EQUIVARIANT HIRZEBRUCH CLASS FOR SINGULAR VARIETIES

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Abstract. We develop an equivariant version of the Hirzebruch class for singular varieties. When the group acting is a torus we apply Localization Theorem of Atiyah-Bott and Berline-Vergne. The localized Hirzebruch class is an invariant of a singularity germ. The singularities of toric varieties and Schubert varieties are of special interest. We prove certain positivity results for simplicial toric varieties. The positivity for Schubert varieties is illustrated by many examples, but it remains mysterious.

The main goal of the paper is to show how a theory of global invariants can be applied to study local objects equipped with an action of a large group of symmetries. The theory of global invariants we are going to discuss is the theory of characteristic classes, more precisely the Hirzebruch class and χ_y-genus. By [Yok94, BSY10] the Hirzebruch class admits a generalization for singular varieties. Let X be an algebraic variety in a compact complex algebraic manifold M. Suppose that X is preserved by a torus $\mathbb{T}$ acting on M. For simplicity assume that the fixed point set $M^{\mathbb{T}}$ is discrete. Then by Localization Theorem of Atiyah-Bott and Berline-Vergne the χ_y-genus of X can be written as a sum of contributions coming from fixed points. The contribution of a fixed point $p \in M^{\mathbb{T}}$ is equal to the equivariant Hirzebruch class restricted to that point and divided by the Euler class of the tangent space at p

$$\chi_y(X) = \sum_{p\in M^{\mathbb{T}}} \frac{td^{\mathbb{T}}_y(X \hookrightarrow M)_{|p}}{eu(p)}\,.$$

The local contributions to χ_y-genus are fairly computable and they are expressed by polynomials in characters of the torus. We will describe all the necessary components of the described construction, give various examples and we will discuss positivity property of the localized Hirzebruch class.

A relation with the Białynicki-Birula decomposition [BB73] will be given in a subsequent paper [Web14]. The reader can find useful to look at the article [Web13], where an elementary and self-contained introduction to equivariant characteristic classes is given.

1991 *Mathematics Subject Classification.* 14C17, 14E15, 14F43, 19L47, 55N91, 14M25, 14M15.

Key words and phrases. Characteristic classes of singular varieties, Hirzebruch class, equivariant cohomology, toric varieties, Schubert varieties.

Supported by NCN grant 2013/08/A/ST1/00804 .

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1. Localization formulae

The χ_y-genus of a smooth compact complex algebraic variety X was defined by Hirzebruch [Hir56] as a formal combination of Euler characteristics of the sheaves of differential forms

$$\chi_y(X) := \sum_{p=0}^{\dim X} \chi(X, \Omega^p_X) y^p \,. \tag{1}$$

The Hirzebruch-Riemann-Roch theorem allows to express that numerical invariant as an integral

$$\chi_y(X) = \int_X td_y(X) = \int_X td(TX)\cdot ch(\Lambda_y(T^*X))\,.$$

The following multiplicative characteristic classes appear in the formula

- the Todd class $td(TX)$ is the multiplicative characteristic class associated to the power series $td(x) = \frac{x}{1-e^{-x}}$
- $ch(\Lambda_y TX^*)$ is the composition of the total λ-operation

$$\Lambda_y(-) = \sum_{p=0}^{\dim X} \Lambda^p(-) y^p$$

  and the Chern character. This way a multiplicative characteristic class is obtained. It is associated to the power series $\varphi(x) = 1 + y\, e^{-x}$.

The equivariant setup demands some explanation but the applied tools are basically standard. Suppose $\mathbb{T} = (\mathbb{C}^*)^r$ acts on a manifold X with finite number of fixed points. The tangent bundle of X is an equivariant $\mathbb{T}$-bundle. The equivariant cohomology $H^*_{\mathbb{T}}(X)$ constructed by Borel is a suitable graded ring, where the equivariant characteristic classes live. In particular we consider the equivariant Hirzebruch class $td^{\mathbb{T}}_y(X)$. The class $td^{\mathbb{T}}_y(X)$ may be nontrivial in arbitrary high gradation, therefore by $H^*_{\mathbb{T}}(X)$ we understand not the direct sum but the product $\prod_{k=0}^{\infty} H^k_{\mathbb{T}}(X)$. The equivariant cohomology has more structure than ordinary cohomology, since it is a module over $H^*_{\mathbb{T}}(pt)$, which is the symmetric algebra spanned by characters $\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. Our main tool is the Localization Theorem for torus action. Essentially it is due to Borel [Bor60, Ch.XII §6], but here is the formulation by Quillen:

Theorem 1.1 ([Qui71], Theorem 4.4)**.** *The restriction map*

$$H^*_{\mathbb{T}}(X) \to H^*_{\mathbb{T}}(X^{\mathbb{T}})$$

is an isomorphism after localization in the multiplicative system generated by nontrivial characters.

Atiyah-Bott [AB84, Formula 3.8] or Berline-Vergne [BV97] Localization Formula (see Theorem 6.3 and the following Integration Formula) allows to express integral of equivariant forms in terms of some data concentrated at the fixed points. We apply the localization formula to the equivariant Hirzebruch class $td^{\mathbb{T}}_y(X)$. By rigidity (see [Mus11]) we obtain just the class of gradation zero, the χ_y-genus

Theorem 1.2.

$$\chi_y(X) = \sum_{p \in X^{\mathbb{T}}} \frac{1}{eu(p)} td^{\mathbb{T}}_y(X)_{|p} \,.$$

Here the Euler class $eu(p) = \prod w_i$ is the product of weights $w_i \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) = H^2_{\mathbb{T}}(pt)$ appearing in the tangent representation T_pX. Note that the χ_y-genus can be written in the form

$$\chi_y(X) = \sum_{p \in X^{\mathbb{T}}} \frac{1}{eu(p)} td^{\mathbb{T}}(X) \cdot ch^{\mathbb{T}}(\Lambda_y(T^*X))_{|p} \,.$$

The local contribution of each fixed point $ch^{\mathbb{T}}(\Lambda_y(T^*X))_{|p}$ is multiplied by

$$\frac{1}{eu(p)} td(X)^{\mathbb{T}}_{|p} = \prod \frac{1}{w_i} \cdot \prod \frac{w_i}{1 - e^{-w_i}} = \prod \frac{1}{1 - e^{-w_i}} \,. \tag{2}$$

This multiplier is equal to the inverse of the Chern character of the class of the skyscraper sheaf $\mathcal{O}_{\{p\}}$. In fact we can localize before applying Chern character and work in the topological equivariant K-theory defined by Segal [Seg68]. The localization in K-theory has a parallel shape comparing with Theorem 1.1

Theorem 1.3 ([Seg68], Proposition 4.1)**.** *The restriction map*

$$K_{\mathbb{T}}(X) \to K_{\mathbb{T}}(X^{\mathbb{T}})$$

is an isomorphism after localization in the multiplicative system[1] *generated by*

$$T_w - 1 \in K_{\mathbb{T}}(pt)$$

for all nonzero characters $w \in Hom(\mathbb{T}, \mathbb{C}^*)$*, where* T_w *is the element of* $K_{\mathbb{T}}(pt)$ *associated to the one dimensional representation with character* w*.*

If X is a compact algebraic variety, then it is enough to localize in the system generated by

$$T_w - 1 \in K_{\mathbb{T}}(pt)$$

for all nonzero characters $w \in Hom(\mathbb{T}, \mathbb{C}^*)$ appearing in the tangent representations at the fixed points.

The equality below is the K-theoretic counterpart of the integration Theorem 1.2:

$$\chi(X, E) = \sum_{p \in X^T} \frac{1}{[\mathcal{O}_p]} [E]_{|p} \tag{3}$$

for any $E \in K_{\mathbb{T}}(X)$. In particular

$$\chi_y(X) = \sum_{p \in X^T} \frac{1}{[\mathcal{O}_p]} [\Lambda_y T^* X]_{|p}\,. \tag{4}$$

Note that from the numerical point of view there is no need to worry in which groups (equivariant K-theory or cohomology) the computation is done. The passage from cohomology to K-theory is reflected by the choice of a new variables, the Chern characters of the line bundles.

A version of the localization theorem in K-theory is discussed in [Gro77, Corollaire 6.12]. The formula is also valid for singular spaces [Bau82]. Baum computes holomorphic Lefschetz number instead of χ_y-genus, but essentially his argument is the same. The holomorphic Lefschetz number

$$\sum_{i=0}^{\dim X} (-1)^i Tr\big(f^* \in End(H^i(X; \mathcal{O}_X))\big)$$

of a finite order automorphism $f : X \to X$ is expressed by the sum of local contributions coming from the fixed points. The contribution at a fixed point p is obtained from the power series

$$\sum_{k=0}^{\infty} Tr\big(f^* \in End(\mathfrak{m}_p^k / \mathfrak{m}_p^{k+1})\big) s^k\,.$$

Here $\mathfrak{m}_p$ is the maximal ideal at p. This series turns out to be a rational function with regular value at $s = 1$. The specialization gives the local contribution to the global

[1] The original formulation of the localization theorem is stronger. Here we apply it for the localization in the dimension ideal. For an arbitrary prime ideal $\mathfrak{p} \subset K_{\mathbb{T}}(pt) = Rep(\mathbb{T})$ there is a maximal group $H \subset \mathbb{T}$, such that $K_{\mathbb{T}}(X)_{\mathfrak{p}} \stackrel{\sim}{\to} K_{\mathbb{T}}(X^H)_{\mathfrak{p}}$. For a smaller ideal $\mathfrak{p}$ we have to take a smaller subgroup H.

Lefschetz number. If the point p is smooth the contribution at p is equal to

$$\prod \frac{1}{1-\lambda_i},$$

where the product is taken over the eigenvalues of f acting on the tangent space T_pX. This is exactly the shape of the formula (2).

Example 1.4. Let $X = \mathbb{P}^1$ with the standard action of $\mathbb{C}^*$. The equivariant ring of a point is equal to the polynomial algebra in one variable t, the generator of $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)$. The weight of the tangent representation at 0 is equal to $w_0 = t$ and the weight of the representation at infinity is equal to $w_\infty = -t$. We find that

$$\begin{aligned}\chi_y(\mathbb{P}^1) &= \frac{1}{w_0}\frac{w_0}{1-e^{-w_0}}\left(1+y\,e^{-w_0}\right) + \frac{1}{w_\infty}\frac{w_\infty}{1-e^{-w_\infty}}\left(1+y\,e^{-w_\infty}\right)\\ &= \frac{1}{1-e^{-t}}\left(1+y\,e^{-t}\right) + \frac{1}{1-e^{t}}\left(1+y\,e^{t}\right)\,.\end{aligned}$$

Let us introduce a new variable $e^{-t} = T$. It can also be considered as an element of $K_{\mathbb{T}}(pt)$ and in fact the calculi can be thought to be performed in the localized K-theory. Now we have

$$\chi_y(\mathbb{P}^1) = \frac{1+y\,T}{1-T} + \frac{1+y\,T^{-1}}{1-T^{-1}} = 1-y$$

Let us remark about the formal setup where the above computations were done. We have started from the equivariant cohomology of a point $H^*_{\mathbb{T}}(pt)$ that is $\mathbb{Q}[t]$ completed in t. We are forced to complete since we consider the infinite power series $td(\pm t) = \frac{t}{1-e^{\pm t}}$ and Chern character $e^{\pm t}$. Equally well the computation might have been done in the equivariant K-theory of a point $K_{\mathbb{T}}(pt) = \mathbb{Z}[T, T^{-1}]$. Formally it is a different object, but from the computational point of view it makes no difference. The K-theory completed in the dimension ideal I (which is generated by $T-1$) and tensored with $\mathbb{Q}$ is isomorphic to $\mathbb{Q}[[t]]$ via the map $T \mapsto e^{-t}$.

The same remarks are valid for tori of bigger dimensions: $\mathbb{T} = (\mathbb{C}^*)^r$. The Chern character is an isomorphism

$$ch^{\mathbb{T}} : R(\mathbb{T})^{\wedge}_I \simeq K_{\mathbb{T}}(pt)^{\wedge}_I \otimes \mathbb{Q} \to H^*_{\mathbb{T}}(pt)^{\wedge}_{\mathfrak{m}} \simeq \mathbb{Q}[[t_1, t_2, \dots, t_r]]\,,$$

where $R(\mathbb{T})$ is the representation ring of $\mathbb{T}$, the dimension ideal is generated by the elements $T_w - 1$ for the nontrivial characters w. (According to our convention $H^*_{\mathbb{T}}(pt) = \prod_{k=0}^{\infty} H^k_{\mathbb{T}}(pt)$ is already completed in the maximal ideal $\mathfrak{m} = H^{>0}_{\mathbb{T}}(pt)$.) The isomorphism $ch^{\mathbb{T}}$ is the composition of the completion isomorphism $K_{\mathbb{T}}(pt)^{\wedge}_I \simeq K(BT)$ (see [AS69] for a broader context) with nonequivariant Chern character $K(BT)\otimes\mathbb{Q} \to H^*(BT) = H^*_{\mathbb{T}}(pt)$.

