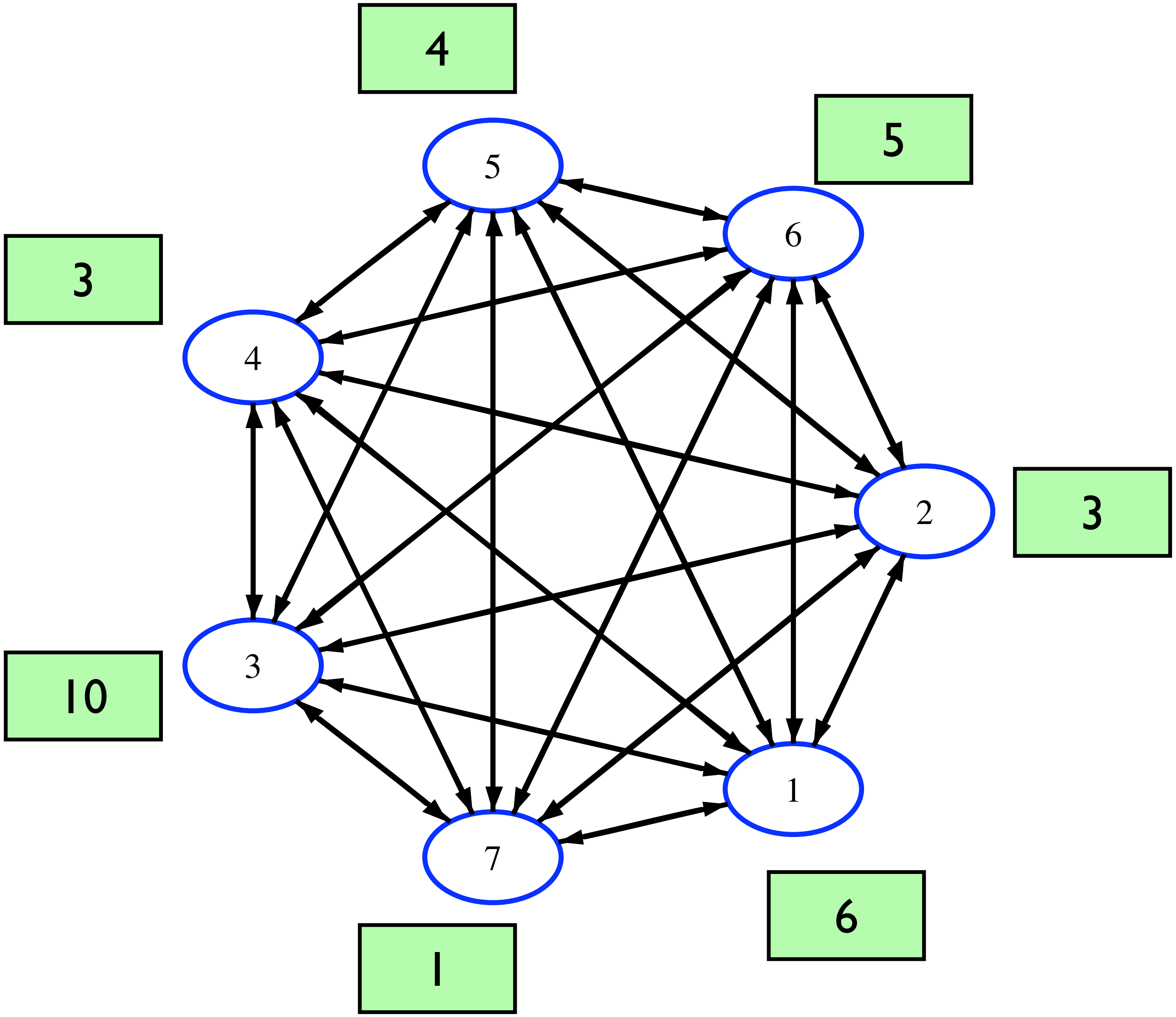
- At each step a pair $(\sigma_i, \sigma_j), i \neq j$ of units is chosen randomly
- Denote $\bar{\sigma}^*$ a configuration resulting from the original configuration by decreasing $\sigma_i$ by one and increasing $\sigma_j$ (letting one unit of potential transfer from $\sigma_i$ to $\sigma_j$ whenever

$$\sigma_i \geq 0$$

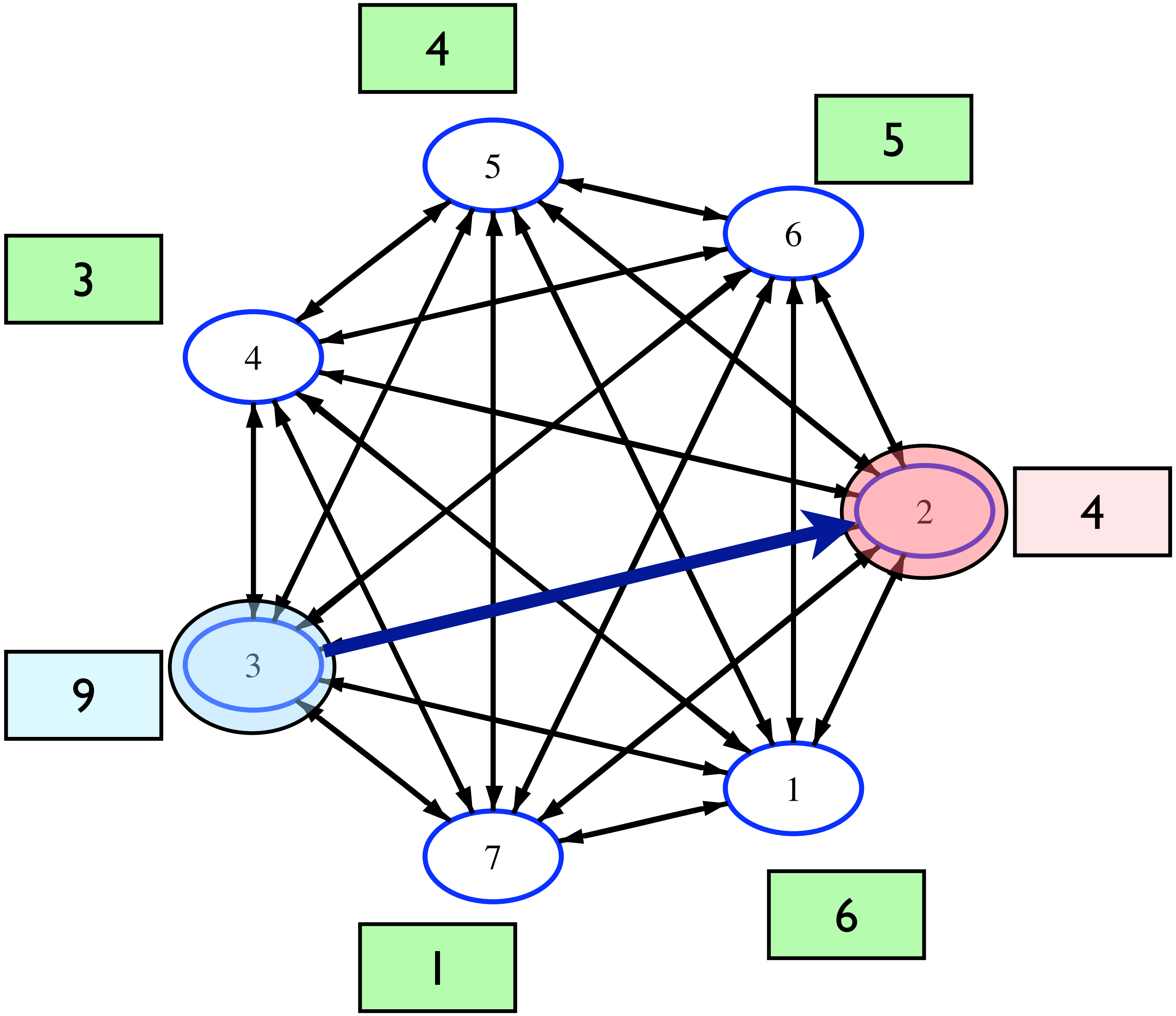
- If $\mathcal{H}(\bar{\sigma}^*) \leq \mathcal{H}(\bar{\sigma})$ the new configuration is accepted, otherwise we accept the new configuration with probability:

