46-th Mathematical Olympiad in Poland
First Round, September–December, 1994

1. Determine all pairs (x, y) of natural numbers, such that the numbers $\dfrac{x+1}{y}$ and $\dfrac{y+1}{x}$ are natural.

2. Given a positive integer $n \geq 2$. Solve the following system of equations:

$$\begin{cases} x_1|x_1| = x_2|x_2| + (x_1-1)|x_1-1| \\ x_2|x_2| = x_3|x_3| + (x_2-1)|x_2-1| \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n|x_n| = x_1|x_1| + (x_n-1)|x_n-1|\,. \end{cases}$$

3. A quadrilateral with sides a, b, c, d is inscribed in a circle of radius R. Prove that if $a^2+b^2+c^2+d^2 = 8R^2$, then either one of the angles of the quadrilateral is right or the diagonals of the quadrilateral are perpendicular.

4. In some school 64 students participate in five different subject olympiads. In each olympiad at least 19 students take part; none of them is a participant of more than three olympiads. Prove that if every three olympiads have a common paticipant, then there are two olympiads having at least five participants in common.

5. Given positive numbers a, b. Prove that the following sentences are equivalent:

(1) $\sqrt{a} + 1 > \sqrt{b}$;

(2) for every $x > 1$, $ax + \dfrac{x}{x-1} > b$.

6. Inside triangle ABC there is chosen a point P. The rays AP, BP, CP intersect the boundary of the triangle in the points A', B', C' respectively. Set

$$u = |AP| : |PA'|, \quad v = |BP| : |PB'|, \quad w = |CP| : |PC'|\,.$$

Express the product uvw in terms of the sum $u + w + v$.

7. (a) Find out, whether there exists a differentiable function $f\colon \mathbf{R} \to \mathbf{R}$, not equaling 0 for all $x \in \mathbf{R}$, satisfying the conditions $2f(f(x)) = f(x) \geq 0$ for all $x \in \mathbf{R}$.

(b) Find out, whether there exists a differentiable function $f\colon \mathbf{R} \to \mathbf{R}$, not equaling 0 for all $x \in \mathbf{R}$, satisfying the conditions $-1 \leq 2f(f(x)) = f(x) \leq 1$ for all $x \in \mathbf{R}$.

8. In a regular pyramid with a regular n-gon as a base, the dihedral angle between a lateral face and the base is equal to α, and the angle between a lateral edge and the base is equal to β. Prove that

$$\sin^2\alpha - \sin^2\beta \leq \operatorname{tg}^2\frac{\pi}{2n}\,.$$

9. Let a and b be positive real numbers with the sum equal to 1. Prove that if a^3 and b^3 are rational, so are a and b.

10. Given a line k and three distinct points on it. Each of these points is the beginning of a pair of the rays – all the rays lie on the same side of the halfplane of the edge k. Each of these three pairs form a convex quadrilateral with another pair (so we have three quadrilaretals formed by these pairs of the rays). Prove that if it is possible to inscribe a circle in two of these quadrilaterals, then it is possible to inscribe a circle in the third one as well.

11. Given are natural numbers $n > m > 1$. We draw m numbers from the set $\{1, 2, \dots, n\}$ one by one without putting the drawn numbers back. Find the expected value of the difference between the largest and the smallest of the drawn numbers.

12. The sequence (x_n) is given by

$$x_1 = \frac{1}{2}, \quad x_n = \frac{2n-3}{2n} \cdot x_{n-1} \quad \text{for } n = 2, 3, \dots\,.$$

Prove that for all natural numbers $n \geq 1$ the following inequality holds

$$x_1 + x_2 + \ldots + x_n < 1.$$