48-th Mathematical Olympiad in Poland

Second Round, February 21–21, 1997

First Day

1. For any real number a determine the number of the ordered triples (x, y, z) of real numbers satisfying the following system of equations

$$\begin{cases} x + y^2 + z^2 = a \\ x^2 + y + z^2 = a \\ x^2 + y^2 + z = a \end{cases}$$

2. Point P lies inside triangle ABC and satisfies the conditions:

$$\angle PBA = \angle PCA = \frac{1}{3}(\angle ABC + \angle ACB)\,.$$

Prove that

$$\frac{AC}{AB + PC} = \frac{AB}{AC + PB}\,.$$

3. Given is a set of n points ($n \geq 2$); no three of the points are collinear. We colour all the line segments with endpoints in this set so that two segments with a common endpoint are of different colours. Determine the least number of colours, for which there exists such a colouring.

Second Day

4. Find all triples of positive integers with the following property: The product of any two of these numbers gives the remainder 1 upon division by the third number.

5. We have thrown k white dice and m black dice. Find the probability that the remainder upon division by 7 of the number on the faces of the white dice is equal to the remainder upon division by 7 of the number on the faces of the black dice.

6. In a cube of the edge 1, eight points are given. Prove that two of the points are the endpoints of a segment of length not greater than 1.