$$\exp(-\beta[\mathcal{H}(\bar{\sigma}^*) - \mathcal{H}(\bar{\sigma})])$$

# Dynamics



# Dynamics



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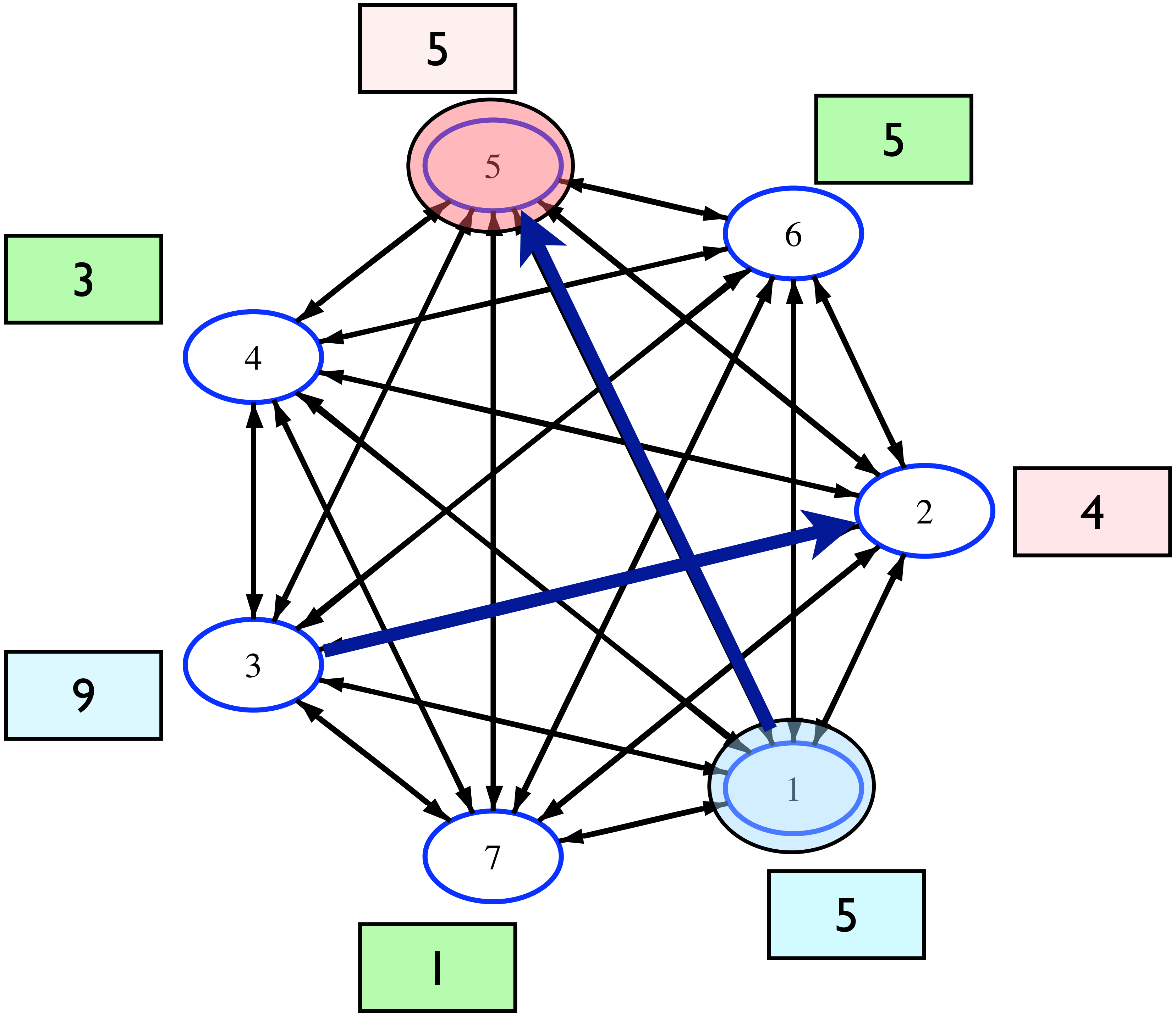