2. Localizing χ_y-genus of singular varieties

Singular spaces enter into the story due to a theorem of Yokura and later by Brasselet-Schürmann-Yokura [Yok94, BSY10] who proved that the Hirzebruch class td_y admits a generalization to singular algebraic varieties. For any map from $X \to M$, both possibly singular algebraic varieties, we associate a homology class with closed supports (Borel-Moore homology)

$$td_y(f:X\to M)\in H_*^{BM}(M)[y].$$

We are interested mainly in inclusions of singular varieties into smooth ambient spaces, therefore due to Poincaré duality we can identify the target group with cohomology. The Brasselet-Schürmann-Yokura class satisfies

- If X is smooth and the map is proper then
$$td_y(f:X\to M)=f_*td_y(X)$$
- If $X=Z\sqcup U$ with Z closed then
$$td_y(f:X\to M)=td_y(f_{|Z}:Z\to M)+td_y(f_{|U}:U\to M)$$

The equivariant version can be developed as well, see the details in §7. It is of special interest when the acting group is a torus $\mathbb{T}$. If M is singular the target group, the equivariant homology, is a bit nonstandard object and eventually we perform all the computations assuming that M is smooth. Since the equivariant cohomology of a point is nontrivial in positive gradations there is a space for nontrivial invariants of germs of singular varieties. By the Atiyah-Bott or Berline-Vergne localization theorem we have

$$\chi_y(X)=\int_M td_y^{\mathbb{T}}(X\hookrightarrow M)=\sum_{p\in M^{\mathbb{T}}}\frac{1}{eu(p)}td_y^{\mathbb{T}}(X\hookrightarrow M)_{|p}\,.$$

The summand $\frac{1}{eu(p)}td_y^{\mathbb{T}}(X\hookrightarrow M)_{|p}$ is considered as the local contribution to χ_y-genus of the point $p\in X^T\subset M^T$. If $p\in X^{\mathbb{T}}$ is a smooth point, then

$$\frac{1}{eu(p)}td_y^{\mathbb{T}}(X\hookrightarrow M)_{|p}=\prod\frac{1+y\,e^{-w_i}}{1-e^{-w_i}}\,,$$

where w_i are the weights of the tangent representation T_pX and Euler class is computed with respect to weights of T_pM. The question remains what are the contributions of singular points? These local genera are analytic invariants of the singularity germs. We would like to understand what properties of a singular point are reflected by this invariant. Assume that $M=\mathbb{C}^n$. The localized Hirzebruch class is of the form

$$td^{\mathbb{T}}(\mathbb{C}^n)\cdot\textit{polynomial in } e^{-w_i} \textit{ and } y\,.$$

The polynomial in the formula is the equivariant Chern character of $f_{\bullet *}\Omega^*_{X_\bullet}$, where $f_\bullet:X_\bullet\to X$ is a smooth hypercovering of X. As a matter of a fact according to [BSY10, Theorem 2.1] the Hirzebruch class factors through K-theory of coherent sheaves on M

$$\begin{array}{ccc} K(Var/M) & \stackrel{td_y(-)}{\longrightarrow} & H_*^{BM}(M)[y] \\ {\scriptstyle mC_*}\searrow & & \nearrow{\scriptstyle td(M)ch(-)} \\ & G(M)\otimes\mathbb{Z}[y]\,. & \end{array}$$

Here $G(M)$ stands for the Grothendieck group of coherent sheaves on M. The same construction is valid for equivariant Hirzebruch class. The class

$$mC_*(X \hookrightarrow M)_{|p} = \left[Rf_{\bullet *}\Omega^*_{X_\bullet}\right]_{|p}$$

in the equivariant Grothendieck group $G_{\mathbb{T}}(pt) = K_{\mathbb{T}}(pt)$ is the main protagonist of this paper. Equally well we can talk about its Chern character in equivariant (co)homology.

Our goal is to clarify the definition, provide examples and formulate some statements for special classes of singularities. Taking the opportunity in §5 we relate the Aluffi construction of Chern-Schwartz-MacPherson classes [Alu99] with the direct definition of the Hirzebruch class for simple normal crossing divisor complements. In §10 we compute the Hirzebruch class for the affine cone of a variety contained in $\mathbb{P}^n$ and give many other examples. The Schubert varieties are of special interest, but for now in §16 we will give just a bunch of computations.

Setting $y = 0$ we arrive to the question what is a relation between $mC_0(id_X) = ch(\mathcal{O}_{X_\bullet})$ with $ch(\mathcal{O}_X)$ for a closed variety X. How far $td^{\mathbb{T}}_y(X \hookrightarrow M)$ is far from $td^{\mathbb{T}}(M)ch^T(\mathcal{O}_X)$? One can treat the difference as an analogue of the Milnor class ([Par88]) which is the difference between Chern-Schwartz-MacPherson class and the expected Fulton-Johnson class [FJ80]. The equality

$$td^{\mathbb{T}}_0(X \hookrightarrow M) = td^{\mathbb{T}}(M)ch^T(\mathcal{O}_X)$$

holds for a class of varieties with rational singularities or more general, almost by the definition, for Du Bois singularities, see [BSY10, Example 3.2]. Then $td^{\mathbb{T}}_0(id_X)$ agrees with Baum-Fulton-MacPherson class. In particular we have equality for

- normal crossing divisors,
- Schubert varieties,
- toric singularities,
- cone hypersurfaces in $\mathbb{C}^n$ of degree $d \le n$.

See §14 for details. In general the difference between $td^{\mathbb{T}}_y$ and its expected value for complete intersections is called the Hirzebruch-Milnor class. Formulas for that class in terms of vanishing cycle sheaves and in terms of a stratification are given in [MSS13].

3. Motivation for positivity

The question of positivity is much more complicated, although it seems to hold for a related class of singularities. At the moment we do not state a reasonable general conjecture. The positivity depends on numerical relations between higher derived images of the sheaves $\Omega^p_{X_\bullet}$. There are natural variables in which $ch^{\mathbb{T}}(\Omega^*_{X_\bullet})_{|p}$ is often a polynomial with nonnegative coefficients. Our choice of variables depends on the ambient space M. The positivity statement is motivated by an easy fact following for example from [PW08].

Theorem 3.1. *Fix a splitting of the torus $\mathbb{T} = (\mathbb{C}^*)^r$. Let $M = \mathbb{C}^n$ be a representation of $\mathbb{T}$ such that all weights of eigenspaces are nonnegative combinations of the* *basis characters. Let X be an invariant subvariety of M. Then the equivariant fundamental class $[X]_{|0} \in H^*_{\mathbb{T}}(pt)$ is a polynomial in the basis characters with nonnegative coefficients.*

The positivity can also be proved using degeneration of X to an union of invariant linear subspaces.

A similar statement holds for K-theory, but one has to assume that the variety X has at most rational singularities. The positivity in K-theory of compact homogeneous spaces G/P demands introducing signs, as it already has appeared in [Bri02, Theorem 1] for the global nonequivariant case. If X is a subvariety in a homogeneous space and it has at worst rational singularities then

$$[\mathcal{O}_X] = \sum_\alpha (-1)^{\dim X - \dim Y_\alpha} c_\alpha [\mathcal{O}_{Y_\alpha}]\,,$$

where the sum is taken over the set of Schubert varieties and the integers c_α are nonnegative. They are the dimensions of certain cohomology groups. The proof essentially relays on Kleiman transversality theorem and the fact that for rational singularities $Tor^*\mathcal{O}_X(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_1})$ is concentrated in one gradation. A similar formula holds in equivariant K-theory, [AGM11, Theorem 4.2]. One can get rid of the sign alternation by multiplying the fundamental classes by $(-1)^{\mathrm{codim} X}$. We will formulate the positivity condition in equivariant cohomology. Let us introduce variables

$$S_i = T_i - 1 = e^{-t_i} - 1 \in H^*_{\mathbb{T}}(pt)\,,$$

for $t_i \in H^2_{\mathbb{T}}(pt)$. These variables have a geometric origin: when we fix a geometric approximation $B\mathbb{T}_m = (\mathbb{P}^m)^r$, then S_i restricts to $-ch(\mathcal{O}_{H_i})$, where $H_i = (\mathbb{P}^m)^{i-1} \times \mathbb{P}^{m-1} \times (\mathbb{P}^m)^{r-i}$ is the coordinate hyperplane in $B\mathbb{T}_m = (\mathbb{P}^m)^r$. The K-theoretic analogue of the Theorem 3.1 is the following statement:

Theorem 3.2. *With the notation and assumptions of Theorem 3.1 suppose that X has rational singularities. Then the cohomology class $(-1)^{\mathrm{codim} X} ch^{\mathbb{T}}([\mathcal{O}_X])_{|0} \in H^*_{\mathbb{T}}(pt)$ is a polynomial in $S_1, S_2, \dots S_r$ with nonnegative coefficients.*

For a proof see §15, Theorem 15.1. We have noticed that if X has mild singularities then a similar property holds for full Hirzebruch class. To start we examine normal crossing divisors and their complements. One easily verifies that positivity holds in variables S_i and $d = -1 - y$. We expect that the positivity is preserved when $X \subset M$ has sufficiently good resolution, but now we cannot formulate and prove a general result. Apparently we need a stronger condition than just rationality. Here we wish to give some examples

- smooth quadratic cone hypersurfaces in $\mathbb{C}^n$,
- simplicial toric singularities (Theorem 13.1),
- du Val surface singularities (we omit explicit computations),
- some examples of Schubert cells.

The equivariant approach has an advantage that we can separate global phenomena from local and study only the local properties of the singularity. In addition, the results of computations are just polynomials, not some abstract classes in a huge unknown object.

The reader can find useful to look at the article [Web13], where an elementary and self-contained introduction to equivariant characteristic classes is given.

4. Recollection of the Hirzebruch class

The χ_y-genus of a smooth and compact complex variety X is a formal combination of the Euler characteristics of the sheaves of differential forms:

$$\chi_y(X) = \sum_{p=0}^{\dim X} \chi(X;\Omega^p_X)y^p = \sum_{p,q=0}^{\dim X} (-1)^q h^{p,q}(X) y^p \in \mathbb{Z}[y]\,,$$

see [Hir56]. To shorten the notation we will write formally

$$\Omega^y_X = \Lambda_y T^*X = \sum_{y=0}^{\dim X} \Omega^p_X\, y^p\,.$$

By the Hirzebruch-Riemann-Roch theorem the χ_y-genus is equal to the integral

$$\int_X td(X)ch(\Omega^y_X)\,,$$

where $td(X) \in H^*(X)$ is the Todd class and $ch(-)$ is the Chern character. (We always consider cohomology with rational coefficients.) The characteristic class $td(X)ch(\Omega^y_X) \in H^*(X)[y]$ is called the Hirzebruch class and denoted by $td_y(X)$. It is a multiplicative characteristic class associated to the formal power series

$$td_y(x) = \frac{x(1+ye^{-x})}{1-e^{-x}} = x + (1+y)\left(\frac{x}{1-e^{-x}} - x\right) = x + (1+y)\,(td(x) - x)$$

$$= x + (1+y)\left(1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \frac{x^6}{30240} - \frac{x^8}{1209600} + \frac{x^{10}}{47900160} + \dots\right)$$

The normalized Hirzebruch (see [Yok94]) class is obtained from the power series

$$\widetilde{td}_y(x) = \frac{x(1+ye^{-(1+y)x})}{1-e^{-(1+y)x}} = \frac{1}{1+y} td_y((1+y)x) =$$

$$= (1+x) - \frac{(1+y)x}{2} + \frac{(1+y)^2x^2}{12} - \frac{(1+y)^4x^4}{720} + \frac{(1+y)^6x^6}{30240} - \frac{(1+y)^8x^8}{1209600} + \dots\,.$$

For any vector bundle E of rank n over a n-dimensional variety we have

$$\int_M \widetilde{td}_y(E) = \int_M td_y(E)$$

therefore

$$\int_X \widetilde{td}_y(TX) = \chi_y(X)\,.$$

In addition for $y=-1$ we recover the total Chern class of the tangent bundle

$$\widetilde{td}_{-1}(TX) = c(TX)\,.$$

We will use unmodified Hirzebruch class having in mind, that normalization is a matter of re-scaling homogeneous components of td_y by a power of $(1+y)$. On the other hand according to Thom the normalized power series are in bijection with multiplicative characteristic classes. In fact for us it is irrelevant which version of td_y we use. In all the computations for a cohomology class $t \in H^2(B\mathbb{T})$ we use a variable $T = e^{-t}$ for unmodified version of the Hirzebruch genus. For the normalized version we would use $T = e^{-(y+1)t}$ and our formulas would remain not effected.

The Hirzebruch class was generalized for singular varieties in [BSY10]. In fact it is well defined on the Grothendieck group $K(Var/M)$ of varieties equipped with a map to a fixed variety M. This means that for any map of possibly singular varieties $f : X \to M$ the class $td_y(f : X \to M)$ is defined and

$$td_y(f : X \to M) = td_y(f_{|Y} : Y \to M) + td_y(f_{|U} : U \to M)$$

for any closed subvariety $Y \subset X$ and $U = X \setminus Y$. The Hirzebruch class takes values in Borel-Moore homology $H_*^{BM}(M)[y]$ (homology with closed supports) or in the Chow group of M. We obtain a transformation of functors

$$td_y : K(Var/-) \to H_*^{BM}(-)$$

defined on the category of complex algebraic varieties with proper maps. This means that if $\phi : M_1 \to M_2$ is a proper map, then the following diagram is commutative:

$$\begin{array}{ccc} K(Var/M_1) & \stackrel{td_y}{\longrightarrow} & H_*^{BM}(M_1) \\ {\scriptstyle \phi_\circ}\downarrow & & \downarrow{\scriptstyle \phi_*} \\ K(Var/M_2) & \underset{td_y}{\longrightarrow} & H_*^{BM}(M_2)\,. \end{array} \tag{5}$$

Here the map $\phi_\circ$ is just the composition sending $f : X \to M_1$ to $\phi \circ f : X \to M_2$. If X is smooth and the map f is proper, then

$$td_y(f : X \to M) = f_*(td_y(X)) = f_*(td(X)ch(\Omega_X^y))\,.$$

If in addition M is smooth then

$$td_y(f : X \to M) = td(M)ch(Rf_*\Omega_X^y)\,.$$

Here we freely use the Poincaré duality isomorphism $H_*^{BM}(M) \simeq H^{2\dim M-*}(M)$. To compute the Hirzebruch class $td_y(f : X \to M)$ for a singular X and possibly non proper map one has to replace X by its *geometric resolution* $X_\bullet$, for example obtained from a cubification in the sense of Guillen and Navarro Aznar [GNA02]. The maps between the members of $X_\bullet$ are not important, we just have to write $[f : X \to M] \in K(Var/M)$ as an alternating sum of classes $\sum_{i=0}^{\dim X}(-1)^i[f_i : X_i \to M]$ with X_i smooth and with the map $f_i : X_i \to M$ being proper. Then

$$td_y(f : X \to M) = \sum_{i=0}^{\dim X}(-1)^i td_y(f_i : X_i \to M)\,.$$

It is convenient to denote the class in K-theory as

$$\Omega^y_{X_\bullet}=\sum_{i=0}^{\dim X}(-1)^i\Omega^y_{X_i}\in\bigoplus_{i=0}^{\dim X}K(X_i)[y]$$

and

$$td_y(f:X\to M)=f_{\bullet *}(td(X_\bullet)\,ch(\Omega^y_{X_\bullet}))\,.$$

5. Hirzebruch class of a SNC divisor complement

The case when X is the complement of a simple normal crossing divisor $D\subset M$ is of particular interest, and it is worth to give an explicit formula in terms of logarithmic forms. A different formula connecting Hirzebruch class with the sheaf of logarithmic forms was given in [MS14, §2].

