45-th Mathematical Olympiad in Poland
First Round, September–December, 1993

1. Prove that the system of equations

$$\begin{cases} a^2 - b = c^2 \\ b^2 - a = d^2 \end{cases}$$

has no integer solutions a, b, c, d.

2. The sequence of functions $f_0, f_1, f_2, \ldots$ is given by the conditions:

$$f_0(x) = |x| \qquad \text{for all} \;\; x \in \mathbf{R}$$
$$f_{n+1}(x) = |f_n(x) - 2| \qquad \text{for} \;\; n = 0, 1, 2, \ldots \quad \text{and all} \;\; x \in \mathbf{R}\,.$$

For each positive integer n, solve the equation $f_n(x) = 1$.

3. Prove that if a, b, c are the lengths of the sides of a triangle, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{1}{a+b-c} + \frac{1}{c+a-b} + \frac{1}{b+c-a}\,.$$

4. Given is a circle with center O, point A inside the circle and a chord PQ which is not a diameter and passing through A. The lines p and q are tangent to the given circle at P and Q respectively. The line ℓ passing through A and perpendicular to OA intersects the lines p and q at K and L respectively. Prove that $|AK| = |AL|$.

5. Prove that if the polynomial $x^3 + ax^2 + bx + c$ has three distinct real roots, so does the polynomial

$$x^3 + ax^2 + \frac{1}{4}(a^2 + b)x + \frac{1}{8}(ab - c)\,.$$

6. The function $f\colon \mathbf{R} \to \mathbf{R}$ is continuous. Prove that if for every real number x, there exists a positive integer n, such that

$$(\underbrace{f \circ f \circ \ldots \circ f}_{n})(x) = 1\,,$$

then $f(1) = 1$.

7. Given convex quadrilateral $ABCD$. We construct the similar triangles APB, BQC, CRD, DSA outside $ABCD$ so that

$$\angle PAB = \angle QBC = \angle RCD = \angle SDA\,, \qquad \angle PBA = \angle QCB = \angle RDC = \angle SAD\,.$$

Prove that if $PQRS$ is a parallelogram, so is $ABCD$.

8. Given positive integers a, b, c such that a^3 is divisible by b, b^3 is divisible by c, c^3 is divisible by a. Prove that $(a + b + c)^{13}$ is divisible by abc.

9. In a conference $2n$ personalities take part. Each person has at least n acquaintaces among the others. Prove that it is possible to quarter the participants into two-person rooms, so that each participant would share the room with his/her acquaintace.

10. Given positive real numbers p, q with $p + q = 1$. Prove that for all positive integers m, n the following inequality holds

$$(1 - p^m)^n + (1 - q^n)^m \ge 1\,.$$

11. A triangle with perimeter $2p$ is inscribed in a circle of radius R and also circumscribed on a circle of radius r. Prove that $p < 2(R + r)$.

12. Prove that the sums of the opposite dihedral angles of a tetrahedron are equal if and only if the sums of the the opposite edges of this tetrahedron are equal.