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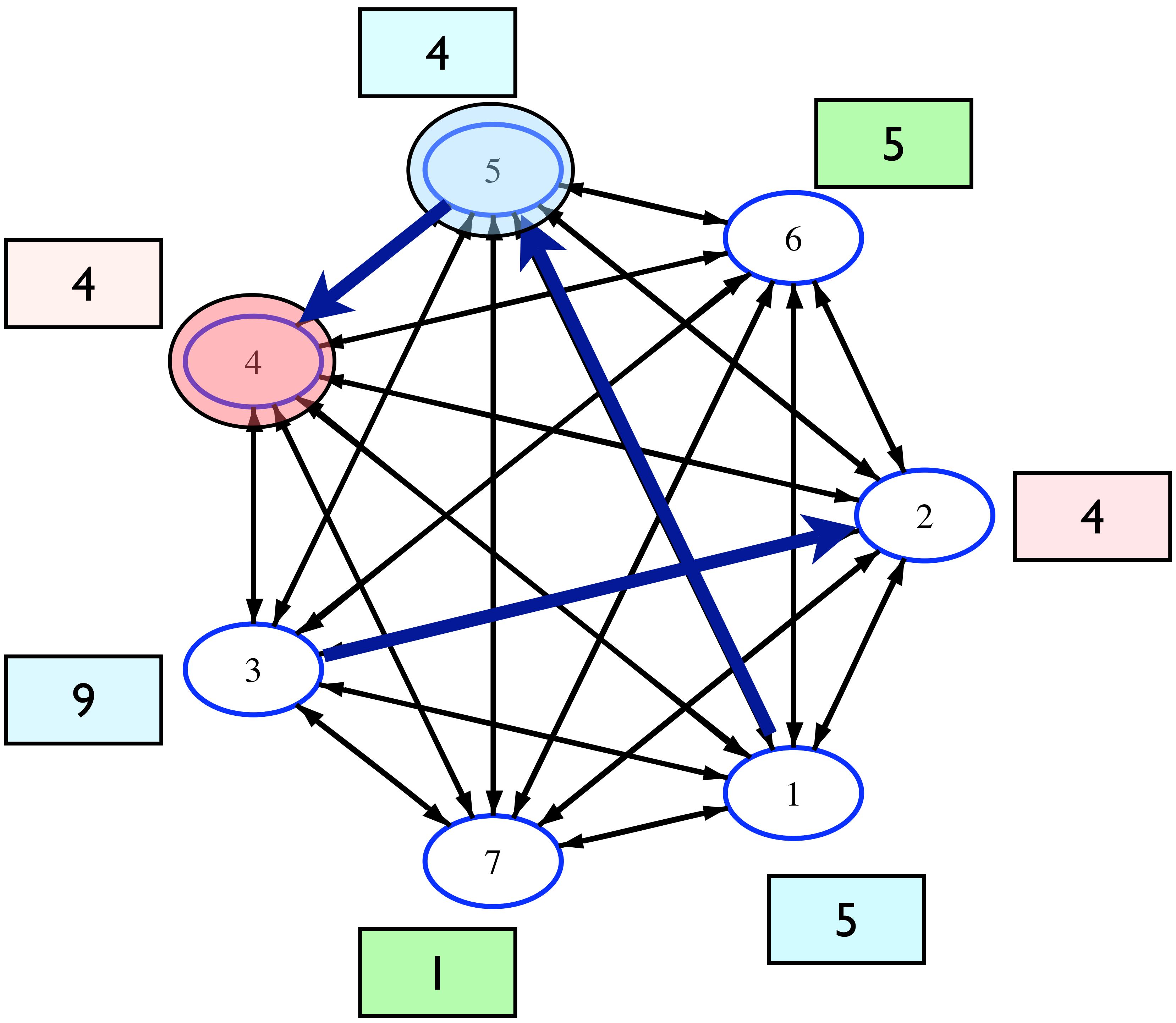

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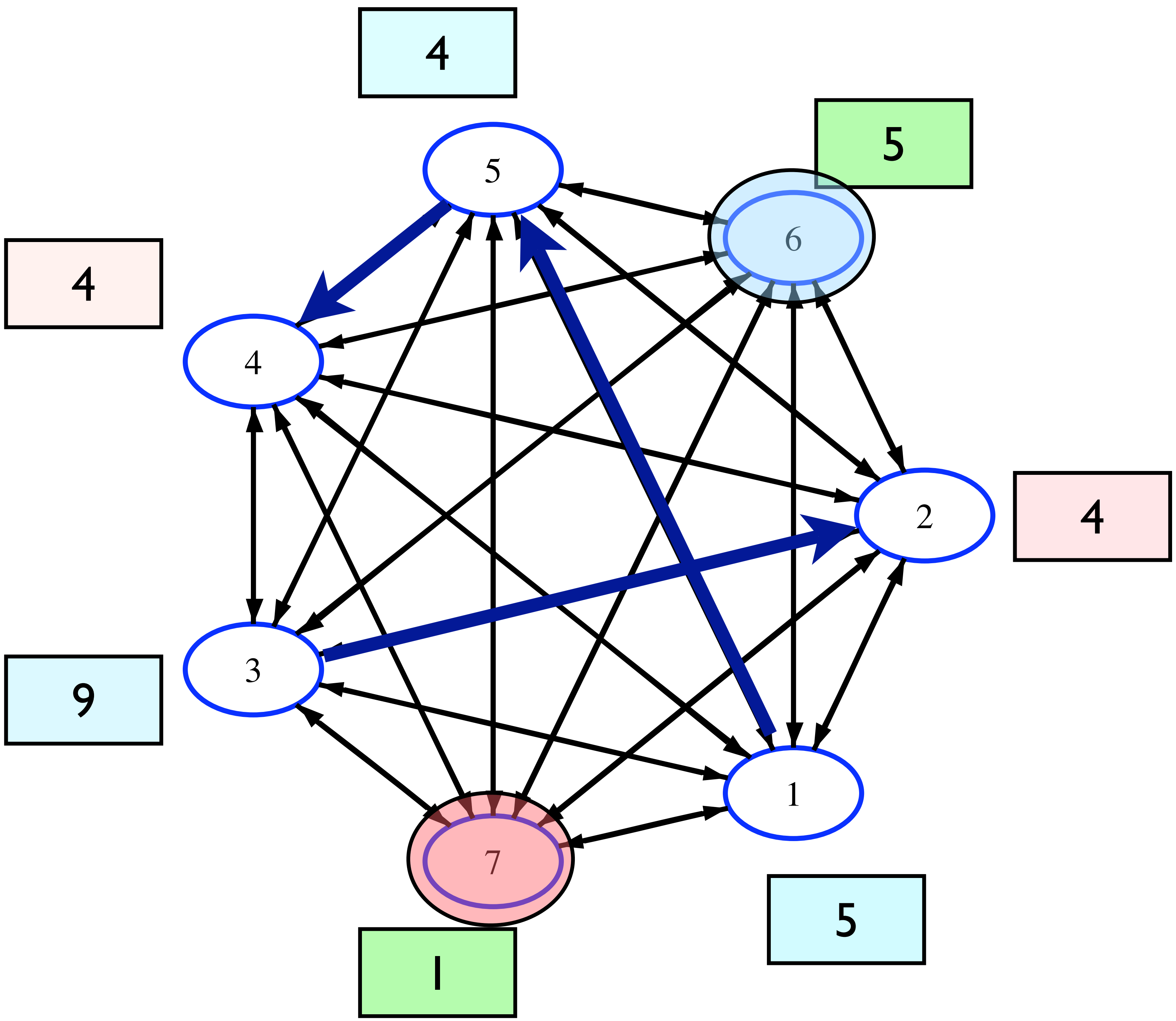

