53-rd Mathematical Olympiad in Poland

Final Round, April 3–4, 2002

First Day

1. Determine all positive integers a, b, c such that the numbers a^2+1 and b^2+1 are prime and the following equality

$$(a^2+1)(b^2+1) = c^2+1$$

holds.

2. In the outside of a triangle ABC two rectangles $ACPQ$ and $BKLC$ are constructed. Assuming that the areas of these rectangles are equal prove that the midpoint of the segment PL, the point C and the circumcenter of the triangle ABC are collinear.

3. On a blackboard three nonnegative integers are written. From these numbers two k, m are chosen and replaced by the numbers $k+m$ and $|k-m|$. The third number remains unchanged. With these new three numbers we proceed similar. The goal is to obtain at least two numbers equal to zero. Find out whether it is possible.

Second Day

4. Prove that for all positive integers $n \geq 3$ and for all positive real numbers $x_1, x_2, \ldots x_n$ at least one of the following inequalities

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}, \quad \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}$$

(where $x_{n+1} = x_1\,, x_{n+2} = x_2\,, x_0 = x_n\,, x_{-1} = x_{n-1}$) holds.

5. A sphere s and a plane π disjoint with s are given. On the plain π three, not collinear points A, B, C are chosen. Through each of these points a tangent line to s is constructed. The contact points of these lines with s are denoted by K, L, M, respectively. A point P lies on s and satisfies the equalities

$$\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}\,.$$

Prove that the circumsphere of the pyramid $ABCP$ is tangent to s.

6. A positive integer k is given. The sequence (a_n) is defined by

$$a_1 = k+1\,, \quad a_{n+1} = a_n^2 - ka_n + k \text{ for } n \geq 1\,.$$

Show that for $m \neq n$ the numbers a_m, a_n are relatively prime.