Problems for the Team Competition Baltic Way 2001

Hamburg, November 4, 2001

1. A set of 8 problems was prepared for an examination. Each student was given 3 of them. No two students received more than one common problem. What is the largest possible number of students?

2. Let $n \geq 2$ be a positive integer. Find whether there exist n pairwise nonintersecting nonempty subsets of $\{1, 2, 3, \ldots\}$ such that each positive integer can be expressed in a unique way as a sum of at most n integers, all from different subsets.

3. The numbers $1, 2, \ldots, 49$ are placed in a 7×7 array, and the sum of the numbers in each row and in each column is computed. Some of these 14 sums are odd while others are even. Let A denote the sum of all the odd sums and B the sum of all even sums. Is it possible that the numbers were placed in the array in such a way that $A = B$?

4. Let p and q be two different primes. Prove that

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \left\lfloor \frac{3p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor = \frac{1}{2}(p-1)(q-1) \quad .$$

(Here $\lfloor x \rfloor$ denotes the largest integer not greater than x.)

5. Let 2001 given points on a circle be colored either red or green. In one step all points are recolored simultaneously in the following way: If both direct neighbors of a point P have the same color as P, then the color of P remains unchanged, otherwise P obtains the other color. Starting with the first coloring F_1, we obtain the colorings $F_2, F_3, \ldots$ after several recoloring steps. Prove that there is a number $n_0 \leq 1000$ such that $F_{n_0} = F_{n_0+2}$. Is the assertion also true if 1000 is replaced by 999?

6. The points A, B, C, D, E lie on the circle c in this order and satisfy $AB \parallel EC$ and $AC \parallel ED$. The line tangent to the circle c at E meets the line AB at P. The lines BD and EC meet at Q. Prove that $|AC| = |PQ|$.

7. Given a parallelogram $ABCD$. A circle passing through A meets the line segments AB, AC and AD at inner points M, K, N, respectively. Prove that

$$|AB| \cdot |AM| + |AD| \cdot |AN| = |AK| \cdot |AC|.$$

8. Let $ABCD$ be a convex quadrilateral, and let N be the midpoint of BC. Suppose further that $\angle AND = 135^\circ$. Prove that $|AB| + |CD| + \dfrac{1}{\sqrt{2}} \cdot |BC| \geq |AD|$.

9. Given a rhombus $ABCD$, find the locus of the points P lying inside the rhombus and satisfying $\angle APD + \angle BPC = 180^\circ$.

10. In a triangle ABC, the bisector of $\angle BAC$ meets the side BC at the point D. Knowing that $|BD| \cdot |CD| = |AD|^2$ and $\angle ADB = 45^\circ$, determine the angles of triangle ABC.

11. The real-valued function f is defined for all positive integers. For any integers $a > 1$, $b > 1$ with $d = \gcd(a, b)$, we have

$$f(ab) = f(d)\left(f\left(\frac{a}{d}\right) + f\left(\frac{b}{d}\right)\right).$$

Determine all possible values of $f(2001)$.

12. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $\sum_{i=1}^{n} a_i^3 = 3$ and $\sum_{i=1}^{n} a_i^5 = 5$. Prove that $\sum_{i=1}^{n} a_i > 3/2$.

13. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_0 = 1$ and $a_n = a_{\lfloor 7n/9 \rfloor} + a_{\lfloor n/9 \rfloor}$ for $n = 1, 2, \ldots$. Prove that there exists a positive integer k with $a_k < \frac{k}{2001!}$.
(Here $\lfloor x \rfloor$ denotes the largest integer not greater than x.)

14. There are $2n$ cards. On each card some real number x, $1 \leq x \leq 2$, is written (there can be different numbers on different cards). Prove that the cards can be divided into two heaps with sums s_1 and s_2 so that $\frac{n}{n+1} \leq \frac{s_1}{s_2} \leq 1$.

15. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive real numbers satisfying $i \cdot a_i^2 \geq (i+1) \cdot a_{i-1} a_{i+1}$ for $i = 1, 2, \ldots$ Furthermore, let x and y be positive reals, and let $b_i = x a_i + y a_{i-1}$ for $i = 1, 2, \ldots$. Prove that the inequality $i \cdot b_i^2 > (i+1) \cdot b_{i-1} b_{i+1}$ holds for all integers $i \geq 2$.

16. Let f be a real-valued function defined on the positive integers satisfying the following condition: For all $n > 1$ there exists a prime divisor p of n such that $f(n) = f(n/p) - f(p)$. Given that $f(2001) = 1$, what is the value of $f(2002)$?

17. Let n be a positive integer. Prove that at least $2^{n-1} + n$ numbers can be chosen from the set $\{1, 2, 3, \ldots, 2^n\}$ such that for any two different chosen numbers x and y, $x + y$ is not a divisor of $x \cdot y$.

18. Let a be an odd integer. Prove that $a^{2^n} + 2^{2^n}$ and $a^{2^m} + 2^{2^m}$ are relatively prime for all positive integers n and m with $n \neq m$.

19. What is the smallest positive odd integer having the same number of positive divisors as 360?

20. From a sequence of integers (a, b, c, d) each of the sequences

$$(c, d, a, b), \quad (b, a, d, c), \quad (a + nc, b + nd, c, d), \quad (a + nb, b, c + nd, d)$$

for arbitrary integer n can be obtained by one step. Is it possible to obtain $(3, 4, 5, 7)$ from $(1, 2, 3, 4)$ through a sequence of such steps?