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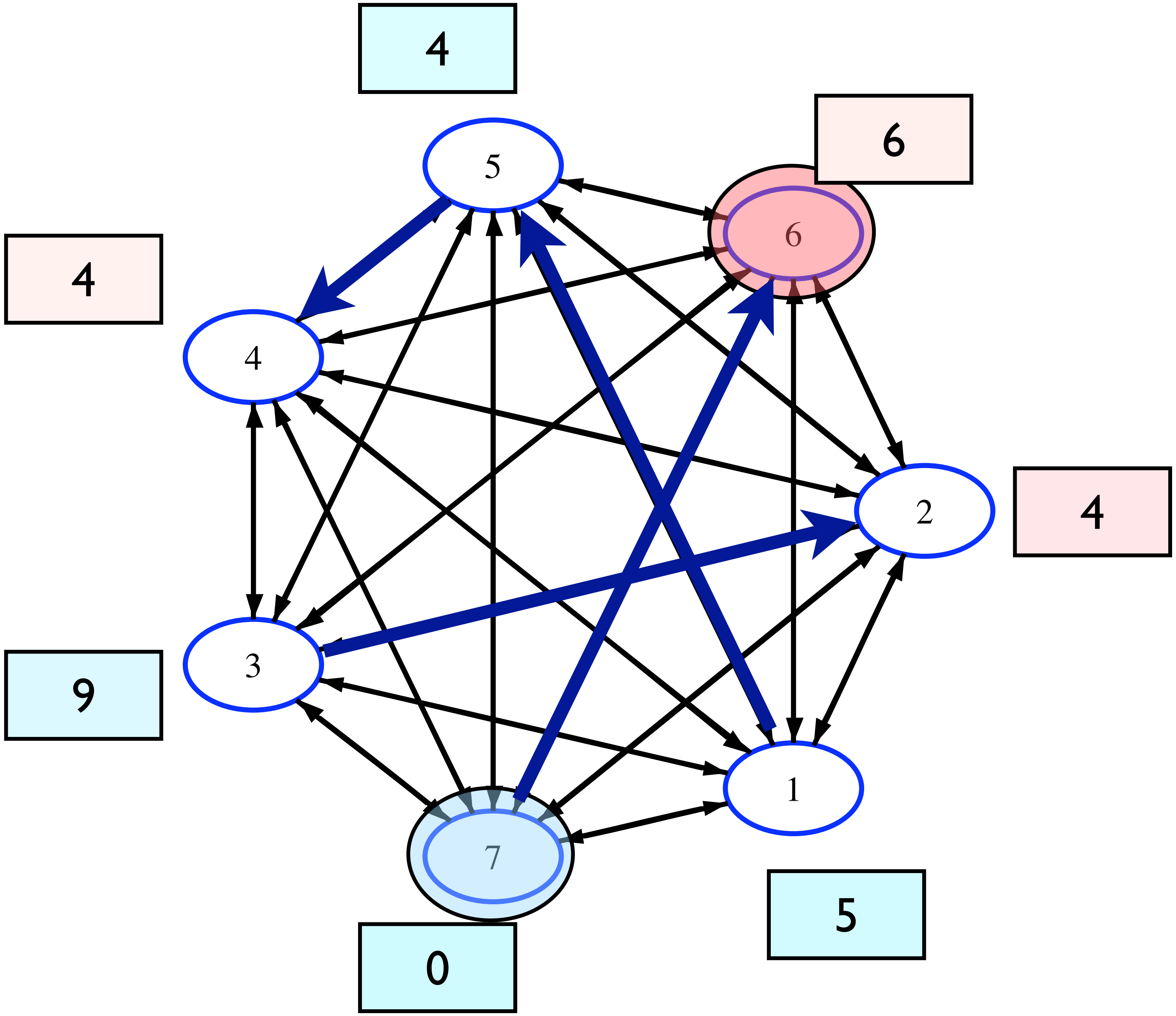

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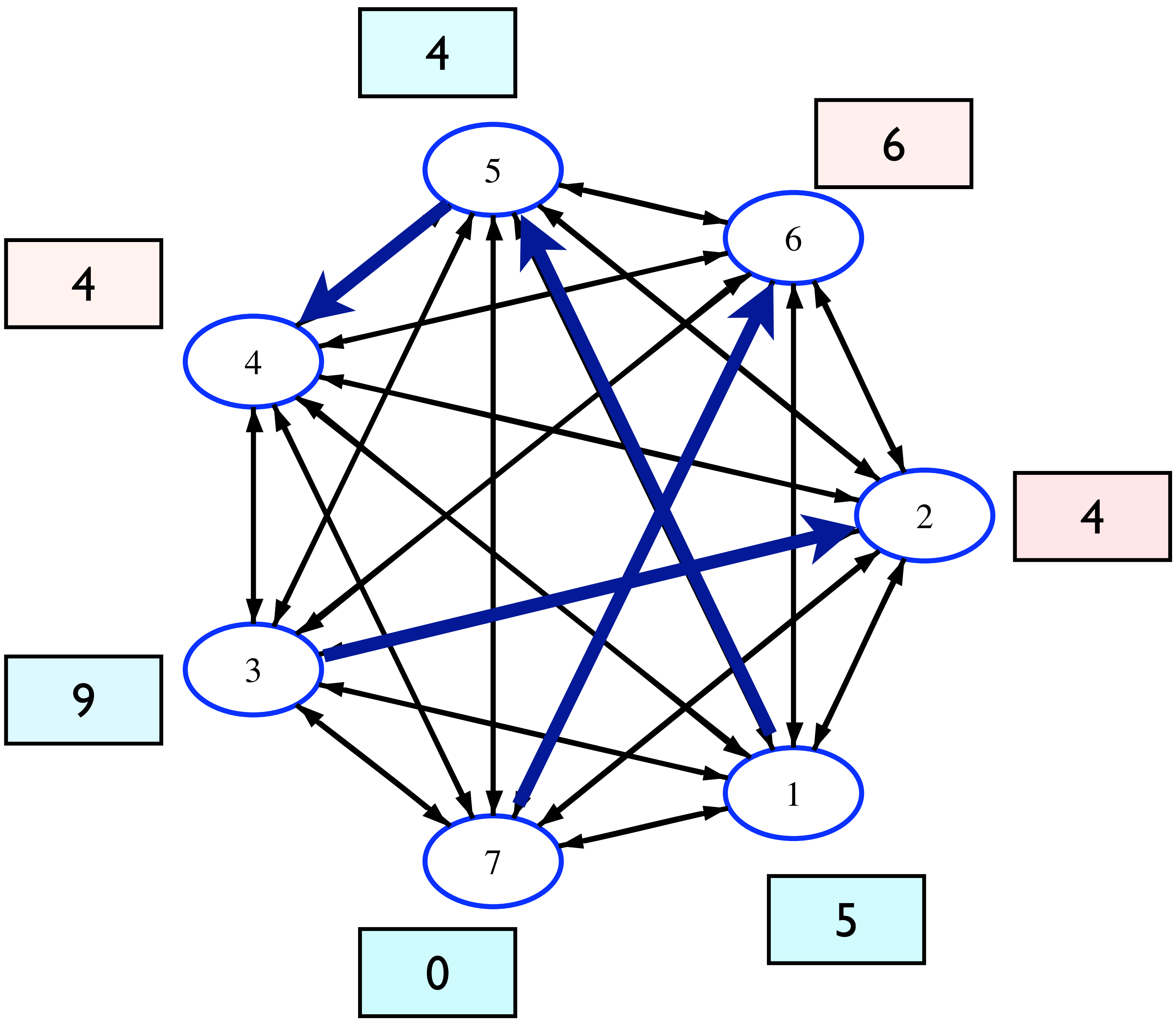


