The 25th Austrian–Polish Mathematics Competition
Pułtusk, June 2002

1. Find all triples (a, b, c) of nonnegative integers such that the number $2^c - 1$ divides the number $2^a + 2^b + 1$.

2. Let $P_1P_2 \ldots P_{2n}$ be a convex polygon with an even number of corners. Prove that there exists a diagonal P_iP_j which is not parallel to any side of the polygon.

3. Let $ABCD$ be a tetrahedron and let S be its center of gravity. A line through S intersects the surface of $ABCD$ in the points K and L. Prove that
$$\frac{1}{3} \leq \frac{KS}{LS} \leq 3\,.$$

4. For each positive integer n find the largest subset $M(n)$ of real numbers possessing the property:
$$n + \sum_{i=1}^{n} x_i^{n+1} \geq n \prod_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i \quad \text{for all } x_1, x_2, \ldots, x_n \in M(n)\,.$$
When does the inequality become an equality?

5. Let A be the set $\{2, 7, 11, 13\}$. A polynomial f with integer coefficients possesses the following property: for each integer n there exists $p \in A$ such that $p|f(n)$. Prove that there exists $p \in A$ such that $p|f(n)$ for all integers n.

6. The diagonals of a convex quadrilateral $ABCD$ intersect in the point E. Let U be the circumcenter of the triangle ABE and H be its orthocenter. Similarly, let V be the circumcenter of the triangle CDE and K be its orthocenter. Prove that E lies on the line UK if and only if it lies on the line VH.

7. Find all real functions f defined on positive integers and satisfying: a) $f(x + 22) = f(x)$, b) $f(x^2y) = (f(x))^2 f(y)$ for all positive integers x and y.

8. Determine the number of real solutions of the system
$$\begin{cases} \cos x_1 = x_2 \\ \cdots\cdots\cdots \\ \cos x_{n-1} = x_n \\ \cos x_n = x_1\,. \end{cases}$$

9. A set P of 2002 persons is given. The family of subsets of P containing exactly 1001 persons has the property: the number of acquaintance pairs in each such subset is the same. (It is assumed that the acquaintance relation is symmetric.) Find the best lower estimation of the acquaintance pairs in the set P.

10. For all real numbers x consider the family $F(x)$ of all sequences $(a_n)_{n\geq 0}$ satisfying the equation
$$a_{n+1} = x - \frac{1}{a_n} \quad (n \geq 0)\,.$$
A positive integer p is called a minimal period of the family $F(x)$ if
a) each sequence $(a_n) \in F(x)$ is periodic with the period p,
b) for each $0 < q < p$ there exists $(a_n) \in F(x)$ such that q is not a period of (a_n).
Prove or disprove that for each positive integer P there exists a real number $x = x(P)$ such that the family $F(x)$ has the minimal period $p > P$.