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SHORT NOTE

Ramsey's theorem for colors from a metric space

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Abstract The Ramsey theorem says that for any countably infinite undirected clique whose edges are colored by a finite number of colors, there is an infinite subclique whose edges are colored by a single color. In this note, we generalize the theorem to a situation where the colors form a compact metric space.

Keywords Ramsey theorem · Metric compact space · Compact semigroups

By $[A]^2$ we denote the family of size two subsets of A, interpreted as edges in the undirected graph with nodes A. If K is a metric space and A is a set, we say that $k \in K$ is the limit of a function $f : [A]^2 \to K$ if for any $\epsilon > 0$, there is some finite subset $C \subseteq A$ such that the image of f, when restricted to $[A - C]^2$, is contained in the ball of radius ϵ and center k.

Theorem 1 *Let K be a metric compact space, B a countably infinite set, and $f : [B]^2 \to K$. Then there exists a $k \in K$ and an infinite subset $A \subseteq B$ such that the restriction of f to $[A]^2$ has limit k.*

Ramsey's theorem is a special case of the above theorem, when K is equipped with a discrete metric. The proof is a simple adaptation of the proof of Ramsey's theorem.

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Proof Without loss of generality we assume that $B = \omega$. By induction, we will construct a sequence of sequences $a^0, a^1, a^2, \ldots \in B^\omega$, such that a^n is a subsequence of a^{n-1}. We write a^n_m for the m-the element of the n-th sequence a^n.

Let $a^0_m = m$. For $n > 0$, let a^n be a subsequence of a^{n-1} such that $a^n_m = a^{n-1}_m$ for $m \le n$, and the sequence

$$f(\{a^n_n, a^n_n\}), f(\{a^n_n, a^n_{n+1}\}), f(\{a^n_n, a^n_{n+2}\}), \ldots$$

is convergent to some k_n. This subsequence exists because K is compact. Moreover, we can choose the sequence a^n so that for all $m > n$ the distance from $f(\{a^n_n, a^n_m\})$ to k_n is at most $1/n$.

Let (q_n) be a sequence of numbers such that k_{q_n} is convergent to some k and moreover, $\delta(k_{q_n}, k) \le 1/n$. Define A as $\{a^{q_n}_{q_n} : n \in \omega\}$.

Let $n < m$. Since a^{q_m} is a subsequence of a^{q_n}, we have that $a^{q_m}_{q_m} = a^{q_n}_z$ for some $z > q_m > q_n > n$.

$$\begin{aligned}
\delta(f(\{a^{q_n}_{q_n}, a^{q_m}_{q_m}\}), k) &\le \delta(f(\{a^{q_n}_{q_n}, a^{q_m}_{q_m}\}), k_{q_n}) + \delta(k_{q_n}, k) \\
&\le \delta(f(\{a^{q_n}_{q_n}, a^{q_m}_{q_m}\}), k_{q_n}) + 1/n \\
&= \delta(f(\{a^{q_n}_{q_n}, a^{q_n}_z)\}, k_{q_n}) + 1/n \\
&\le 1/n + 1/n = 2/n
\end{aligned}$$

Since $\delta(f(\{a^{q_n}_{q_n}, a^{q_m}_{q_m}\}), k) < 2/n$, the set A satisfies our claim. □

The theorem assumes that the space K is metric. We show a space K which is compact but not metric, and where the statement of the theorem fails. Let K be any compact space where not every sequence has a convergent subsequence. For instance, K can be an uncountable product of unit intervals. Take some sequence $a_1, a_2, \ldots$ in K which does not have any convergent subsequence, and define $f : [\omega]^2 \to K$ by $f(\{i, j\}) = a_i$. The theorem, when applied to this function f, would imply that there is a converging subsequence of $a_1, a_2, \ldots$.

By induction on k, one can generalize the theorem to functions f defined on k-element subsets.

In the following corollary, we see what happens when K has a semigroup structure. If $x \in K^\omega$ is a sequence of elements from a semigroup K, we write $x[i..j)$ for the multiplication of $x_i \cdots x_{j-1}$.

Corollary 1 *If in Theorem 1, the space K has a semigroup structure with continuous multiplication, $B = \omega$, and f is obtained from a sequence $x \in K^\omega$ by setting $f(\{i, j\}) = x[i..j)$, then the limit k is idempotent, i.e. $k = k \cdot k$.*

Proof Apply Theorem 1, yielding a set $A = \{u_1 < u_2 < \ldots\} \subseteq \omega$. If the action is continuous, then k is idempotent, since

$$k \cdot k = \lim_{n\to\infty} x\Big|^{u_n}_{u_{n+1}} \cdot \lim_{n\to\infty} x\Big|^{u_{n+1}}_{u_{n+2}} = \lim_{n\to\infty} x\Big|^{u_n}_{u_{n+1}} \cdot x\Big|^{u_{n+1}}_{u_{n+2}} = \lim_{n\to\infty} x\Big|^{u_n}_{u_{n+2}} = k. \quad \square$$

In some papers, e.g. [1], a *compact semigroup* is a semigroup S with compact topology such that the mapping $t \mapsto s \cdot t$ is required to be continuous for each $s \in S$.

This assumption is weaker than our assumption from Corollary 1 that the action in S is continuous.

However, the weaker assumption is not sufficient for idempotence of s in Corollary 1. Indeed, consider the semigroup $S = \{0, 1, 2, \ldots, \omega, \omega + 1\}$, with the action $a \oplus b = \min(a + b, \omega + 1)$, and the distance $\delta(a, b) = |f(a) - f(b)|$, where $f(n) = 1/(n + 1)$, $f(\omega) = 0$, $f(\omega + 1) = -1$. This semigroup is a compact semigroup in the meaning from [1]. Now, let $x_n = n$. If we apply Corollary 1 to x, s has to be ω, but ω is not idempotent: $\omega \oplus \omega = \omega + 1$.

References

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