# Spike flow model

- Spike flow model is similar to a Boltzmann Machine, it is also possible to encode combinatorial problems within this framework
- Since weights are chosen from normal distribution, the number of negative and positive weights is approximately equal
- The energy minimum is therefore highly non trivial
- Each edge used to exchange potential is labelled, by

$$F_{i \to j}$$

we denote the number of times the edge *(i, j)* was used

# Spike flow model

- Total amount of "potential" is preserved under the dynamics
- All states having a fixed amount of potential are reachable, therefore the system forms an irreducible and aperiodic Markow chain
- It is easy to see, that system stationary distribution has the following form:

$$\mathbb{P}_n(\bar{\sigma}) = \begin{cases} \frac{\exp(-\beta\mathcal{H}(\bar{\sigma}))}{\sum_{\bar{\sigma}',\ \sum_i \sigma'_i = n} \exp(-\beta\mathcal{H}(\bar{\sigma}'))}, & \text{if } \sum_i \sigma_i = n, \\ 0, & \text{otherwise} \end{cases}$$

# Spike flow model

- Note that for T large enough that the system gets close to its equilibrium and that each edge is processed a statistically significant number of times we have:

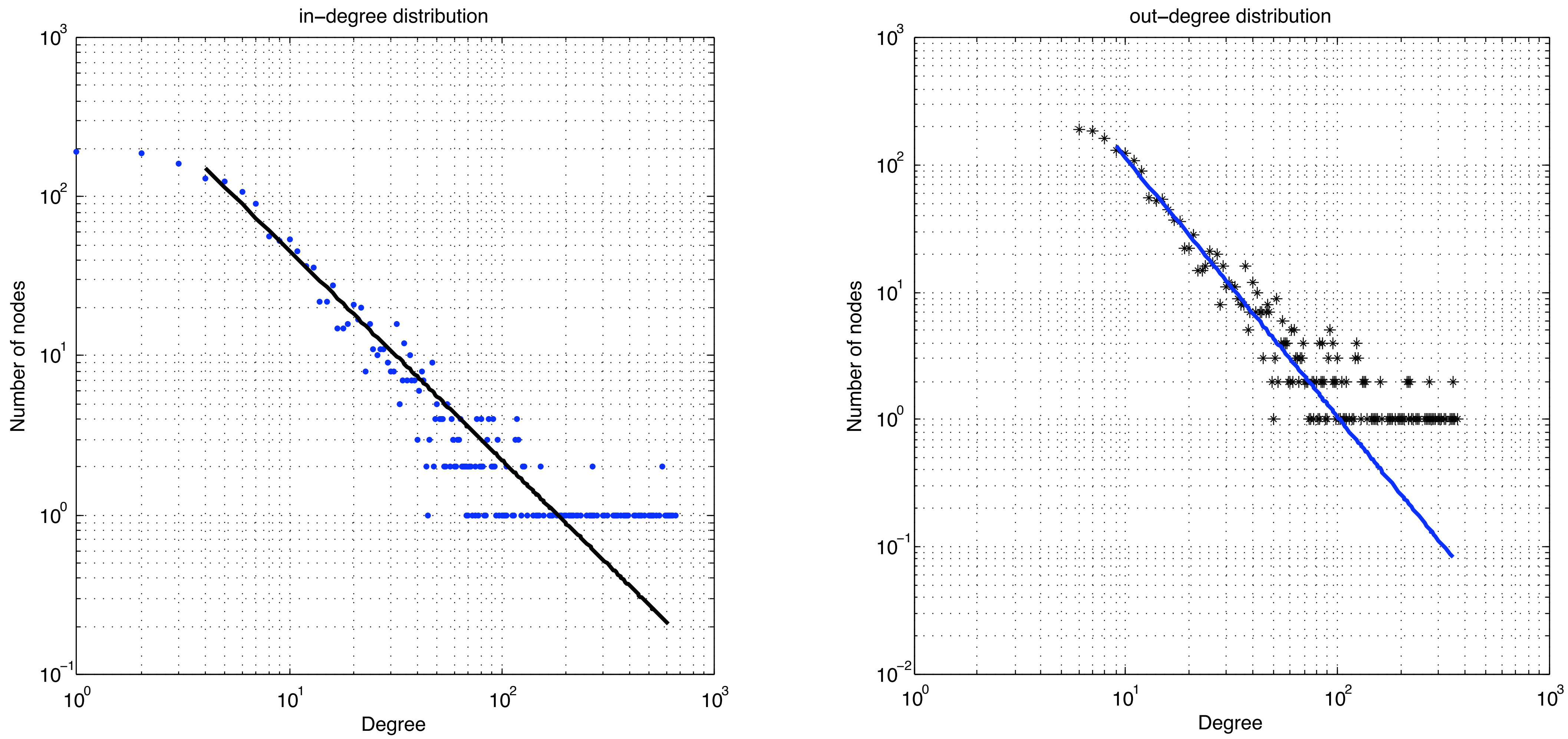
$$F_{i\to j}/T \approx \frac{1}{N(N-1)}\mathbb{E}_n \pi_{i\to j}(\bar{\sigma})$$

- With $\mathbb{E}_n$ standing for expectation operator under distribution $\mathbb{P}_n$ and $\pi_{i\to j}(\bar{\sigma})$ for a transition probability from i to j given the state $\bar{\sigma}$ .
- In particular $F_{i\to j}/T$ becomes a deterministic function of the network and n for large T.