Let

$$D=\bigcup_{k=1}^m D_k\subset M$$

be the decomposition of D into smooth components. For a subset $I\subset\{1,2,\dots,m\}$ let

$$D_I=\bigcap_{i\in I}D_i\,.$$

By inclusion-exclusion formula the K-theoretic class $mC_*(X\hookrightarrow M)\in K(M)[y]$ is equal to

$$\iota_{\bullet *}\Omega^y_{X_\bullet}=\sum_{I\subset\{1,2,\dots,m\}}(-1)^{|I|}\iota_{I*}\Omega^y_{D_I}\in K(M)\,.$$

Here $\iota_I:D_I\to M$ denotes the inclusion. We will find another expression for $\iota_{\bullet *}\Omega^y_{X_\bullet}$. Let

$$L\Omega^p_I=\iota_{I*}\Omega^p_{D_I}(\log D_{>I})$$

be the sheaf of differential forms on D_I with logarithmic poles along the divisor $D_{>I}=\bigcup_{J>I}D_J$ introduced in [Del71]. We write

$$L\Omega^y_I=\sum_{p=0}^{\dim D_I}L\Omega^p_I\,y^p\,.$$

Theorem 5.1. *Let $y=-1-\delta$. With the notation introduced for a simple normal crossing divisor complement $X=M\setminus D$ we have*

$$\iota_{\bullet *}\Omega^y_{X_\bullet}=\sum_{I\subset\{1,2,\dots,m\}}\delta^{|I|}\,L\Omega^y_I\in K(M)[\delta]\,. \tag{6}$$

The Hirzebruch class of X is equal to

$$td_y(X\hookrightarrow M)=\sum_{I\subset\{1,2,\dots,m\}}\delta^{|I|}\,td(M)\,ch(L\Omega^y_I)\,. \tag{7}$$

Remark 5.2. Another way of expressing $\Omega^y_{X_\bullet}$ in terms of sheaves of logarithmic forms was given in [MS14, Prop.2.2]:

$$\Omega^y_{X_\bullet} = \mathcal{O}_M(-D) \otimes \Omega^y_M(\log D) \tag{8}$$

and

$$td_y(X \hookrightarrow M) = td(M)\, ch\big(\mathcal{O}_M(-D) \otimes \Omega^y_M(\log D)\big)\,. \tag{9}$$

The formulas (8) and (6) are equivalent as we explain below. Moreover the formula (9) can be generalized is the setup of mixed Hodge modules, see [MSS13, Proposition 5.2.1]. Let's compare two expressions (8) and (6) for $\Omega^y_{X_\bullet}$, say under assumption that D consists of one component. We will skip ι in the notation:

$$\Omega^y_{X_\bullet} = \Omega^y_M(\log D) + \delta\Omega^y_D = \mathcal{O}_M(-D) \otimes \Omega^y_M(\log D)\,.$$

Since in K-theory $\mathcal{O}_M(-D) = \mathcal{O}_M - \mathcal{O}_D$ we have

$$\Omega^y_M(\log D) + \delta\Omega^y_D = (\mathcal{O}_M - \mathcal{O}_D) \otimes \Omega^y_M(\log D)\,.$$

It follows

$$\delta\Omega^y_D = -\mathcal{O}_D \otimes \Omega^y_M(\log D)\,.$$

In general in K-theory there is an equality

$$\mathcal{O}_M(-D) = \sum_{I\subset\{1,2,\dots,m\}} (-1)^{|I|}\mathcal{O}_{D_I}\,.$$

and one can show (for example inductively from the equation of the formulas (8) and (6)) that

$$(-\delta)^{|I|} L\Omega^y_I = \mathcal{O}_{D_I} \otimes \Omega^y_M(\log D)\,.$$

We recall that the weight filtration and the whole mixed Hodge structure for an open smooth variety were constructed via the logarithmic complex by Deligne [Del71]. Theorem 5.1 or [MS14, Prop 2.2] clarifies the relation of the Hirzebruch class with the mixed Hodge structure. The general point of view was presented in [BSY10, §4], but the considered case is fairly explicit. More general approach reletting mixed Hodge modules and theory of characteristic classes is described in the survey [Sch11].

Proof of Theorem 5.1. We will skip ι in the notation. The sheaf of logarithmic differential forms is equipped with the weight filtration. The associated quotients of the weight filtration in $L\Omega^p_\emptyset = \Omega^p_M(\log D)$ are equal to the sheaves of forms on the intersections of divisor components (with shifted gradation), therefore we have

$$L\Omega^y_\emptyset = \sum_{I\subset\{1,2,\dots,m\}} y^{|I|}\,\Omega^y_{D_I}\,. \tag{10}$$

Similarly

$$L\Omega^y_J = \sum_{I\supset J} y^{|I|-|J|}\,\Omega^y_{D_I}\,. \tag{11}$$

From the equations (10) and (11) we find that

$$\Omega^y_J = \sum_{I\supset J}(-y)^{|I|-|J|}\, L\Omega^y_{D_I}\ . \tag{12}$$

By the inclusion-exclusion formula we have

$$\Omega^y_{X_\bullet} = \sum_{J\subset\{1,2,\dots,m\}} (-1)^{|J|}\Omega^y_{D_J} = \sum_{J\subset\{1,2,\dots,m\}} (-1)^{|J|}\sum_{I\supset J}(-y)^{|I|-|J|}\, L\Omega^y_{D_I} =$$

$$= \sum_{I\subset\{1,2,\dots,m\}} \left(\sum_{J\subset I}(-1)^{|J|}\,(-y)^{|I|-|J|}\right) L\Omega^y_{D_I} =$$

$$= \sum_{I\subset\{1,2,\dots,m\}} \left(\sum_{k=0}^{|I|}\binom{|I|}{k}(-1)^k\,(-y)^{|I|-k}\right) L\Omega^y_{D_I} = \sum_{I\subset\{1,2,\dots,m\}} (-1-y)^{|I|} L\Omega^y_{D_I}\,.$$

□

Example 5.3. If $D\subset M$ is a smooth divisor then we have the residue exact sequences

$$0\to\Omega^*_M\hookrightarrow\Omega^*_M(\log D)\stackrel{res}{\to}\Omega^{*-1}_D\to 0\,.$$

Therefore in K-theory we have

$$L\Omega^y_\emptyset = \Omega^y_M(\log D) = \Omega^y_M + y\Omega^y_D\,. \tag{13}$$

The decomposition of the Theorem 5.1 takes form of the sum of two components, logarithmic and residual part:

$$\Omega^y_{X_\bullet} = L\Omega^y_\emptyset + \delta L\Omega^y_{\{1\}}\,,$$

Indeed

$$L\Omega^y_\emptyset \oplus \delta L\Omega^y_{\{1\}} = \Omega^y_M + y\Omega^y_D - (1+y)\Omega^y_D = \Omega^y_M - \Omega^y_D\,.$$

Let us now derive from formula (9) the Aluffi expression $c^{SM}(1\!\!1_X)$ via logarithmic tangent bundle.

Corollary 5.4. *The formula (9) specializes to the Aluffi formula* [Alu99] *for Chern-Schwartz-MacPherson class*

$$c^{SMC}(1\!\!1_X) = c(TM(-\log D))$$

by taking $y=-1$, that is $\delta=0$.

Proof. Suppose x_i for $i=1,2,\dots,\dim(M)$ are the Chern roots of TM and ξ_i for $i=1,2,\dots,\dim(M)$ are the Chern roots of $TM(-\log D)$. By (9)

$$\widetilde{td}_y(X\hookrightarrow M) = \prod_{i=1}^{\dim(M)} \frac{x_i}{1-e^{-\delta x_i}}\cdot e^{-\delta[D]}\cdot \prod_{i=1}^{\dim(M)}(1-(1+d)e^{\delta\xi_i})\,.$$

Let us compute what is the limit of this characteristic class with $\delta\to 0$. We check that

$$\lim_{\delta\to 0} e^{-\delta[D]} = 1\,,\qquad \lim_{\delta\to 0}\frac{x}{1-e^{\delta x}}(1-(1+\delta)\,e^{\delta\xi}) = 1+\xi\,.$$

Therefore

$$\lim_{y\to -1} \widetilde{td}_y(M) = \prod_{i=1}^{\dim(M)} (1+\xi_i) = c(TM(-\log D))\,.$$

□

Remark 5.5. We can easily derive the Aluffi formula from (7) and see which summands contribute to the Chern-Schwartz-MacPherson class. Indeed

$$\widetilde{td}_y(X \hookrightarrow M) = \sum_{I\subset\{1,2,\dots,m\}} \delta^{|I|}\, \widetilde{td}(M)\, ch(\Omega^y_{D_I}(\log D_{>I}))\,, \tag{14}$$

and $ch(\Omega^y_M(\log D))$ is the product of the factors $1 + y\, e^{-(y+1)\xi_i} = 1-(1+\delta)e^{\delta\xi_i}$. It follows that

$$\lim_{\delta\to 0} \widetilde{td}(M)\, ch(\Lambda_y T^*M(-\log D)) = c(TM(-\log D))$$

The remaining summands in the formula (14) vanish in the limit since they converge to

$$\lim_{\delta\to 0} \delta^{|I|}\widetilde{td}(D_I)\, ch(\Omega^y D_I(\log D_{>I})) = 0^{|I|} c(TD_I(-\log D_{>I})) = 0$$

for $|I| > 0$.

Let us examine the specialization $y = 0$.

Corollary 5.6. *With the introduced notation for a simple normal crossing divisor complement $X = M \setminus D$ we have*

$$td_0(X \hookrightarrow M) = td(M)\, ch(\mathcal{O}_M(-D)) = td(M)e^{-D}\,.$$

Proof. If $y = 0$, then $\delta = -1$ and

$$\Omega^y_{X_\bullet} = \sum_{I\subset\{1,2,\dots,m\}} \delta^{|I|}\, L\Omega^y_I = \sum_{I\subset\{1,2,\dots,m\}} (-1)^{|I|}\mathcal{O}_{D_I} = \mathcal{O}_M - \mathcal{O}_D$$

and from the exact sequence

$$0 \to \mathcal{O}_M(-D) \to \mathcal{O}_M \to \mathcal{O}_D \to 0$$

we obtain the result. □

Note that we have

$$td_0(D \hookrightarrow M) = td(M)(1 - ch(\mathcal{O}_M(-D))) = td(M)ch(\mathcal{O}_D)\,,$$

which agrees with the image of the Baum-Fulton-MacPherson class of D in $H_*(M)$, see [Ful98, 18.3.5]. This is not always the case. Only the varieties with mild singularities have this property, see §14. We will show various explicit examples after having introduced the local Hirzebruch class for varieties with torus action.

6. Localization of equivariant homology

Our goal is to study singularities locally, but the characteristic classes are global objects. Assume that an algebraic group is acting on an algebraic variety and a singular point is fixed. Then when we localize the characteristic class at this point, we obtain some nontrivial information. There is a technical inconvenience: the Hirzebruch class is defined in the homology of the target space M. If M is smooth, then homology can be replaced by cohomology, and its equivariant version is well developed and widely known. If M is singular, the Hirzebruch class naturally lives in equivariant homology. This theory is less developed but it is present in the literature ([BL94, GKM98, BZ03]). Equally well one can work with equivariant Chow groups [EG98]. Another definition of equivariant homology can be found in [AFP14, §3.3]. We will briefly recall the general theory not assuming that M is smooth. We will concentrate on the case when a torus is acting.

The definition of equivariant homology by approximation given in [BZ03] or in [EG98] is the most convenient for us. We fix a decomposition of the torus $\mathbb{T} = (\mathbb{C}^*)^r$. The universal $\mathbb{T}$-bundle $E\mathbb{T} \to B\mathbb{T}$ is approximated by the sequence of finite dimensional algebraic varieties

$$E\mathbb{T}_m = (\mathbb{C}^{m+1} \setminus \{0\})^r \to B\mathbb{T}_m = (\mathbb{P}^m)^r \,.$$

The equivariant homology of a $\mathbb{T}$-variety M is defined by

$$H_{\mathbb{T},k}(M) = \lim_{\substack{\longleftarrow \\ m}} H^{BM}_{k+2rm}(E\mathbb{T}_m \times_{\mathbb{T}} M) \,.$$

The limit is taken with respect to the Gysin maps

$$H^{BM}_{k+2rm}(E\mathbb{T}_m \times_{\mathbb{T}} M) \to H^{BM}_{k+2r(m-1)}(E\mathbb{T}_{m-1} \times_{\mathbb{T}} M) \,,$$

and for a fixed gradation it stabilizes. The equivariant homology may be nonzero in gradations $\leq 2\,\mathrm{dim} M$, also in negative gradations. The equivariant homology is isomorphic to the equivariant cohomology with coefficient in the dualizing sheaf $\mathcal{D}_M$, which is an equivariant sheaf in the sense of [BL94]. Equivariant homology is a module over the equivariant cohomology of the point

$$H^*_{\mathbb{T}}(pt) = H^*(B\mathbb{T}) = Sym^*(\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q})$$

via the pullback $H^*(B\mathbb{T}) \to H^*(B\mathbb{T}_m) \to H^*(E\mathbb{T}_m \times_{\mathbb{T}} M)$ composed with the cap product. If M is smooth, then $\mathcal{D}_M = \mathbb{C}_M[2\,\mathrm{dim} M]$ and

$$\cap[M] : H^{2\,\mathrm{dim} M - k}_{\mathbb{T}}(M) \to H_{\mathbb{T},k}(M)$$

is an isomorphism. The localization theorem as stated in [GKM98, Theorem 6.2] says that up to a $H^*_{\mathbb{T}}(pt)$-torsion all the information about $H_{\mathbb{T},k}(M)$ is encoded at the fixed points: the restriction map

$$H_{\mathbb{T},*}(M) = H^{-*}_{\mathbb{T}}(M; \mathcal{D}_M) \to H^{-*}_{\mathbb{T}}(M^{\mathbb{T}}; (\mathcal{D}_M)_{|M^{\mathbb{T}}}) \,,$$

has torsion kernel and cokernel. The image is of special interest when the fixed points are isolated. Nevertheless for singular variety the groups $H^{-*}_{\mathbb{T}}(\{p\}; (\mathcal{D}_M)_{|p}) = H_{\mathbb{T},*}(M, M \setminus \{p\})$ might be complicated.

