44-th Mathematical Olympiad in Poland
First Round, September–December, 1992

1. Solve the following equation in real numbers:

$$\frac{(x^2-1)(|x|+1)}{x+\operatorname{sgn} x} = [x+1].$$

2. Given is a natural number $n \geq 3$. Solve the system of equations:

$$\begin{cases} \tan x_1 + 3\cot x_1 = 2\tan x_2 \\ \tan x_2 + 3\cot x_2 = 2\tan x_3 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \tan x_n + 3\cot x_n = 2\tan x_1\,. \end{cases}$$

3. Given is a hexagon $ABCDEF$ with a center of symmetry. The lines AB and EF meet at the point A', the lines BC and AF meet at the point B', and the lines AB and CD meet at the point C'. Prove that $AB \cdot BC \cdot CD = AA' \cdot BB' \cdot CC'$.

4. Determine all functions $f\colon \mathbf{R} \longrightarrow \mathbf{R}$ such that

$$f(x+y) - f(x-y) = f(x)\cdot f(y) \qquad \text{for } x, y \in \mathbf{R}.$$

5. Given is a halfplane with points A and C on its edge. For every point B on this halfplane consider the squares $ABKL$ and $BCMN$ lying outside of the triangle ABC. Prove that all the lines LM (as the point B varies) have a common point.

6. The sequence (x_n) is determined by the conditions:

$$x_0 = 1992, \qquad x_n = -\frac{1992}{n}\cdot\sum_{k=0}^{n-1} x_k \qquad \text{for } n \geq 1.$$

Find $\displaystyle\sum_{n=0}^{1992} 2^n x_n\,.$

7. Given are the points $A_0 = (0,0,0)$, $A_1 = (1,0,0)$, $A_2 = (0,1,0)$, $A_3 = (0,0,1)$ in the space. Let P_{ij} $(i,j \in \{0,1,2,3\})$ be the point determined by the equality: $\overrightarrow{A_0P_{ij}} = \overrightarrow{A_iA_j}$. Find the volume of the smallest convex polyhedron which contains all the points P_{ij}.

8. Given is a positive integer $n \geq 2$. Determine the maximum value of the sum of natural numbers $k_1, k_2, \dots, k_n$ satisfying the condition:

$$k_1^3 + k_2^3 + \dots + k_n^3 \leq 7n\,.$$

9. Prove that for all real numbers a, b, c the inequality

$$(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2) \leq (a+b-c)^2(b+c-a)^2(c+a-b)^2$$

holds.

10. Let $\mathcal{C}$ be a cube and let $f\colon \mathcal{C} \to \mathcal{C}$ be a surjection with

$$|PQ| \geq |f(P)f(Q)|$$

for all $P, Q \in \mathcal{C}$. Prove that f is an isometry.

11. Given is an $n \times n$ chessboard. With the same probability, we put six pawns on its six cells. Let p_n denotes the probability that there exists a row or a column containing at least two pawns. Find $\lim\limits_{n\to\infty} np_n$.

12. Prove that the polynomial $x^n + 4$ can be expressed a product of two polynomials (each with degree less than n) with integer coefficients, if and only if n is divisible by 4.