# Simulation setup

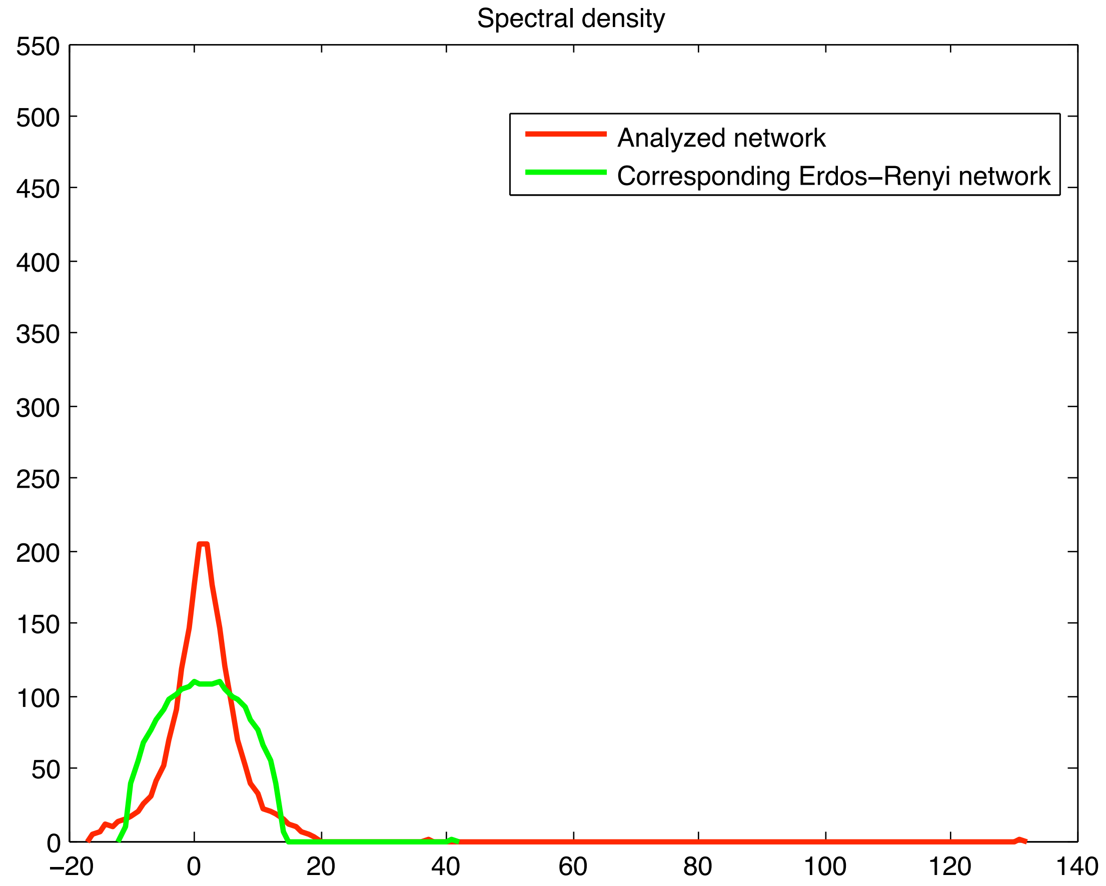
- ~ 2000 nodes
- low temperature regime (mostly energy driven transitions)
- ~ 10*2000*2000 steps
- about 10 runs for every instance to avoid random fluctuations
- Initially the potential is distributed uniformly, every node holds 5 units of "potential"