Example 6.1. Suppose that $\mathbb{T}=\mathbb{C}^*$ and $p\in M$ is an isolated fixed point. Let U be a conical neighbourhood of p, which is invariant with respect to $S^1\subset\mathbb{C}^*$. Then

$$H^{-*}_{\mathbb{T}}(\{p\};(\mathcal{D}_M)_{|p})=H_{\mathbb{T},*}(\overline{U},\partial U)\,.$$

The exact sequence of the pair $(\overline{U},\partial U)$ gives us some information about the cohomology of the point with coefficients in $\mathcal{D}_M$. In this exact sequence we have $H_{\mathbb{T},*}(\overline{U})=H^{-*}_{\mathbb{T}}(pt)$ and $H_{\mathbb{T},*}(\partial U)=H_*(\partial U/S^1)$, since the action of S^1 on ∂U has finite isotropy groups. For $k\leq \dim M$ we have

$$0\to H_{\mathbb{T},2k+1}(\overline{U},\partial U)\to H_{2k}(\partial U/S^1)\to\mathbb{Q}\to H_{\mathbb{T},2k}(\overline{U},\partial U)\to H_{2k-1}(\partial U/S^1)\to 0\,.$$

In general if the fixed points are isolated, then the difference between $H^*_{\mathbb{T}}(\{p\})$ and $H^*_{\mathbb{T}}(\{p\};(\mathcal{D}_M)_{|p})$ is measured by $H_{\mathbb{T},*}(\partial U;(\mathcal{D}_M)_{|\partial U})$. This is a torsion $H^*_{\mathbb{T}}(pt)$-module since there are no fixed points of $\mathbb{T}$ on ∂U. Finally, the localization theorem [GKM98, Theorem 6.2] for isolated fixed points and the dualizing sheaf $\mathcal{D}_M$ takes form:

Theorem 6.2 (Localization in equivariant homology)**.** *For any $\mathbb{T}$-variety with isolated fixed points*

$$\mathcal{S}^{-1}H_{\mathbb{T},*}(M)\simeq\bigoplus_{p\in M^T}\mathcal{S}^{-1}H^{-*}_{\mathbb{T}}(p)\,,$$

where $\mathcal{S}$ is the multiplicative system generated by nonzero characters

$$w\in H^2_{\mathbb{T}}(pt)=Hom(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}\,.$$

The situation simplifies when M is smooth. One can explicitly write down the inverse map of the restriction $H^*_{\mathbb{T}}(M)\to H^*_{\mathbb{T}}(M^{\mathbb{T}})$. Then $H_{\mathbb{T},*}(M,M\setminus\{p\})$ is a free $H^*_{\mathbb{T}}(pt)$ module with one generator in the gradation $2\dim M$. Let us identify $H_{\mathbb{T},*}(M,M\setminus\{p\})$ with $H^*_{\mathbb{T}}(pt)$ by Poincaré duality. In addition, when M is smooth and complete then $H_{\mathbb{T},*}(M)=H^{2\dim M-*}_{\mathbb{T}}(M)$ is a free module over $H^*_{\mathbb{T}}(pt)$ ([GKM98, Theorem 14.1]), hence the restriction to the fixed points is injective. Atiyah-Bott formula [AB84, Formula 3.8], [BV82] (or [EG98] for Chow groups) is a recipe how to recover the class from its restriction to the fixed points:

Theorem 6.3 (Localization Formula)**.** *Let M be a smooth and complete algebraic variety. Assume that $M^{\mathbb{T}}$ is discrete and let $i_p:\{p\}\to M$ be the inclusion. Then for any equivariant class $a\in H^*_{\mathbb{T}}(M)$*

$$a=\sum_{p\in M^{\mathbb{T}}}i_{p*}\frac{i_p^*(a)}{eu(p)}\,,$$

where $eu(p)\in H^{2\dim M}_T(\{p\})$ is the product of weights appearing in the tangent representation T_pM.

The resulting *Integration Formula* for isolated fixed points takes form

$$\int_M a=\sum_{p\in M^{\mathbb{T}}}\frac{i_p^*(a)}{eu(p)} \tag{15}$$

Remark 6.4. Some generalizations for singular spaces are available, but additional assumption about the fixed point set are needed [EG98, Proposition 6].

7. Equivariant version of Hirzebruch class

Similarly to [BZ03, Ohm06] we propose a definition of the equivariant Hirzebruch class of an equivariant map $X \to M$. It is based on the observation that the varieties $B\mathbb{T}_m = (\mathbb{P}^m)^r$ which approximates $B\mathbb{T}$ and $E\mathbb{T}_m \times_{\mathbb{T}} X$ approximating Borel construction have their own Hirzebruch classes.

Definition 7.1 (Of the equivariant Hirzebruch class)**.**

$$td_y^{\mathbb{T}}(X \to M) = \lim_{\substack{\longleftarrow \\ m}} \left(td_y(E\mathbb{T}_m \times_{\mathbb{T}} X \to E\mathbb{T}_m \times_{\mathbb{T}} M) \cap (\pi^* td_y(B\mathbb{T}_m))^{-1}\right) ,$$

where $\pi : E\mathbb{T}_m \times_{\mathbb{T}} M \to B\mathbb{T}_m$ is the canonical projection. Note that since we pass to the limit with finite dimensional skeleta of $B\mathbb{T}$, we obtain a class which might be nontrivial in infinitely many gradations:

$$td_y^{\mathbb{T}}(X \to M) \in \prod_{k=-\infty}^{2\dim M} \Big(H_{\mathbb{T},k}(M) \otimes \mathbb{Q}[y]\Big) .$$

We omitting completion in the notation and write $H_{\mathbb{T},*}(X)$ for this group.

The limit in the definition stabilizes and for a bounded range of gradations. It is enough to perform all computations for a sufficiently large m. If M is smooth then

$$H_{\mathbb{T},2\dim M-*}(X) \simeq H^*_{\mathbb{T}}(M) = \prod_{k=0}^{\infty} H^k_{\mathbb{T}}(M)$$

by Poincaré duality. If $X = M$ then the Hirzebruch class $td_y(X \to M) \in H^*_{\mathbb{T}}(M)[y]$ can be computed as the characteristic class of the equivariant tangent bundle. Note that only finitely many powers of y appear in the expression for that characteristic class. For arbitrary singular X the class $td_y^{\mathbb{T}}(X \to M)$ is a combination of finitely many classes $td_y^{\mathbb{T}}(X_i \to M)$ for smooth X_i, therefore also in that case only finitely many powers of y appear. This means that

$$td_y^{\mathbb{T}}(X \to M) \in \left(\prod_{k=-\infty}^{2\dim M} H_{\mathbb{T},k}(M)\right) \otimes \mathbb{Q}[y] .$$

If $M = pt$ we do not obtain any additional information. The action of $\mathbb{T}$ is not reflected by χ_y-genus:

Theorem 7.2 (Rigidity for singular varieties)**.** *The equivariant Hirzebruch class $td_y^{\mathbb{T}}(X \to pt) \in H_{\mathbb{T},*}(pt)[y]$ is trivial in nonzero gradations and*

$$td_y^{\mathbb{T}}(X \to pt) = \chi_y(X) \in H_{\mathbb{T},0}(pt)[y] = \mathbb{Q}[y] .$$

Remark 7.3. This property for smooth varieties was studied in [Mus11, Tot07]. The series $td_y(x) \in \mathbb{Q}[[x]]$ gives rise to an universal rigid characteristic class. To prove that $td_y(x)$ is universal suppose a formal series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ defines a rigid characteristic class, i.e. the equivariant f-genus of a smooth variety X vanishes in positive gradations. Rigidity implies some relations between coefficients of f. It is easy to check, that setting $X = \mathbb{P}^1$ or $\mathbb{P}^2$ with the standard torus actions the conditions imposed on f leave only three degrees of freedom: f is determined by the coefficients a_0, a_1 and a_2. It follows that up to a constant or re-scaling x the series $f(x)$ has to be equal $td_{y_0}(x)$ for some $y_0 \in \mathbb{Q}$.

We derive rigidity for singular algebraic varieties from the motivic properties of the Hirzebruch class.

Proof of Theorem 7.2. The canonical projection $E\mathbb{T}_m \times_{\mathbb{T}} X \to E\mathbb{T}_m \times_{\mathbb{T}} pt = B\mathbb{T}_m$ is a locally trivial fibration in Zariski topology therefore in $K(Var/B\mathbb{T}_m)$ the classes $[E\mathbb{T}_m \times_{\mathbb{T}} X \to B\mathbb{T}_m]$ and $[B\mathbb{T}_m \times X \to B\mathbb{T}_m]$ are equal. By the product property of the Hirzebruch class we have

$$td_y(E\mathbb{T}_m \times_{\mathbb{T}} X \to B\mathbb{T}_m) = \chi_y(X) td_y(B\mathbb{T}_m)\,.$$

The conclusion follows. □

We note that if X admits a decomposition into affine spaces as often happens for $\mathbb{T}$-varieties, then $td_y(X \to pt)$ is easy. Namely:

$$td_y^{\mathbb{T}}(X \to pt) = \chi_y(X) = \sum_{i=0}^{\dim X} a_i(-y)^i\,,$$

where a_i is the number of i dimensional cells. Similarly when X is a toric variety, then

$$td_y^{\mathbb{T}}(X \to pt) = \chi_y(X) = \sum_{i=0}^{\dim X} b_i(-(y+1))^i\,,$$

where b_i is the number of i dimensional orbits. On the other hand the local contribution coming from a singular point to the global class might be fairly complicated.

Suppose, that M is smooth and complete. Assume, that $\mathbb{T}$ has only a finite number of fixed points on M. Then according to Atiyah-Bott or Berline-Vergne formula (15)

$$\chi_y(X) = \sum_{p \in M^{\mathbb{T}}} \frac{td_y^{\mathbb{T}}(f : X \to M)_{|p}}{eu(p)} = \sum_{p \in M^{\mathbb{T}}} \frac{ch^{\mathbb{T}}(mC_*(f : X \to M))_{|p}}{ch^{\mathbb{T}}(\mathcal{O}_{\{p\}})}\,. \tag{16}$$

As it was explained in the introduction, instead of computations in cohomology we can apply localization theorem for equivariant K-theory, [Seg68]. This alternative point of view does not influence the computations, which are done in the ring of Laurent series.

Let us give an example which shows how the local Hirzebruch class at a singular point can be computed by global Localization Theorem 1.2, provided that at the remaining fixed points the variety are smooth. In certain cases this method can be applied in much more general situations. It was applied in [Web12] to compute Chern-MacPherson-Schwartz classes of determinantal variety.

Example 7.4. Consider the torus $\mathbb{T} = (\mathbb{C}^*)^2$ acting on $\mathbb{P}^3$ by the formula

$$(\xi_1, \xi_2) \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : \xi_1^2 x_1 : \xi_2^2 x_2 : \xi_1 \xi_2 x_3] .$$

The action preserves the projective cone over the quadric

$$X = \{[x_0 : x_1 : x_2 : x_3] \subset \mathbb{P}^3 \,|\, x_1 x_2 - x_3^2 = 0\} .$$

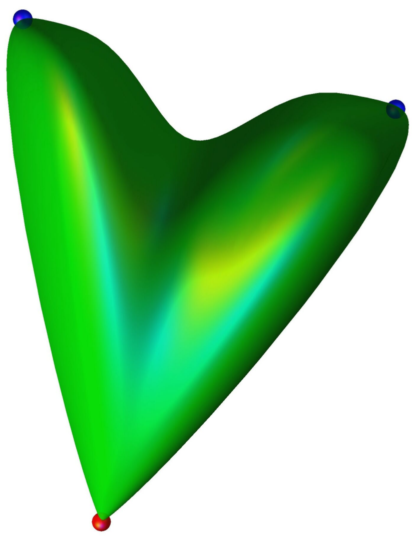
Since X has a decomposition into affine cells it is immediate to compute the χ_y-genus

$$\chi_y(X) = 1 - y + y^2 .$$

The variety X has three fixed points: $[0 : 1 : 0 : 0]$ and $[0 : 0 : 1 : 0]$ are smooth and $[1 : 0 : 0 : 0]$ is singular (the A_1 singularity). Let us denote the characters $\mathbb{T} \to \mathbb{C}^*$

$$t_1 : (\xi_1, \xi_2) \mapsto \xi_1, \qquad t_2 : (\xi_1, \xi_2) \mapsto \xi_2 .$$

In the standard affine neighbourhood of the point $[0 : 1 : 0 : 0]$ there are the coordinates $\frac{x_0}{x_1}, \frac{x_2}{x_1}, \frac{x_3}{x_1}$. The tangent representation of $\mathbb{T}$ at $[0 : 1 : 0 : 0]$ has the characters $-2t_1\,,\ 2(t_2 - t_1)\,,\ t_2 - t_1$. The weight at the point $[0 : 0 : 1 : 0]$ differ by a switch of subscripts. The equation of X is $\frac{x_2}{x_1} + \left(\frac{x_3}{x_1}\right)^2 = 0$. The linear part has weight $2(t_2 - t_1)$, therefore $[X]_{|[0:1:0:0]} = 2(t_2 - t_1)$.



