Problems for the Team Competition Baltic Way 1999

Reykjavik, November 6, 1999

1. Determine all real numbers a, b, c, d that satisfy the following equations

$$\begin{cases} abc + ab + bc + ca + a + b + c = 1 \\ bcd + bc + cd + db + b + c + d = 9 \\ cda + cd + da + ac + c + d + a = 9 \\ dab + da + ab + bd + d + a + b = 9\,. \end{cases}$$

2. Determine all positive integers n with the property that the third root of n is obtained by removing its last three decimal digits.

3. Determine all positive integers $n \geq 3$ such that the inequality

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1 \leq 0$$

holds for all real numbers $a_1, a_2, \ldots, a_n$ which satisfy $a_1 + a_2 + \ldots + a_n = 0$.

4. For all positive real numbers x and y let

$$f(x,y) = \min\left(x, \frac{y}{x^2+y^2}\right).$$

Show that there exist x_0 and y_0 such that $f(x,y) \leq f(x_0,y_0)$ for all positive x and y, and find $f(x_0,y_0)$.

5. The point (a,b) lies on the circle $x^2+y^2=1$. The tangent to the circle at this point meets the parabola $y = x^2+1$ at exactly one point. Find all such points (a,b).

6. What is the least number of moves it takes a knight to get from one corner of an $n \times n$ chessboard, where $n \geq 4$, to the diagonally opposite corner?

7. Two squares on an 8×8 chessboard are called adjacent if they have a common edge or common corner. Is it possible for a king to begin in some square and visit all squares exactly once in such a way that all moves except the first are made into squares adjacent to an even number of squares already visited?

8. We are given 1999 coins. No two coins have the same weight. A machine is provided which allows us with one operation to determine, for any three coins, which one has the middle weight. Prove that the coin that is the 1000th by weight can be determined using no more than 1000000 operations and that this is the only coin whose position by weight can be determined using this machine.

9. A cube with edge length 3 is divided into 27 unit cubes. The numbers $1, 2, \ldots, 27$ are distributed arbitrarily over the unit cubes, with one number in each cube. We form the 27 possible row sums (there are nine such sums of three integers for each of the three directions parallel with the edges of the cube). At most how many of the 27 row sums can be odd?

10. May the points of a disc of radius 1 (including its circumference) be partitioned into three subsets in such a way that no subset contains two points separated by a distance 1?

11. Prove that for any four points in the plane, no three of which are collinear, there exists a circle such that three of the four points are on the circumference and the fourth point is either on the circumference or inside the circle.

12. In a triangle ABC it is given that $2AB = AC + BC$. Prove that the incentre of ABC, the circumcenter of ABC, and the midpoints of AC and BC are concyclic.

13. The bisectors of the angles A and B of the triangle ABC meet the sides BC and CA at the points D and E, respectively. Assuming that $AE + BD = AB$, determine the angle C.

14. Let ABC be an isosceles triangle with $AB = AC$. Points D and E lie on the sides AB and AC, respectively. The line passing through B and parallel to AC meets the line DE at F. The line passing through C and parallel to AB meets the line DE at G. Prove that

$$\frac{[DBCG]}{[FBCE]} = \frac{AD}{AE},$$

where $[PQRS]$ denotes the area of the quadrilateral $PQRS$.

15. Let ABC be a triangle with $\angle C = 60°$ and $AC < BC$. The point D lies on the side BC and satisfies $BD = AC$. The side AC is extended to the point E where $AC = CE$. Prove that $AB = DE$.

16. Find the smallest positive integer k which is representable in the form $k = 19^n - 5^m$ for some positive integers m and n.

17. Does there exist a finite sequence of integers $c_1, c_2, \ldots, c_n$ such that all the numbers $a + c_1$, $a + c_2$, $\ldots$, $a + c_n$ are primes for more than one but not infinitely many different integers a?

18. Let m be a positive integer such that $m \equiv 2 \pmod 4$. Show that there exists at most one factorization $m = ab$ where a and b are positive integers satisfying

$$0 < a - b < \sqrt{5 + 4\sqrt{4m + 1}}\,.$$

19. Prove that there exist infinitely many even positive integers k such that for every prime p the number $p^2 + k$ is composite.

20. Let a, b, c and d be prime numbers such that $a > 3b > 6c > 12d$ and $a^2 - b^2 + c^2 - d^2 = 1749$. Determine all possible values of $a^2 + b^2 + c^2 + d^2$.