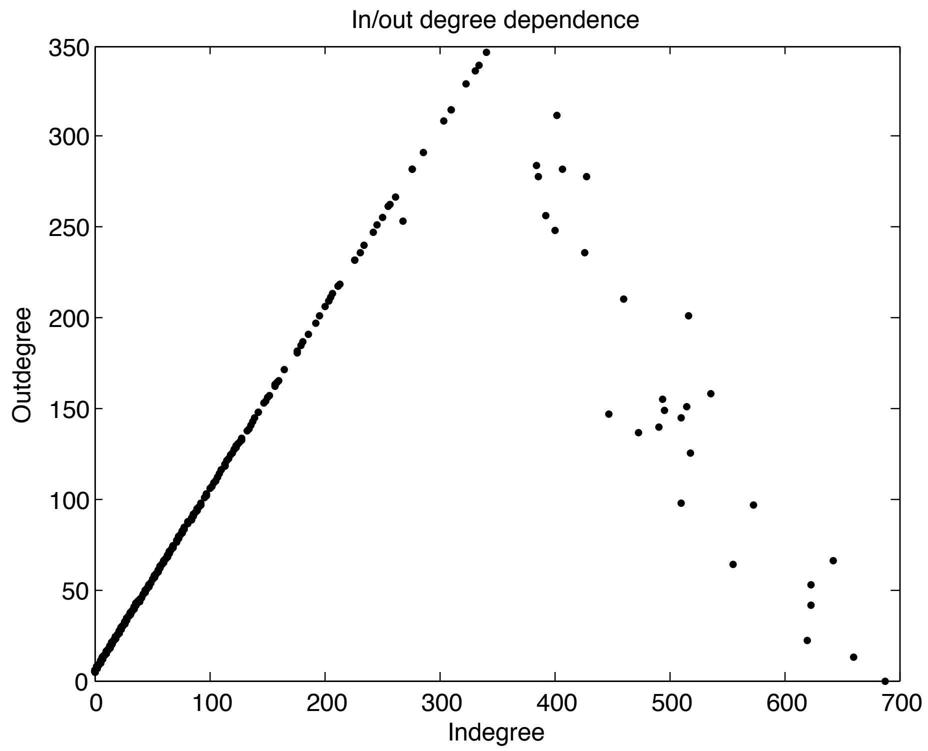
# Results - degree distribution

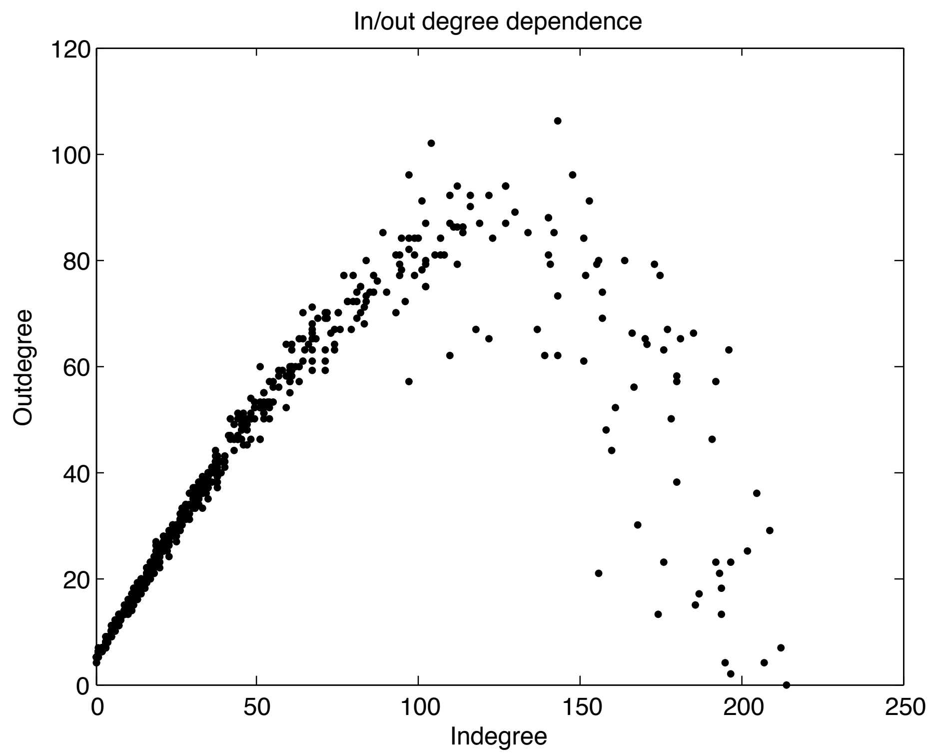



$\gamma_{in} \approx 1.3$ $\gamma_{out} \approx 2.0$

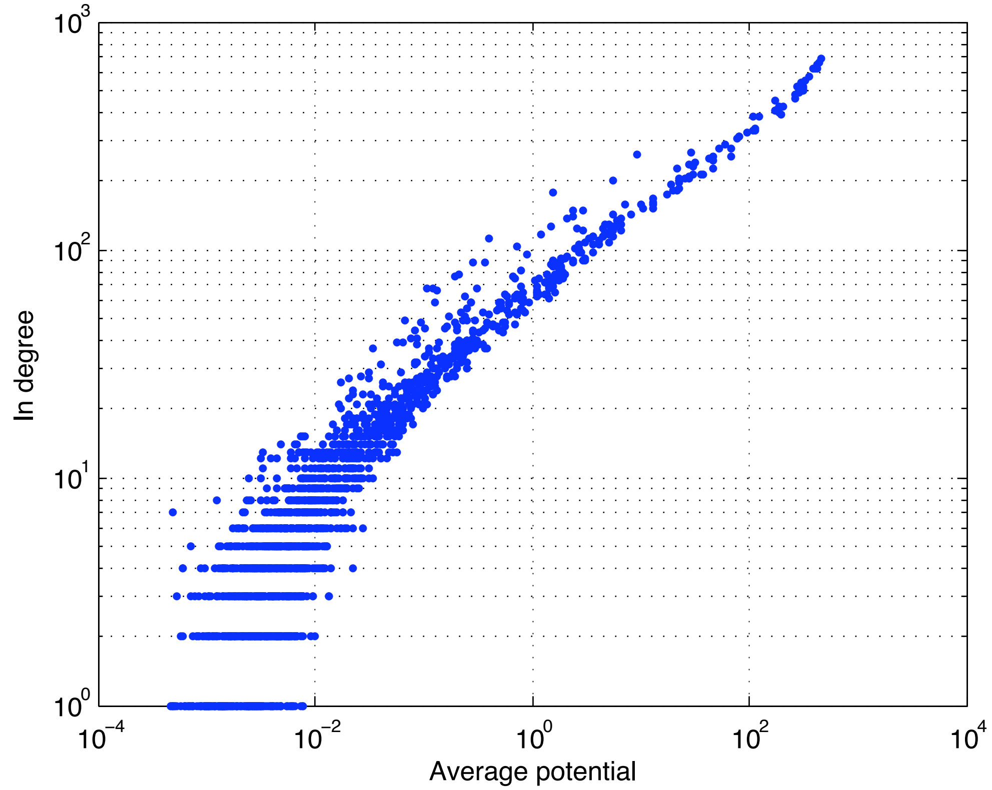
# Results in/out degree

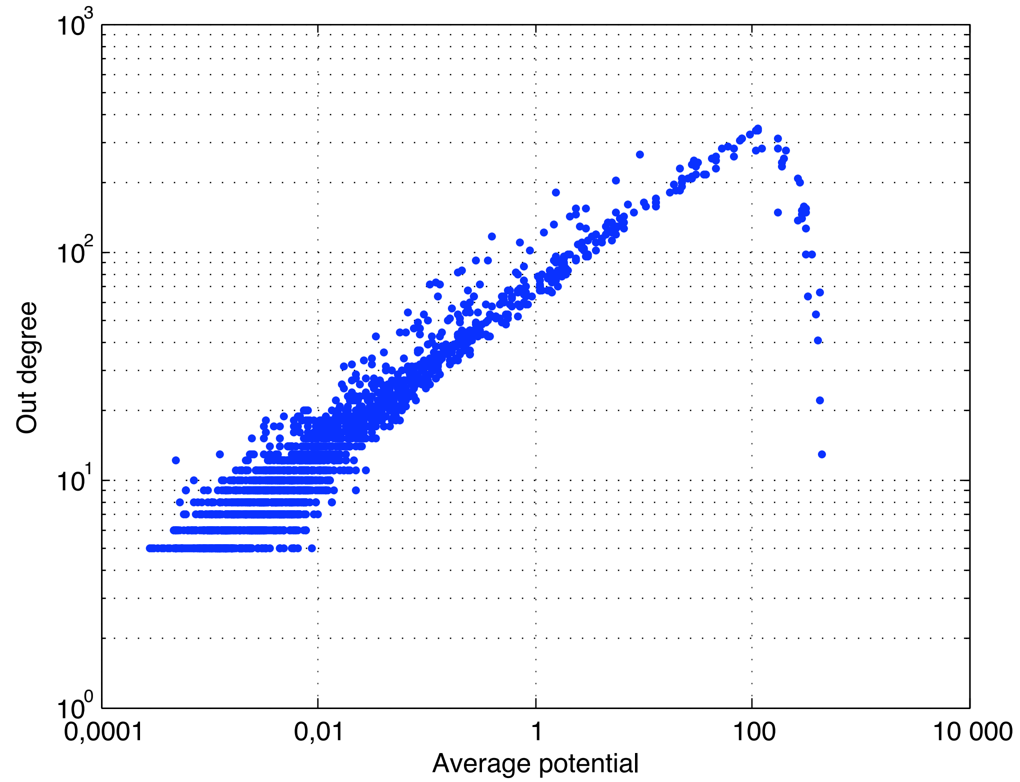


# Results - average potential vs. degree



# Results - clustering

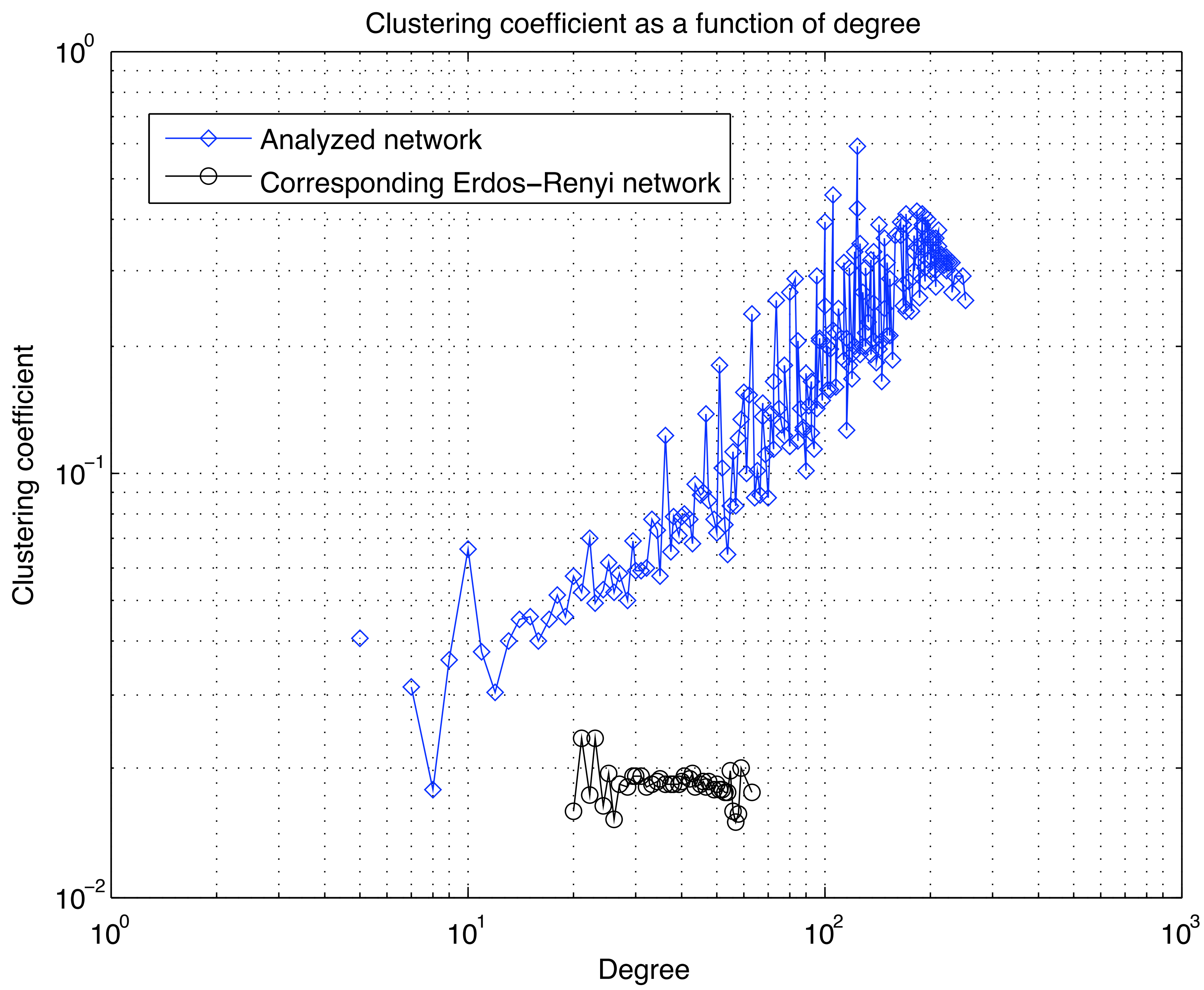
- Clustering coefficient:

$$C_i = \frac{2|\{e_{j,k}\}|}{d_i(d_i - 1)}$$

- Alternative definition:

$$C_i = \frac{2\lambda_G(i)}{d_i(d_i - 1)}$$

# Results - clustering



# Results

- The graph structure is clearly scale-free
- This is a result of a preferential attachment - **the more potential unit gains, the more likely it will exchange it with its neighbours**
- The model to some extent resembles activity of SNN, perhaps on the level of neuronal groups, not on the level of single neurons*
- The model doesn't grow, but there are multiple edges, which prevent the model from saturation

# Conclusions

- A simple model unifying scale-free graph theory and neural networks has been introduced
- The model bears a lot of common features with Boltzmann Machine,
- The differences include unbounded state space, and a fact that at each step exactly two units are affected
- Numerical simulations support the expectation of a scale-free nature of the graph

# Further research

- The preferential attachment in spike flow model results from a state memory (amount of potential), of a single unit. We claim that single neurons do not posses enough of such “memory”, which explains empirical results on C. elegans worm.
- * Some interesting results were obtained for Spiking Neural Network, see our paper *“Emergence of Scale-Free Graphs in Dynamical Spiking Neural Networks”*, Accepted to 2007 International Joint Conference on Neural Networks, Orlando 2007
- Spike flow model itself may be used to encode combinatorial problems

# Questions

# Questions

Mahalo
and
Aloha!