The Hirzebruch class at the point $[0 : 1 : 0 : 0]$ is equal to

$$td_y^{\mathbb{T}}(X)_{|[0:1:0:0]} = \frac{-2t_1(1 + ye^{2t_1})}{1 - e^{2t_1}} \cdot \frac{(t_2 - t_1)(1 + ye^{t_1 - t_2})}{1 - e^{t_1 - t_2}} .$$

The image under the inclusion into $M=\mathbb{P}^3$ is equal

$$td_y^{\mathbb{T}}(X\to M)_{|[0:1:0:0]}=2(t_2-t_1)\frac{-2t_1(1+ye^{2t_1})}{1-e^{2t_1}}\cdot\frac{(t_2-t_1)(1+ye^{t_1-t_2})}{1-e^{t_1-t_2}}\,.$$

Similarly we compute $td_y^{\mathbb{T}}(X)_{|[0:0:1:0]}$. The variables t_i cancel out in the formula (16), it remains only $T_i=e^{-t_i}$. After this substitution we have

$$y^2-y+1=$$
$$=\frac{\left(1+y\frac{1}{T_1{}^2}\right)\left(1+y\frac{T_2}{T_1}\right)}{\left(1-\frac{1}{T_1{}^2}\right)\left(1-\frac{T_2}{T_1}\right)}-\frac{\left(1+y\frac{1}{T_2{}^2}\right)\left(1+y\frac{T_1}{T_2}\right)}{\left(1-\frac{1}{T_2{}^2}\right)\left(1-\frac{T_1}{T_2}\right)}+\frac{ch^{\mathbb{T}}(mC_*(f:X\to M))_{|[0:0:0:1]}}{(1-T_1^2)(1-T_2^2)(1-T_1T_2)}\,.$$

Simplifying the expression we find the formula for $ch^{\mathbb{T}}(mC_*(f:X\to M))_{|[0:0:0:1]}$:

$$(1-T_1{}^2T_2{}^2)+y(T_1+T_2)^2(1-T_1T_2)+y^2T_1T_2(1-T_1{}^2T_2{}^2) \tag{17}$$

In particular here for $y=0$

$$ch^{\mathbb{T}}(mC_0(f:X\to M))=ch^{\mathbb{T}}(1-\mathcal{O}(-X))=ch^{\mathbb{T}}(\mathcal{O}_X).$$

8. Local Hirzebruch class for SNC divisor

We want to describe $td_y^{\mathbb{T}}(X\hookrightarrow M)$ of a singularity germ of a subvariety $X\subset M$. Let us concentrate on the case when M is smooth and X is an open subset. To compute this local invariant directly we resolve singularities, that is we find a proper map $f:\widetilde{M}\to M$ such that $X\simeq\widetilde{X}=f^{-1}(X)$ and $\widetilde{M}\setminus\widetilde{X}$ is a simple normal crossing divisor. Then $td_y^{\mathbb{T}}(X\hookrightarrow M)=f_*(td_y^{\mathbb{T}}(\widetilde{X}\hookrightarrow\widetilde{M}))$. Therefore it is crucial to understand the situation when $D=M\setminus X$ already is SNC divisor.

The local Hirzebruch class is invariant with respect to analytic changes of coordinates, so we can assume that we have coordinates preserved by the torus action. The divisor D which is a union of coordinate hyperplanes defined by the equation $\prod_{i=1}^k x_i=0$. The weights of $\mathbb{T}$ acting on coordinates will be denoted by $w_1,w_2,\dots,w_n$. It is immediate to write down the Hirzebruch classes of the basic constructible sets:

Proposition 8.1. *Let $D=\{0\}\subset\mathbb{C}=M$, $X=\mathbb{C}\setminus D$, and let $\mathbb{T}=\mathbb{C}^*$ acts on $\mathbb{C}$ with the weight w. Set $\delta=-1-y$, $S=e^{-iw}$. Then*

$$td_y^{\mathbb{T}}(D\hookrightarrow\mathbb{C})_{|0}=w \tag{18}$$

$$td_y^{\mathbb{T}}(\mathbb{C})_{|0}=w\frac{1+y\,e^{-w}}{1-e^{-w}}=w\frac{\delta+S+\delta S}{S} \tag{19}$$

$$td_y^{\mathbb{T}}(X\hookrightarrow\mathbb{C})_{|0}=w\frac{(1+y)\,e^{-w}}{1-e^{-w}}=w\frac{\delta(1+S)}{S} \tag{20}$$

$$td^{\mathbb{T}}(\mathbb{C})ch^{\mathbb{T}}(L\Omega^y_\emptyset)_{|0}=w\frac{1+y}{1-e^{-w}}=w\frac{\delta}{S} \tag{21}$$

Proof. The formula (18) holds since $w \in H^2(B\mathbb{T}) = Hom(\mathbb{T}, \mathbb{C})$ is the Euler class of the normal bundle of D. The formula (19) is by the straight forward substitution. To prove (20) we use additivity.

$$td_y^{\mathbb{T}}(X \hookrightarrow \mathbb{C})_{|0} = td_y^{\mathbb{T}}(\mathbb{C})_{|0} - td_y^{\mathbb{T}}(D \hookrightarrow \mathbb{C})_{|0} = w\left(\frac{\delta + S + \delta S}{S} - 1\right) = w\frac{\delta(1+S)}{S}\,.$$

The formula (21) follows from (13)

$$\begin{aligned} td^{\mathbb{T}}(\mathbb{C})ch^{\mathbb{T}}(L\Omega^y_{\emptyset})_{|0} &= td_y^{\mathbb{T}}(\mathbb{C})_{|0} + y\, td_y^{\mathbb{T}}(D \hookrightarrow \mathbb{C})_{|0} \\ &= w\left(\frac{\delta + S + \delta S}{S} - (\delta+1)\right) = w\frac{\delta}{S} \end{aligned}$$

□

By the product property of the Hirzebruch class we obtain

Corollary 8.2. *Let $D = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid \prod_{i=1}^k x_i = 0\}$ be a simple normal crossing divisor and $X = M \setminus D$. Set $S_i = e^{-w_i} - 1$ and $\delta = -1 - y$, where w_i is the weight of $\mathbb{T}$ acting on i-th coordinate. For arbitrary n and k we have*

$$td_y^{\mathbb{T}}(\mathbb{C}^n)_{|0} = eu(0)\prod_{i=1}^n \frac{\delta + S_i + \delta S_i}{S_i} \tag{22}$$

$$td_y^{\mathbb{T}}(X \hookrightarrow \mathbb{C}^n)_{|0} = eu(0)\delta^k \prod_{i=1}^k \frac{1+S_i}{S_i} \prod_{j=k+1}^n \frac{\delta + S_j + \delta S_j}{S_j} \tag{23}$$

$$td^{\mathbb{T}}(\mathbb{C}^n)ch^{\mathbb{T}}(L\Omega^y_{\emptyset})_{|0} = eu(0)\delta^k \prod_{i=1}^k \frac{1}{S_i} \prod_{j=k+1}^n \frac{\delta + S_j + \delta S_j}{S_j} \tag{24}$$

Multiplying by $\frac{1}{eu(0)}\prod_{i=1}^n S_i = (-1)^n td^{\mathbb{T}}(\mathbb{C}^n)^{-1}$ we obtain an expression for

$$(-1)^n ch^{\mathbb{T}} mC_*(X \hookrightarrow M)_{|0}$$

which is a polynomial with nonnegative coefficients in δ and S_i. We will examine various examples and we will see that the positivity in the S_i and δ variables is preserved for a large class of singularities.

9. Whitney umbrella: an example of computation via resolution

Example 9.1. Consider the torus $\mathbb{T} = (\mathbb{C}^*)^2$ acting on $\mathbb{C}^3$ with weights

$$w_1 = t_1 + t_2\,,\; w_2 = t_1\,,\; w_3 = 2t_2\,.$$

The weights are nonnegative combinations of the weights t_i. It follows that for any invariant subvariety its fundamental class in equivariant cohomology is a nonnegative combination of the monomials in t_i (Theorem 3.1). We will observe a similar effect for the Hirzebruch class of the Whitney umbrella

$$X = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_1^2 - x_2^2 x_3 = 0\}\,.$$

Let
$$Z=\{(x,y,z)\in\mathbb{C}^3\,|\,x=0,y=0\}\quad\text{and}\quad X^o=X\setminus Z\,.$$
and let
$$f:\widetilde{X}=\mathbb{C}^2\to\mathbb{C}^3\,,\quad f(u,v)=(uv,u,v^2)\,,$$
be the resolution of the Whitney umbrella, $\widetilde{X}^o=f^{-1}X^o=\{u\neq 0\}$. We have
$$\begin{aligned}td^{\mathbb{T}}_y(X\to\mathbb{C}^3)&=td^{\mathbb{T}}_y(X^o\to\mathbb{C}^3)+td^{\mathbb{T}}_y(Z\to\mathbb{C}^3)\\&=f_*td^{\mathbb{T}}_y(X^o\to\mathbb{C}^2)+td^{\mathbb{T}}_y(Z\to\mathbb{C}^3)\\&=2(t_1+t_2)\frac{t_1(1+y)\,e^{-t_1}}{1-e^{-t_1}}\frac{t_2(1+y\,e^{-t_2})}{1-e^{-t_2}}+(t_1+t_2)t_1\frac{2t_2(1+y\,e^{-2t_2})}{1-e^{-2t_2}}\,.\end{aligned}$$
In the variables $T_i=e^{-t_i}$
$$\frac{td^{\mathbb{T}}_y(X\to\mathbb{C}^3)}{2t_1t_2(t_1+t_2)}=\frac{1+T_1T_2+y(T_1+2T_1T_2+T_2^2)+y^2(T_1T_2+T_1T_2^2)}{(1-T_1)(1-T_2^2)}$$
and in the variables S_i
$$\frac{S_1S_2(2+S_2)+\delta\cdot(S_1+2S_2+4S_1S_2+S_2^2+2S_1S_2^2)+\delta^2(1+S_1)(1+S_2)(2+S_2)}{S_1(S_2^2+S_2)}\,.$$
For the complement of the Whitney umbrella we obtain
$$\frac{\delta(1+S_1)(1+S_2)\left(S_1S_2(2+S_2)+\delta S_2\,(1+3S_1+S_2+2S_1S_2)+\delta^2(1+S_1)(1+S_2)^2\right)}{(S_1S_2+S_1+S_2)S_1(S_2^2+S_2)}$$
The expressions have nonnegative coefficients and multiplying by the factor
$$(S_1S_2+S_1+S_2)S_1(S_2^2+S_2)=(-1)^3td^{\mathbb{T}}(\mathbb{C}^3)^{-1}$$
we obtain the formula for $mC_*(X\hookrightarrow\mathbb{C}^3)$ with globally predicted signs $(-1)^3$.

There is another issue which is the same as in the normal crossing case: for $y=0$ the Todd class of the Whitney umbrella is equal to
$$td^{\mathbb{T}}(\mathbb{C}^3)\cdot(1-e^{2(t_1+t_2)})=td^{\mathbb{T}}(\mathbb{C}^3)\cdot ch^{\mathbb{T}}(\mathcal{O}_X)\,.$$

10. Conical singularities

First let us recall the basic calculational properties of Todd class. Stably the tangent bundle $T\mathbb{P}^{n-1}$ is equal to $\mathcal{O}(1)^{\oplus n}$ therefore $td(\mathbb{P}^{n-1})=\left(\frac{h}{1-e^{-h}}\right)^n$, where $h=\mathcal{O}(1)$. We have
$$1=\int_{\mathbb{P}^{n-1}}td(T\mathbb{P}^{n-1})=\text{coefficient of }h^{n-1}\text{ in }\frac{h^n}{(1-e^{-h})^n}=Res_{h=0}\frac{1}{(1-e^{-h})^n}$$
For convenience let us set $U=e^{-h}-1$. We have
$$Res_{h=0}\frac{1}{U^n}=(-1)^n$$
for $n>0$.

We note the following easy facts

Lemma 10.1.

$$Res_{h=0}\frac{(1+U)^k}{U^n} = -\binom{k-1}{n-1}.$$

This computation can be done elementary, but having in mind Riemann-Roch theorem and Serre duality we see immediately

$$Res_{h=0}\frac{(1+U)^k}{(-U)^n} = \chi(\mathbb{P}^{n-1};\mathcal{O}(-k)) = (-1)^{n-1}\chi(\mathbb{P}^{n-1};\mathcal{O}(k-n))$$

which is equal to $(-1)^{n-1}h^0(\mathbb{P}^{n-1};\mathcal{O}(k-n)) = \dim\big(Sym^{k-n}(\mathbb{C}^n)\big)$ for large $k \geq n$, but the formula for Euler characteristic holds for all k.

Lemma 10.2. *Let S be an independent variable. For $0 \leq k < n$ we have*

$$Res_{h=0}\frac{(1+U)^{k+1}}{U^n(S-U)} = -\frac{(1+S)^k}{S^n}. \tag{25}$$

Proof.

$$\begin{aligned}
Res_{h=0}\frac{(1+U)^{k+1}}{U^n(S-U)} &= Res_{h=0}\left(\sum_{i=0}^{\infty}\frac{(1+U)^{k+1}}{S^{i+1}U^{n-i}}\right)\\
&= -\sum_{i=0}^{n-1}\binom{k}{n-i-1}\frac{1}{S^{i+1}}\\
&= -\frac{1}{S^n}\sum_{i=0}^{n-1}\binom{k}{n-i-1}S^{n-i-1}\\
&= -\frac{(1+S)^k}{S^n}
\end{aligned}$$

□

The equality (25) for $k \geq n$ is much more complicated, since summing the terms $\binom{k}{n-i-1}S^{n-i+1}$ up to $n-1$ we do not have the full binomial expansion of $(1+S)^k$.

Let $Y \subset \mathbb{P}^{n-1}$ be a subvariety (in fact Y can be any constructible subset) We give a formula for the Hirzebruch class of the affine cone over Y with the vertex removed. Let $\mathbb{T} = \mathbb{C}^*$ acting on $\mathbb{C}^n$ by multiplication of coordinates.

