The 19-th Austrian–Polish Mathematics Competition
Poland, June 26–28, 1996

1. Let $k \geq 1$ be a positive integer. Prove that there exist exactly 3^{k-1} natural numbers n with the following properties:

(a) n has exactly k digits in decimal representation,

(b) all the digits of n are odd,

(c) n is divisible by 5,

(d) the number $m = n/5$ has k odd digits (in decimal representation).

2. A convex hexagon $ABCDEF$ satisfies the following conditions:

(a) opposite sides are parallel, i.e. $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$,

(b) the distances between opposite sides are equal, i.e. $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between parallel lines g and h,

(c) the angles FAB and CDE are right.

Prove that the angle between the diagonals BE and CF is equal to $45°$.

3. The polynomials $P_n(x)$ are defined recursively

$$P_0(x) = 0\,, \qquad P_1(x) = x \qquad \text{and} \qquad P_n(x) = xP_{n-1}(x) + (1-x)P_{n-2}(x) \quad \text{for } n \geq 2.$$

For each integer $n \geq 1$ find all real numbers x with $P_n(x) = 0$.

4. Real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

5. A convex polyhedron $\mathcal{P}$ and a sphere $\mathcal{S}$ are placed in the space so that $\mathcal{S}$ cuts off from each edge AB of $\mathcal{P}$ the segment XY with

$$|AX| = |XY| = |YB| = \tfrac{1}{3}|AB|\,.$$

Prove that there exists a sphere tangent to all the edges of $\mathcal{P}$.

6. Given natural numbers k and n with $1 < k < n$. Solve in real numbers the system of equations

$$x_i^3 \cdot (x_i^2 + x_{i+1}^2 + \ldots + x_{i+k-1}^2) = x_{i-1}^2 \qquad \text{for} \quad 1 \leq i \leq n$$

with n unknowns $x_1, x_2, \ldots, x_n$.

Note: $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$, etc.

7. Prove that there don't exist non-negative integers k and m with $k! + 48 = 48(k+1)^m$.

8. Prove that there doesn't exist a polynomial $P(x)$ of degree 998 with real coefficients such that for all x the following equality holds: $P(x)^2 - 1 = P(x^2 + 1)$.

9. We have rectangular blocks, none of which being a cube. The lengths of edges of the blocks are positive integers. For any triple (a, b, c) of positive integers such that the equality $a = b = c$ does not hold, we have a suffiecient number of the blocks of dimensions $a \times b \times c$. Let the box $10 \times 10 \times 10$ be filled up with our blocks.

(a) Assume that we have used at least 100 blocks. Prove that there exist at least two blocks with the same dimensions placed parallelly, i.e. such that if AB is an edge of one block and $A'B'$ of another and $AB \parallel A'B'$, then $|AB| = |A'B'|$.

(b) Prove the same for a number of the used blocks less than 100. The less the number, the better solution!