45-th Mathematical Olympiad in Poland

Second Round, February, 1994

First Day

1. Determine all polynomials $P(x)$ of degree 5, with real coefficients, such that $(x-1)^3 \mid P(x)+1$ and $(x+1)^3 \mid P(x)-1$.

2. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i \,.$$

Let $b_1, b_2, \ldots, b_n$ be real numbers satisfying $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. Prove that

$$\sum_{i=1}^{n} b_i \leq \prod_{i=1}^{n} b_i \,.$$

3. A section of a cube, passing through a center of the cube, is a cyclic hexagon. Prove that this hexagon is regular.

Second Day

4. To each vertex of a cube there is assigned a number 1 or -1. To each face there is assigned a product of the four assigned numbers to the vertices of this face. Determine the set of all possible values that may be attained by the sum of all the 14 assigned numbers.

5. The circle o inscribed in a triangle ABC is tangent to the sides AB and BC of $\triangle ABC$ in the points P and Q. The line PQ intersects the angle bisector of $\angle BAC$ in the point S. Prove that this angle bisector is perpendicular to the line SC.

6. Given is a prime p. Prove that the following two sentences are equivalent:

(1) there exists an integer n with $p \mid n^2 - n + 3$;

(2) there exists an integer m with $p \mid m^2 - m + 25$.