Proposition 10.3. *Let $Y \subset \mathbb{P}^{n-1}$. Suppose that*

$$td_y(Y \hookrightarrow \mathbb{P}^{n-1}) = h^n\frac{f(U)}{U^n}$$

for a polynomial $f \in \mathbb{Z}[y][U]$ which has degree in U less or equal than $n-1$. The Hirzebruch class of the cone without the vertex $X = C_Y \setminus \{0\} \subset \mathbb{C}^n$ is given by the formula

$$td_y^{\mathbb{T}}(X \hookrightarrow \mathbb{C}^n) = t^n\delta\left(\chi_y(Y) - \frac{f(S)}{S^n}\right)$$

where $S = e^{-t}-1$.

This proposition is an extension of the formula for the Chern-Schwartz-MacPherson class given in [Web12], which in turn originates from [AM11, Lemma 3.10].

Proof. For simplicity we assume that $f(U) = (1+U)^k$, for $k = 0, 1, \dots, n-1$. These polynomials form a basis of $H^*(\mathbb{P}^{n-1})$. Let $\widetilde{\mathbb{C}}^n$ denote the blowup of $\mathbb{C}^n$ at 0 and let $E = \mathbb{P}^{n-1}$ be the exceptional divisor, its first Chern class is equal to $t-h$. Denote by $\widetilde{X}$ the proper transform of X in $\widetilde{\mathbb{C}}^n$. Hence

$$td_y^{\mathbb{T}}(\widetilde{X} \hookrightarrow \widetilde{\mathbb{C}}^n)_{|E} = td_y^{\mathbb{T}}(Y \hookrightarrow \mathbb{P}^n) \cdot (td_y^{\mathbb{T}}(N_E\widetilde{\mathbb{C}}^n) - [E]_{|E})\,,$$

and the characteristic class $td_y^{\mathbb{T}}$ applied to the normal bundle of the exceptional divisor $td_y^{\mathbb{T}}(N_E\widetilde{\mathbb{C}}^n)$ is equal to

$$td_y^{\mathbb{T}}(N_E\widetilde{\mathbb{C}}^n) = (t-h)\frac{1+y\,e^{h-t}}{1-e^{h-t}} = (t-h)\frac{1+y(1+S)/(1+U)}{1-(1+S)/(1+U)}\,.$$

As in Example 5.3 we decompose:

$$td_y^{\mathbb{T}}(\widetilde{X} \hookrightarrow \widetilde{\mathbb{C}}^n)_{|E} = [LOG] + [RES]\,,$$

$$[LOG] = td_y^{\mathbb{T}}(Y \hookrightarrow \mathbb{P}^n) \cdot \left(td_y^{\mathbb{T}}(N_E\widetilde{\mathbb{C}}^n) - (1+\delta)[E]_{|E}\right)\,, \tag{26}$$

$$[RES] = td_y^{\mathbb{T}}(Y \hookrightarrow \mathbb{P}^n) \cdot \delta[E]_{|E})\,. \tag{27}$$

Since

$$\frac{1+y(1+S)/(1+U)}{1-(1+S)/(1+U)} - (1-\delta) = \frac{\delta(1+U)}{S-U},$$

the logarithmic part of $td_y^{\mathbb{T}}(\widetilde{X} \hookrightarrow \widetilde{\mathbb{C}}^n)_{|E}$ is equal to

$$[LOG] = \delta(t-h)\frac{1+U}{S-U}td_y^{\mathbb{T}}(Y \hookrightarrow \mathbb{P}^n)$$

and the residual part is just push-forward of the original Hirzebruch class multiplied by δ

$$[RES] = \delta(t-h)\,td_y^{\mathbb{T}}(Y \hookrightarrow \mathbb{P}^n)\,.$$

This decomposition resembles the decomposition of one dimensional space in Proposition 8.2 (21). We will compute the Hirzebruch class using functoriality (5) and the local integration formula

$$\frac{1}{t^n}td_y^{\mathbb{T}}(X \hookrightarrow \mathbb{C}^n)_{|\{0\}} = \int_E \frac{1}{t-h}\left(td_y^{\mathbb{T}}(\widetilde{X} \hookrightarrow \widetilde{\mathbb{C}}^n)_{|E}\right) = \int_{\mathbb{P}^{n-1}} \frac{[LOG]+[RES]}{t-h}$$

The integral of the logarithmic part is computed due to Lemma 10.2:

$$\int_{\mathbb{P}^{n-1}} \frac{[LOG]}{t-h} = \delta\,Res_{h=0}(1+U)^k\frac{1+U}{S-U} = -\frac{(1+S)^k}{S^n}\,.$$

The second ingredient is

$$\int_{\mathbb{P}^{n-1}} \frac{[RES]}{t-h} = \int_E \delta\,td(\mathbb{P}^{n-1})(1+U)^k = \delta\,\chi_y(Y)\,.$$

The formula of the Proposition follows. ∎

Let us concentrate on the case $y = 0$. Now let X be the closed cone over a smooth hypersurface $Y \subset \mathbb{P}^{n-1}$ of degree d. The Todd class of Y is equal to

$$td(\mathbb{P}^{n-1})ch^{\mathbb{T}}(\mathcal{O}_Y) = \frac{h^n}{(-U)^n}(1-(1+U)^d)\,,$$

hence

$$td((\mathbb{P}^{n-1}\setminus Y)\hookrightarrow \mathbb{P}^{n-1}) = \frac{h^n}{(-U)^n}(1+U)^d\,,$$

and from Proposition 10.3 for $d < n$

$$td^{\mathbb{T}}((\mathbb{C}^n\setminus X)\hookrightarrow \mathbb{C}^n) = \frac{t^n}{(-S)^n}(1+S)^d\,.$$

We conclude that

$$td^{\mathbb{T}}(X\hookrightarrow \mathbb{C}^n) = \frac{t^n}{(-S)^n}(1-(1+S)^d)\,.$$

We have used the assumption of Proposition 10.3 that $d < n$, and by Lemma 10.2 the formula holds also for $d = n$.

Corollary 10.4. *Let $d \le n$, then*

$$td^{\mathbb{T}}(X\hookrightarrow \mathbb{C}^n) = td^{\mathbb{T}}(\mathbb{C}^n)\cdot ch^{\mathbb{T}}(\mathcal{O}_X)\,.$$

In fact as explained in [BSY10, Example 3.2] and discussed in §14 the conclusion of Corollary holds for Du Bois singularities.

We note another phenomenon which is valid only for the degree $d = 2$:

Proposition 10.5. *If $X = Q_n \subset \mathbb{C}^n$ is a quadratic cone then both*

$$td^{\mathbb{T}}((\mathbb{C}^n\setminus Q_n)\hookrightarrow \mathbb{C}^n) \quad \text{and} \quad td^{\mathbb{T}}(Q_n\hookrightarrow \mathbb{C}^n)$$

when expanded in the variables S and δ have nonnegative coefficients. The same holds for the logarithmic part of $td^{\mathbb{T}}(\mathbb{C}^n\setminus Q_n\hookrightarrow \mathbb{C}^n)$.

Proof. The proof is by induction. We check directly that for $n > 2$ (we omit straightforward computations, see [MW15])

$$td^{\mathbb{T}}_y(Q_n\to \mathbb{C}^n) = \frac{\delta S(2+S)(\delta+S+\delta S)^{n-2}}{S^n} + (1+\delta)td^{\mathbb{T}}_y(Q_{n-2}\hookrightarrow \mathbb{C}^{n-2})$$

$$\begin{aligned} td^{\mathbb{T}}_y((\mathbb{C}^n\setminus Q_n)\to \mathbb{C}^n) = &\frac{\delta^2(1+S)^2(\delta+S+\delta S)^{n-2}}{S^n}\\ &+(1+\delta)td^{\mathbb{T}}_y((\mathbb{C}^{n-2}\setminus Q_{n-2})\to \mathbb{C}^{n-2})\end{aligned}$$

as functions in S and δ. The positivity of the classes $td^{\mathbb{T}}_y(Q_n\to\mathbb{C}^n)$ and $td^{\mathbb{T}}_y((\mathbb{C}^n\setminus Q_n)\to \mathbb{C}^n)$ follows. The logarithmic part of $td^{\mathbb{T}}_y((\mathbb{C}^n\setminus Q_n)\to\mathbb{C}^n)$ is just the sum of terms with negative exponents of S, therefore it has nonnegative coefficients as well. □

Example 10.6. We have said that for the quadratic cones the class $td_y^{\mathbb{T}}(Q_n \hookrightarrow \mathbb{C}^n)$ is positive in δ and S variables. Another example of a homogenous hypersurface which has the positive Hirzebruch class is the cone over an elliptic curve E:

$$td_y^{\mathbb{T}}(Cone(E) \hookrightarrow \mathbb{C}^3) = \frac{1}{S^2}(3\delta^2 + 3\delta^2 S + S^2)\,,$$

$$td_y^{\mathbb{T}}((\mathbb{C}^3 \setminus Cone(E)) \hookrightarrow \mathbb{C}^3) = \frac{\delta}{S^3}(1+S)(\delta^2 + 2\delta^2 S + 3S^2 + 3\delta S^2 + \delta^2 S^2))\,.$$

All other hypersurfaces in higher dimension spaces or of higher degrees do not have positive Hirzebruch classes.

11. Toric varieties

Let us come back to the general situation considered in §5 of the SNC divisor complement, but with the additional assumption that the fixed points of the torus action are contained in the 0-dimensional stratum of the divisor:

Lemma 11.1. *Suppose that M is a smooth variety of dimension n, $X = M \setminus D$ is the complement of a SNC divisor. Assume*

$$M^{\mathbb{T}} = \bigcup_{|I|=n} D_I$$

with the notation of §5. Then

$$td_y^{\mathbb{T}}(X \hookrightarrow M) = (y+1)^n td^{\mathbb{T}}(X \hookrightarrow M)\,,$$

up to $H^*_{\mathbb{T}}(pt)$-torsion.

Note that we study the Hirzebruch class mainly in two cases:

- global class in $H^*_{\mathbb{T}}(M)$ for M smooth and complete,
- the restriction to a fixed point.

In both situations there is no $H^*_{\mathbb{T}}(pt)$-torsion.

Proof. By Localization Theorem 1.1 it is enough to verify the equality at the fixed points. Indeed, if $p \in D_I = \bigcap_{i\in I} D_i$

$$td_y^{\mathbb{T}}(X \hookrightarrow M)_{|p} = \prod_{i\in I} \frac{w_i(y+1)e^{-w_i}}{1-e^{-w_i}} = (y+1)^n \prod_{i\in I} \frac{w_i e^{-w_i}}{1-e^{-w_i}}$$

by formula (20), where $w_i = [D_i]_{|p}$. □

The above Lemma extends to the case of a general pair (M, D), provided that it has a resolution with desired properties. In particular we obtain for toric varieties:

Corollary 11.2. *Let X be a toric variety, then*

$$td_y^T(id_X) = \sum_{\text{orbits}} (1+y)^{\dim \mathcal{O}_\sigma} td^T(O_\sigma)\,.$$

Proof. For the basic properties of toric varieties we refer the reader to [Ful93]. We argue that for each orbit $\mathcal{O}_\sigma$ we have

$$td_y^T(\mathcal{O}_\sigma \hookrightarrow X) = (1+y)^{\dim \mathcal{O}_\sigma} td^T(O_\sigma)\,.$$

The claim is true for smooth complete toric varieties by Lemma 11.1, since each orbit is a SNC divisor complement in its closure. The general case follows since each toric variety can be completed and resolved in the category of toric varieties. □

The nonequivariant version of Corollary 11.2 appeared in [MS14]. The Euler-Maclaurin formula for the Todd class of a closed orbit was given by many authors, see e.g. [BV82] (for simplicial cones) and by Brylinski-Zhang [BZ03] in general. One has to sum the lattice points in the dual cones

$$td^{\mathbb{T}}(id_{X_\Sigma}) = \sum_{\sigma\in\Sigma'} \sum_{m\in\sigma^\vee} e^{-m}[\mathcal{O}_\sigma]\,,$$

where the sum is taken over the cones of maximal dimension indexed by $\Sigma' \subset \Sigma$. The formula makes sense under the identification of the dual lattice $\mathrm{Hom}(\mathbb{T},\mathbb{C}^*)$ with $H^2_{\mathbb{T}}(pt)$.

To compute the class $td_y^{\mathbb{T}}(\mathcal{O}_\tau \hookrightarrow X)$ of an orbit we have to modify the formula: the closure of the orbit is defined by the fan which combinatorially is the link of τ. From the Hirzebruch class of $\overline{\mathcal{O}_\sigma}$ one has to subtract the classes of the boundary. We obtain:

$$td^{\mathbb{T}}(\mathcal{O}_\tau \hookrightarrow X) = \sum_{\substack{\sigma\,\in\,\Sigma' \\ \sigma\succ\tau}} \sum_{m\in int(\sigma^\vee\cap\tau^\perp)} e^{-m}[\mathcal{O}_\sigma]\,.$$

We sum the contributions coming from orbits to obtain a global formula

$$td_y^{\mathbb{T}}(id_X) = \sum_{\tau\in\Sigma}(1+y)^{\dim\mathcal{O}_\tau} \sum_{\substack{\sigma\,\in\,\Sigma' \\ \sigma\succ\tau}} \sum_{m\in int(\sigma^\vee\cap\tau^\perp)} e^{-m}[\mathcal{O}_\sigma]\,.$$

The local Hirzebruch class is given by a modified Euler-Maclaurin sum:

Theorem 11.3. *The restriction of the Hirzebruch class to the fixed point corresponding to a maximal cone $\sigma \in \Sigma'$ is equal*

$$td_y^{\mathbb{T}}(id_X))|p_\sigma = \sum_{\sigma\succ\tau}(1+y)^{\dim\mathcal{O}_\tau} \sum_{m\in int(\sigma^\vee\cap\tau^\perp)} e^{-m}[\mathcal{O}_\sigma]\,.$$

Example 11.4. Suppose $\mathbb{T} = (\mathbb{C}^*)^4$ acts on $\mathbb{C}^4$ with weights t_3-t_1, t_4-t_1, t_3-t_2, t_4-t_2 (in fact a quotient 3-dimensional torus acts effectively). Let X be defined by the equation $x_1x_4 = x_2x_3$. It is the affine cone over $\mathbb{P}^1\times\mathbb{P}^1 \subset \mathbb{P}^3$. The variety X is an affine toric variety associated to a cone $\sigma_X \in \mathbb{Z}^3$. The dual cone of X

$$\sigma_X^\vee \subset \{(t_1,t_2,t_3,t_4)\in\mathbb{Z}^4 \,:\, t_1+t_2+t_3+t_4=0\} \simeq \mathbb{Z}^3$$

is spanned by the rays t_3-t_1, t_4-t_1, t_3-t_2, t_4-t_2. (This is [Ful93, Example of §1.3, p.18] with the notation $e_1^*=t_3-t_1$, $e_3^*=t_4-t_1$, $e_1^*+e_2^*=t_3-t_2$, $e_2^*+e_3^*=t_4-t_2$.)

