44-th Mathematical Olympiad in Poland

Final Round, Warszawa, April 4–5, 1993

First Day

1. Solve in rational numbers t, w, x, y, z the following system of equations

$$\begin{cases} 2xy = t^2 - w^2 + z^2 \\ 2xz = t^2 - y^2 + w^2 \\ 2yz = t^2 - w^2 + x^2 . \end{cases}$$

2. The point O is a center of a circle k which is inscribed in a nonisosceles trapezoid $ABCD$ whose long base AB has the midpoint M. The short base CD is tangent to the circle k at point E. The line OM intersects the base CD at point F. Prove that $|DE| = |FC|$ if and only if $|AB| = 2|CD|$.

3. Denote by $g(k)$ the greatest odd divisor of the positive integer k and set

$$f(k) = \begin{cases} \dfrac{k}{2} + \dfrac{k}{g(k)} & \text{for an even } k, \\ 2^{(k+1)/2} & \text{for an odd } k. \end{cases}$$

The sequence (x_n) is defined by $x_1 = 1$, $x_{n+1} = f(x_n)$. Prove that the number 800 appears in this sequence exactly once. Determine the number n, for which $x_n = 800$.

Second Day

4. Given is a convex polyhedron whose all faces are triangular. We colour the vertices of this polyhedron using three colours. Prove that the number of faces having the vertices of all the three colours is even.

5. Determine all functions $f\colon \mathbf{R} \to \mathbf{R}$ satisfying the following conditions

$$f(-x) = -f(x), \quad f(x+1) = f(x) + 1 \quad \text{for } x \in \mathbf{R},$$
$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2} \quad \text{for } x \neq 0.$$

6. Find out, whether one can compute the volume of a tetrahedron knowing the areas of its faces and the radius of the circumscribed sphere (i.e. whether the volume of a tetrahedron is a function of the areas of its faces and the circumradius).