47-th Mathematical Olympiad in Poland

Third Round, March 29–30, 1996

First Day

1. Determine all pairs (n, r), where n is a positive integer, and r is real, for which the polynomial $(x+1)^n - r$ is divisible by $2x^2 + 2x + 1$.

2. Inside given triangle ABC there is chosen point P satisfying the conditions:

$$\angle PBC = \angle PCA < \angle PAB\,.$$

The line BP intersects the circumcircle of triangle ABC at points B and E. The circumcircle of triangle APE intersects the line CE at points E and F. Prove that the points A, P, E, F are the consecutive vertices of a quadrilateral and that the ratio of the areas of the quadrilateral $APEF$ and the triangle ABP does not depend on the choice of the point P.

3. Given an integer $n \geq 2$ and positive numbers $a_1, a_2, \ldots, a_n$ with the sum equal to 1.

(a) Prove that for any positive numbers $x_1, x_2, \ldots, x_n$ with the sum equal to 1, holds the following inequality:

$$2\sum_{i<j} x_i x_j \leq \frac{n-2}{n-1} + \sum_{i=1}^{n} \frac{a_i x_i^2}{1-a_i}\,.$$

(b) Determine all numbers $x_1, x_2, \ldots, x_n$ for which the above inequality turns into the equality.

Second Day

4. In a tetrahedron $ABCD$ hold the following equalities:

$$\angle BAC = \angle ACD \qquad \text{and} \qquad \angle ABD = \angle BDC\,.$$

Prove that the edges AB and CD have the same length.

5. For a natural number $k \geq 1$ denote by $p(k)$ the least prime number which is not a divisor of k. If $p(k) > 2$, then we define $q(k)$ to be the product of all primes less than $p(k)$; if $p(k) = 2$, we put $q(k) = 1$. Define the sequence (x_n) by the formulas

$$x_0 = 1, \qquad x_{n+1} = \frac{x_n p(x_n)}{q(x_n)} \quad \text{for} \quad n = 0, 1, 2, \ldots\,.$$

Determine all positive integers n with $x_n = 111\,111$.

6. From a collection of all permutations f of the set $\{1, 2, \ldots, n\}$ satisfying the condition

$$f(i) \geq i - 1 \qquad \text{for} \quad i = 1, 2, \ldots, n$$

we choose one at random. Let p_n be the probability that the chosen permutation satisfies

$$f(i) \leq i + 1 \qquad \text{for} \quad i = 1, 2, \ldots, n\,.$$

Determine all positive integers n with $p_n > 1/3$.