The 21-st Austrian–Polish Mathematics Competition
Poland, June 29 – July 1, 1998

1. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1^2 + x_2^2 \leq 1$. Prove the inequality
$$(x_1y_1 + x_2y_2 - 1)^2 \geq (x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1)\,.$$

2. Consider n points $P_1, P_2, \ldots, P_n$ lying in that order on a straight line. We colour each of these n points using one of the following colours: white, red, green, blue and violet. A colouring is called admissible if for each two consecutive points P_i, P_{i+1} $(i = 1, 2, \ldots, n-1)$ either both are of the same colour or at least one of them is white. How many admissible colourings are there?

3. Find all pairs of real numbers (x, y) satisfying the following system of equations
$$2 - x^3 = y\,, \quad 2 - y^3 = x\,.$$

4. Let m, n be positive integers. Denote
$$S_m(n) = \sum_{1 \leq k \leq n} \left[\sqrt[k^2]{k^m} \right]$$
($[x]$ is the biggest integer not bigger than x). Prove, that
$$S_m(n) \leq n + m \cdot (\sqrt[4]{2^m} - 1)\,.$$

5. Determine all pairs (a, b) of positive integers such that the equation
$$x^3 - 17x^2 + ax - b^2 = 0$$
has three integer roots (not necessarily different).

6. Different points A, B, C, D, E, F lie on the circle k in that order. The tangents to the circle k in the points A and D and the lines BF and CE intersect in one point P. Prove that the lines AD, BC and EF are either parallel or intersect in one point.

7. Consider all pairs (a, b) of natural numbers such that the product $a^a \cdot b^b$ written in decimal system ends with exactly 98 zeros. Find the pair (a, b) for which the product ab is the smallest.

8. Let $n > 2$ be a given natural number. Consider a square net on the plain. In each unit square of the net a natural number is written. The polygons with area equal to n, whose sides are contained in the lines forming the net, are called admissible. The sum of all the numbers written in the squares contained in a polygon is called value of the polygon. Prove that if the values of any two congruent admissible polygons are equal then all the numbers written in the unit squares of the net are equal.
Remark. We recall that the symmetric image Q of the polygon P is congruent to P.

9. Let K, L, M be the midpoints of the sides BC, AC, AB of triangle ABC. The points A, B, C divide the circumcircle of triangle ABC on three arcs $\widehat{AB}, \widehat{BC}, \widehat{CA}$. Let X be the point of the arc $\widehat{BC}$ such that $BX = XC$. Analogously, let Y be the point of the arc $\widehat{AC}$ such that $AY = YC$ and Z be the point of the arc $\widehat{AB}$ such that $AZ = ZB$. Denote by R the circumradius of triangle ABC and by r the inradius of triangle ABC. Prove that
$$r + KX + LY + MZ = 2R\,.$$