44-th Mathematical Olympiad in Poland

Second Round, February, 1993

First Day

1. Prove that for any positive numbers x, y, u, v the inequality

$$\frac{xy+xv+uy+uv}{x+y+u+v} \geq \frac{xy}{x+y} + \frac{uv}{u+v}$$

holds.

2. Given is a circle with center O and a point P lying outside of this circle. Let l be a line passing through the point P and cutting the given circle at points A and B. Let C be the symmetric point to A with respect to the line OP and let m be the line connecting the points B and C. Prove that all the lines m, determined by the various lines l, have a common point.

3. On the edge OA_1 of a tetrahedron $OA_1B_1C_1$ there are chosen points A_2, A_3, such that $OA_1 > OA_2 > OA_3 > 0$. Let B_2, B_3 be the points of the edge OB_1, and C_2, C_3 – the points of the edge OC_1 such that the planes $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$ are parallel. Let V_i $(i = 1, 2, 3)$ be the volume of the tetrahedron $OA_iB_iC_i$ and V be the volume of the tetrahedron $OA_1B_2C_3$. Prove that

$$V_1 + V_2 + V_3 \geq 3V.$$

Second Day

4. Let (x_n) be the sequence of natural numbers such that:

$$x_1 = 1 \quad \text{and} \quad x_n < x_{n+1} \leq 2n \qquad \text{for } n = 1, 2, 3 \ldots.$$

Prove that for every natural number k, there exist the subscripts r and s, such that $x_r - x_s = k$.

5. On the sides BC, CA, AB of a triangle ABC there are chosen points D, E, F (respectively), such that inradii of the triangles AEF, BFD, CDE are all equal to r_1. Inradii of the triangles DEF and ABC are equal to r_2 and r respectively. Prove that $r_1 + r_2 = r$.

6. Given is a continuous function $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ satisfying the condition:

$$f(1000) = 999\,, \qquad f(x) \cdot f(f(x)) = 1 \qquad \text{for } x \in \mathbf{R}.$$

Find $f(500)$.