We compute Euler-Maclaurin sum for each face of the cone and multiply it by a power of $1+y$. Let $T_{ij}=e^{t_i-t_j}$. The sums are equal to:

- 1 for the vertex.
- For the rays:
$$(1+y)\left(\sum_{i=1}^\infty T_{13}^i+\sum_{i=1}^\infty T_{14}^i+\sum_{i=1}^\infty T_{23}^i+\sum_{i=1}^\infty T_{24}^i\right)= \\ =(1+y)\left(\tfrac{T_{13}}{1-T_{13}}+\tfrac{T_{14}}{1-T_{14}}+\tfrac{T_{23}}{1-T_{23}}+\tfrac{T_{24}}{1-T_{24}}\right).$$
- Similarly for the two dimensional faces
$$(1+y)^2\left(\tfrac{T_{13}}{1-T_{13}}\tfrac{T_{14}}{1-T_{14}}+\tfrac{T_{13}}{1-T_{13}}\tfrac{T_{23}}{1-T_{23}}+\tfrac{T_{14}}{1-T_{14}}\tfrac{T_{24}}{1-T_{24}}+\tfrac{T_{23}}{1-T_{23}}\tfrac{T_{24}}{1-T_{24}}\right).$$
- To compute the sum of the entries in the interior we divide the cone into two simplicial cones. The first one is spanned by t_3-t_1,t_4-t_1,t_3-t_2, the second by t_4-t_2,t_4-t_1,t_3-t_2. The common face is spanned by t_4-t_1,t_3-t_2, see [Ful93, p.49]. The rays of these cones generate the corresponding semigroups of integral points, therefore the summand corresponding to the interior of the cone is equal to
$$(1+y)^3\left(\tfrac{T_{13}}{1-T_{13}}\tfrac{T_{14}}{1-T_{14}}\tfrac{T_{23}}{1-T_{23}}+\tfrac{T_{42}}{1-T_{42}}\tfrac{T_{14}}{1-T_{14}}\tfrac{T_{23}}{1-T_{23}}+\tfrac{T_{14}}{1-T_{14}}\tfrac{T_{23}}{1-T_{23}}\right).$$

Let $S_{ij}=T_{ij}-1$ and substitute $(y+1)$ by $-\delta$. The localized Hirzebruch class of X is equal to

$$1+\delta\left(\frac{1+S_{13}}{S_{13}}+\frac{1+S_{14}}{S_{14}}+\frac{1+S_{23}}{S_{23}}+\frac{1+S_{24}}{S_{24}}\right)+ \tag{28}$$

$$+\delta^2\left(\frac{1+S_{13}}{S_{13}}\frac{1+S_{14}}{S_{14}}+\frac{1+S_{13}}{S_{13}}\frac{1+S_{23}}{S_{23}}+\frac{1+S_{14}}{S_{14}}\frac{1+S_{24}}{S_{24}}+\frac{1+S_{23}}{S_{23}}\frac{1+S_{24}}{S_{24}}\right)+$$

$$+\delta^3\left(\frac{1+S_{13}}{S_{13}}\frac{1+S_{14}}{S_{14}}\frac{1+S_{23}}{S_{23}}+\frac{1+S_{42}}{S_{42}}\frac{1+S_{14}}{S_{14}}\frac{1+S_{23}}{S_{23}}-\frac{1+S_{14}}{S_{14}}\frac{1+S_{23}}{S_{23}}\right).$$

After simplification the last summand is equal to

$$\delta^3\frac{(1+S_{14})(1+S_{23})(S_{13}+S_{24}+S_{13}S_{24})}{S_{13}S_{14}S_{23}S_{24}}.$$

12. A_n SINGULARITIES

We consider here the surface singularities of type A_n they are important but easy from the computational point of view. On one hand they form the first series in the list of simple singularities, on the other hand they are quotient singularities of $\mathbb{C}^2$ by a cyclic group contained in $SL_2(\mathbb{C})$, and finally they admit an action of a 2-dimensional torus and the computations can be done by means to Theorem 11.3.

Let us define A_{n-1} singularity as the quotient $\mathbb{C}^2$ by $\mathbb{Z}_n\subset\mathbb{C}^*$ which acts on $\mathbb{C}^2$ by the formula $\xi(z_1,z_2)=(\xi z_1,\xi^{-1}z_2)$. The ring of invariants is generated by the monomials $x_1:=z_1^n$, $x_2:=z_2^n$, $x_3:=z_1z_2$. Another description of this singularity is given by the zeros of the polynomial

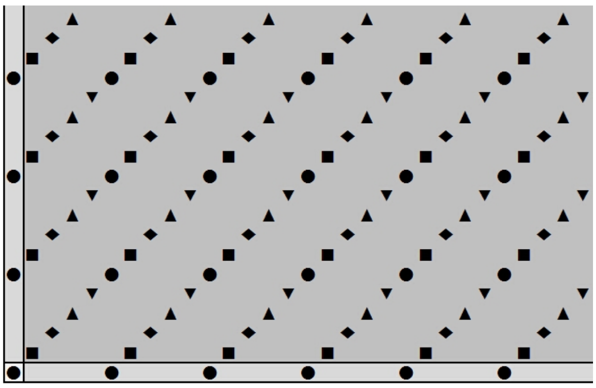
$$X=\{(x_1,x_2,x_3)\in\mathbb{C}^3\,|\,x_1x_2=x_3^n\}$$

in $\mathbb{C}^3$. The torus $\mathbb{T}=(\mathbb{C}^*)^2$ acting on $\mathbb{C}^3$ with weights $w_1=n\,t_1$, $w_2=n\,t_2$, $w_3=t_1+t_2$ preserves X. Let

$$H=\{(t_1,t_2)\in\mathbb{T}\;:\;t_1t_2=1\,,\;t_1^n=1\}\simeq\mathbb{Z}/n\,.$$

The quotient torus $\mathbb{T}'=\mathbb{T}/H$ acts on X effectively. The affine toric variety X corresponds to a rational cone $\sigma\subset Hom(\mathbb{C}^*,\mathbb{T}')\otimes\mathbb{Q})=Hom(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}$. Its dual cone $\sigma^\vee\subset Hom(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}$ is spanned by the weights $w_1=n\,t_1$ and $w_1=n\,t_2$. The lattice in $Hom(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}$ corresponding to the quotient torus is generated by $n\,t_1$, $n\,t_2$ and t_1+t_2. According to Theorem 11.3 we sum up

1	$0-$ dimensional orbit
$(y+1)\sum_{k=1}^\infty e^{-k\,n\,t_1}+(y+1)\sum_{\ell=1}^\infty e^{-\ell\,n\,t_2}$	$1-$ dimensional orbits
$(y+1)^2\sum_{k,\ell}^\infty e^{-k\,n\,t_1+\ell\,n\,t_2}\ \sum_{i=1}^n e^{-i(t_1+t_2)}$	$2-$ dimensional orbit.



We introduce new variables $S_{ni}=e^{-n\,t_i}-1$, $S_{12}=e^{-(t_1+t_2)}-1$ and as always $\delta=-1-y$. Then

$$\frac{td_y^{\mathbb{T}}(X\hookrightarrow\mathbb{C}^3)_{|0}}{eu(0)}=1+\delta\left(\frac{1+S_{n1}}{S_{n1}}+\frac{1+S_{n2}}{S_{n2}}\right)+\delta^2\frac{1}{S_{n1}S_{n2}}\sum_{i=1}^n(S_{12}+1)^i$$

We see that the class of the A_{n-1} singularity in $\mathbb{C}^3$ is of the form $\frac{1}{S_{n1}S_{n2}S_{12}}$ multiplied by a polynomial in δ, S_{n1}, S_{n2} and S_{12} with nonnegative coefficients. One can check that in our case the local Hirzebruch class of the complement is positive as well. Divided by the Euler class it is equal to

$$\delta\frac{1+S_{12}}{S_{12}}+\delta^2\left(\frac{1+S_{n1}}{S_{n1}}\frac{1+S_{n2}}{S_{n2}}+\frac{1+S_{12}}{S_{12}}\right)+\delta^3\frac{1+S_{n1}}{S_{n1}}\frac{1+S_{n2}}{S_{n2}}\frac{1+S_{12}}{S_{12}}$$

The remaining du Val singularities, D_n series and E_n singularities are invariant with respect to actions of one dimensional torus. By a direct computation one finds that the Hirzebruch classes of these singularities are positive in variables $\delta=-1-y$ and $S=T-1$.

13. Positivity for simplicial toric varieties

The method of summation of lattice points applied in the case of A_n can be generalized[2].

Theorem 13.1. *Let X be the toric variety defined by a simplicial cone $\sigma \subset Hom(\mathbb{C}^*, \mathbb{T})$. We fix an equivariant embedding $X \to \mathbb{C}^N$ given by a choice of generators*

$$w_1, w_2, \dots, w_N \in Hom(\mathbb{T}, \mathbb{C}^*) \cap \sigma^\vee .$$

We assume that $w_1, w_2, \dots, w_n$ are the primitive vectors spanning the rays of $\sigma^\vee$. Then the Hirzebruch class of the open orbit $td_y^{\mathbb{T}}(\mathcal{O} \hookrightarrow \mathbb{C}^N)$ is of the form

$$\delta^n \prod_{i=1}^{r} \frac{1}{S_{w_i}} \cdot P(\{S_{w_i}\}_{i=1,2,\dots,N}) ,$$

where $P(\{S_{w_i}\}_{i=1,2,\dots,N})$ is a polynomial expression with nonnegative coefficients depending on the variables $S_{w_i} = e^{-w_i} - 1$.

Proof. The vectors $w_1, w_2, \dots, w_n$ generate a sublattice $\Lambda \subset Hom(\mathbb{T}, \mathbb{C}^*)$. The quotient group $(Hom(\mathbb{T}, \mathbb{C}^*)/\Lambda$ is finite. Let A be the closed cube spanned by $w_1, w_2, \dots, w_n$ and

$$A_0 = A \cap int(\sigma^\vee) \cap Hom(\mathbb{T}, \mathbb{C}^*) .$$

Then

$$Hom(\mathbb{T}, \mathbb{C}^*) \cap int(\sigma^\vee) = \bigsqcup_{w \in A_0} \bigsqcup_{k_1, k_2, \dots, k_n > 0} \left\{ w + \sum_{i=1}^{n} k_j w_i \right\} .$$

The Euler-Maclaurin sum is equal to

$$\sum_{w \in A_0} e^{-w} \prod_{i=1}^{n} \frac{1}{1 - e^{-w_i}} = (-1)^n \sum_{w \in A_0} (S_w + 1) \prod_{j=1}^{n_i} \frac{1}{S_{w_i}} .$$

Since $S_{w+w'} = S_w S_{w'} + S_w + S_{w'}$, the above formula can be rewritten in terms of the generators of $Hom(\mathbb{T}, \mathbb{C}^*) \cap \sigma^\vee$. Applying Theorem 11.3 we arrive to the conclusion. Note that since $y + 1 = -\delta$ the signs cancel. □

If σ is not simplicial, then we can apply the same method of summation. The dual cone $\sigma^\vee$ can be divided into a finite number of open simplicial cones. Some of these the cones will be of lower dimension. Multiplying by $(-\delta)^n$ we do not get rid of sign alternation (see the minus in Example 11.4, formula (28)). Indeed in the dimension four there exist examples of toric singularities with Hirzebruch class, which is not positive. One can check that the cone over the suspension of a pentagon (precisely the cone spanned by $P_1 = (0,0,1,1)$, $P_2 = (1,0,1,1)$, $P_3 = (2,1,1,1)$, $P_4 = (1,2,1,1)$, $P_5 = (0,1,1,1)$, $R_1 = (1,1,2,1)$ and $R_2 = (1,1,0,1)$) has non-positive Hirzebruch class.

Instead in the dimension three we have:

---

[2] I thank Oleg Karpenkov for driving my attention to this method of summation

Proposition 13.2. *Let X be the toric variety defined by a three dimensional cone $\sigma \subset Hom(\mathbb{C}^*, \mathbb{T})$. Suppose that the dual cone is spanned by the primitive vectors $w_1, w_2, \dots, w_k$ and $w_{k+1}, w_{k+2}, \dots, w_N$ are remaining generators of $\sigma^\vee$. Then the Hirzebruch class of the open orbit $td^{\mathbb{T}}_y(\mathcal{O} \hookrightarrow \mathbb{C}^N)$ is of the form*

$$\delta^3 \prod_{i=1}^{k} \frac{1}{S_{w_i}} \cdot P(\{S_{w_i}\}_{i=1,2,\dots,N}) \,,$$

where $P(\{S_{w_i}\}_{i=1,2,\dots,N})$ is a polynomial expression with nonnegative coefficients depending on the variables $S_{w_i} = e^{-w_i} - 1$.

Proof. The cone $\sigma^\vee$ is the cone over a k-polygon P. This polygon can be divided into triangles in a way that no additional vertex is introduced. Intersection of two triangles is an edge. Consequently, the polygon P is the disjoint union $int(P) = \bigsqcup_{i=1}^{k-2} P_i$, where P_1 is an open triangle and P_i for $i > 1$ are triangles with one edge added. Modifying the proof of Theorem 13.1 we show that each piece of decomposition gives a nonnegative contribution to $td^{\mathbb{T}}_y(X)$. Precisely, for the cones $\tau_i = cone(P_i)$, $i > 1$ we choose A_0 to contain the corresponding edge monomials of τ_i, not its opposite in the interior. □

14. Comparison with Baum-Fulton-MacPherson class

The equality

$$td^{\mathbb{T}}_0(X \hookrightarrow \mathbb{C}^n) = td^{\mathbb{T}}(\mathbb{C}^n) ch^{\mathbb{T}}(\mathcal{O}_X) \tag{29}$$

does not hold always. The simplest example is the cusp of the type A_{2n}.

Example 14.1. Let $X = \{x^2 = y^{2n+1}\} \subset \mathbb{C}^2$ with the torus $\mathbb{C}^*$ action having weights $(2n+1)t$, $2t$. Then

$$td^{\mathbb{T}}_0(X \hookrightarrow \mathbb{C}^2) = f_*(td^{\mathbb{T}}(\mathbb{C}))\,,$$

where $f : \mathbb{C} \to \mathbb{C}^2$, $f(z) = (z^{2n+1}, z^2)$ is the normalization and $\mathbb{C}$ is equipped with the action of $\mathbb{C}^*$ of weight t. The fundamental class of X in $H^*_{\mathbb{T}}(\mathbb{C}^2)$ is equal to $2(2n+1)t$. Hence

$$td^{\mathbb{T}}_0(X \hookrightarrow \mathbb{C}^2) = 2(2n+1)t \frac{t}{1-T}\,,$$

while Baum-Fulton-MacPherson class is equal to

$$td(\mathbb{C}^2) ch^{\mathbb{T}} \mathcal{O}_X = td(\mathbb{C}^2) ch^{\mathbb{T}}\big(\mathcal{O}_{\mathbb{C}^2} - \mathcal{O}_{\mathbb{C}^2}(-X)\big) = \frac{(2n+1)t}{1-T^{2n+1}} \frac{2t}{1-T^2} \big(1 - T^{2(2n+1)}\big)\,.$$

Also taking the affine cone over a smooth hypersurface in $\mathbb{P}^{n-1}$ of degree $d > n$ (applying Proposition 10.3) we obtain counterexamples which are normal. We treat the equality (29) as a special property of the singularity germ. On the level of K-theory we simply ask if

$$mC_0(id_X) = [\mathcal{O}_X]\,.$$

If it is the case then Baum-Fulton-MacPherson-Todd class [BFM75] of X coincides with $td_0(id_X)$. The same holds in the equivariant setup.

Let us recall some conditions for the equality. There exists a characterization of the Du Bois singularities which fits to our situation the best. It is given in [Sch07].

Suppose that X is a subvariety in a smooth ambient space, $f : \widetilde{M} \to M$ is a proper map such that $f_{|f^{-1}(M\setminus X)}$ is an isomorphism and $D = f^{-1}(X)$ is a smooth divisor with normal crossings (we say f is a resolution of the pair (M, X)). Then X has Du Bois singularities if and only if the natural map

$$\mathcal{O}_X \to Rf_*\mathcal{O}_D$$

is a quasi-isomorphism. Then

$$mC_0(M \setminus X \hookrightarrow M) = f_* mC_0(\widetilde{M} \setminus X \hookrightarrow \widetilde{M}) = [Rf_*\mathcal{O}_{\widetilde{M}}] - [Rf_*\mathcal{O}_D] = [\mathcal{O}_M] - [\mathcal{O}_X]\,.$$

Hence

$$mC_0(X \hookrightarrow M) = [\mathcal{O}_X]\,.$$

(Compare [BSY10, Example 3.2].) If X has rational singularities i.e. X is normal and $Rf_*\mathcal{O}_{\widetilde{X}} = f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, then by [Kov99] it has at most Du Bois singularities. Most of the examples we have considered here have rational singularities: Schubert varieties by [Ram85, Theorem 4], toric varieties by [Oda88, Cor. 3.9]. One can add to this list the cones over a smooth hyperplane in $\mathbb{P}^{n-1}$ (considered in Corollary 10.4) provided that the degree is smaller than n. (If $d = n$ the equality still holds, but the singularity is not rational.) If X is a surface with rational singularities, then the exceptional divisor of X is a tree of rational curves. If $f : \widetilde{X} \to X$ is a resolution of rational singularities of higher dimension, then the associated complex of intersections of divisors is $\mathbb{Q}$-acyclic, [ABW13, Theorem 3.1]. But only from rationality we do not get enough information about derived direct images of $L\Omega^y_{D_I}$. There should exist a natural class of varieties for which the positivity discussed below holds.

15. Question of positivity

Let $\mathbb{T} = (\mathbb{C}^*)^r$ be a torus acting on $\mathbb{C}^n$ with weights which are nonnegative combinations of the basis characters. Suppose that $X \subset \mathbb{C}^n$ is an invariant subvariety. As it was explained in the introduction we develop the class

$$mC_*(X \hookrightarrow \mathbb{C}^n) \in G_{\mathbb{T}}(\mathbb{C}^n)[y] = K_{\mathbb{T}}(\mathbb{C}^n)[y] = K_{\mathbb{T}}(pt)[y]$$

in the basis of monomials in $\delta = -1 - y$ and $S_i = T^i - 1$. Let us assume that X is an open subset and let $f : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ be a resolution of the pair $(\mathbb{C}^n, X)$. Then using the notation of §5

$$mC_*(X) = \sum_I \delta^{|I|} Rf_*(L\Omega^y_I)\,.$$

Examples considered by us (quadratic cones of Proposition 10.5, simplicial toric varieties) show that all sheaves (or their Chern characters)

$$Rf_*(L\Omega^y_I) = \sum_{p=0}^{\dim X - |I|} (-1-\delta)^p Rf_*(L\Omega^p_I)$$

have nonnegative coefficients in the expansion. On the other hand further decomposition into sheaves $(-1-\delta)^p Rf_*(L\Omega^p_I)$ does not preserve positivity. A counterexample is the cone over a smooth quadric. It suggests that the sheaf $Rf_*L\Omega^y_I = f_*\Lambda_{-1-\delta}(T^*X_I(-\log D))$ should be treated as a single object.

We would like to point out that the positivity of the class $\alpha \in K_{\mathbb{T}}(\mathbb{C}^n)$ can be understood in various ways

(1) α is represented by an effective sheaf
(2) α is represented by a sum of the sheaves $(-1)^{\mathrm{codim} Y}[\mathcal{O}_Y]$ where Y are subvarieties in $\mathbb{C}^n$
(3) α is represented by a sum of the sheaves $(-1)^{\mathrm{codim} Y}[\mathcal{O}_Y]$ where Y are subvarieties in $\mathbb{C}^n$ with rational singularities.
(4) α is a polynomial in S_w with nonnegative coefficients, where w are the weights of the action of $\mathbb{T}$ on $\mathbb{C}^n$.

We note that

Theorem 15.1. *The condition 3) implies that α is a polynomial in S_{t_i} with nonnegative coefficients, where t_i is the positive basis of characters fixed in the beginning.*

Proof. Let $Y \subset \mathbb{C}^n$ be an invariant subvariety with rational singularities. Fix an approximation of the classifying space $B\mathbb{T}_m = (\mathbb{P}^m)^r$ with the universal $\mathbb{T}$-bundle $E\mathbb{T}_m = \boxtimes_{i=1}^r \mathcal{O}^*(1)$. Then the associated bundle obtained via Borel construction $ET_{\mathfrak{m}} \times_{\mathbb{T}} \mathbb{C}^n$ is globally generated. Let s be a generic section. Then $s^{-1}(ET_{\mathfrak{m}} \times_{\mathbb{T}} Y)$ has expected dimension, and it has at most rational singularities. Let $X_I \subset (\mathbb{P}^m)^r$ be the closure of a cell of the standard decomposition of the product of projective spaces. By our choice of variables $[\mathcal{O}_{X_I}] = (-1)^{\mathrm{codim} X_I} S_I$ under the identification of $K_{\mathbb{T}}(\mathbb{C}^n)$ with $K_{\mathbb{T}}(pt)$ given by the section s. Applying [Bri02, Theorem 1] for the homogeneous space $B\mathbb{T}_m = (\mathbb{P}^m)^r$ we find that

$$[\mathcal{O}_{ET_{\mathfrak{m}} \times_{\mathbb{T}} Y}] = \sum (-1)^{\mathrm{codim} Y - \mathrm{codim} X_I} c_I [\mathcal{O}_{X_I}]\,,$$

where the numbers c_I are nonnegative. □

16. Schubert varieties and cells

The singularities of Schubert varieties in flag manifolds G/B or more general in homogeneous spaces G/P were studied by many authors, see for example the monograph [BL00], especially §4.4, or [WY08]. The maximal torus $T \subset G$ has a discrete fixed point set in G/P. Each fixed point corresponds to a Schubert cell. Whenever a Schubert cell contains another cell in the closure, one can ask what is the singularity at the boundary point. Even the question of smoothness has turned out to be difficult, see [BL00, §4.6]. Localized Hirzebruch class of the boundary point is an invariant which describes in a way the singularity. For example if the point is smooth, then the local Hirzebruch class decomposes into linear factors. We have computed localized Hirzebruch classes in some cases and noticed certain phenomenon. Various computation experiments show that the class

$$mC_*(X \hookrightarrow \mathbb{C}^n))_{|p} = [f_{\bullet *}\Omega^y_{X_\bullet}]_{|p}$$

is positive for Schubert varieties. The Schubert varieties are sums of open cells and the positivity holds for cells as well. The expansion should be in variables δ and S_i for a positive basis in the sense of [AGM11]. We have already given an example of the cone over a quadric in $\mathbb{P}^1$, Example 7.4 formula (17). The Hirzebruch class written in variables δ and $S_w = e^{-w} - 1$ is equal to

$$\frac{S_{2t_1} + S_{2t_2} + \delta(2S_{t_1+t_2} + S_{t_1+t_2}^2 + S_{2t_1}S_{2t_2}) + \delta^2(S_{t_1+t_2}+1)(S_{t_1+t_2}+2)}{S_{2t_1}S_{2t_2}}\,.$$

This is the singularity of the codimension one Schubert variety in the Lagrangian Grassmannian $LG(2)$. The class of the complement is equal to:

$$\frac{\delta(S_{t_1+t_2}+1)S_{2t_1}S_{2t_2} + \delta^2(S_{t_1+t_2}+1)(S_{t_1+t_2}^2 + S_{2t_1}S_{2t_2} + S_{t_1+t_2}) + \delta^3(S_{t_1+t_2}+1)^3}{S_{2t_1}S_{2t_2}S_{t_1+t_2}}\,.$$

It is remarkable that the Hirzebruch class can be expressed by the variables associated to the characters of the representation, although they do not span whole equivariant cohomology of a point. This phenomenon is easy to explain: resolving the singularity of X we construct new $\mathbb{T}$-varieties for which the weights of the tangent spaces are combinations of the original weights.

The method of [Web12] allows to compute the Hirzebruch class of the codimension one Schubert variety in the classical Grassmannian $Gr_n(\mathbb{C}^{2n})$. Locally this Schubert variety is equal to the determinantal variety consisting of singular quadratic matrices. The case $n = 2$ is the quadratic singularity in $\mathbb{C}^4$, which is toric. The Hirzebruch class was computed in Example 11.4, and the formula for the open complement is the following:

$$\frac{\delta^2(1+S_{13})(1+S_{24})}{S_{13}S_{14}S_{23}S_{24}}\times$$
$$\times\Big(S_{14}S_{23} + S_{13}S_{24} + \delta(S_{14} + S_{23} + 2S_{14}S_{23} + S_{13}S_{24})) + \delta^2(1+S_{13})(1+S_{24})\Big)\,.$$

The formula is not symmetric since there is a relation between variables

$$(1+S_{13})(1+S_{24}) = (1+S_{23})(1+S_{14})\,.$$

For bigger n the formulas are quite complicated. Below we give the result for $n = 3$ restricted to one dimensional torus acting diagonally.

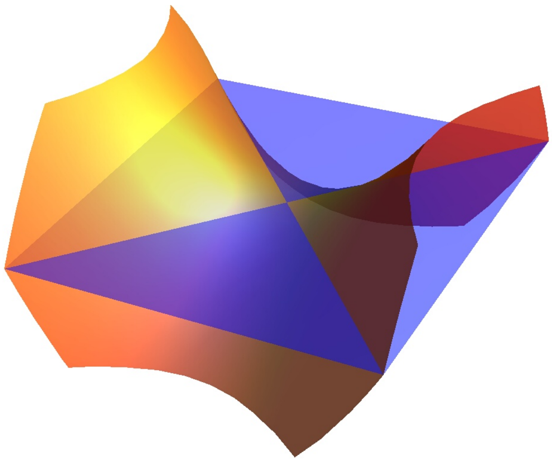
$$\frac{(\delta+1)^3(S+1)^3}{S^9}\Big(6S^6 + 9\delta S^5(2+3S) + 5\delta^2 S^4(6+15S+10S^2)+$$
$$+\delta^3 S^3(30+105S+123S^2+49S^3) + 9\delta^4 S^2(1+S)^3(2+3S)+$$
$$+\delta^5 S(1+S)^3(6+15S+8S^2) + \delta^6(1+S)^6\Big)$$

Let us give another example.

Example 16.1. Let X be the open cell in the flag variety $Fl(3) = GL_3(\mathbb{C})/B$. The 0-dimensional Schubert cell which corresponds to the standard flag has a neighbourhood (the opposite open cell) which can be identified with the set of lower-triangular matrices with 1's at the diagonal

$$\begin{pmatrix} 1 & 0 & 0 \\ x_{21} & 1 & 0 \\ x_{31} & x_{32} & 1 \end{pmatrix}.$$

The three dimensional diagonal torus is acting on $Fl(3)$ preserving the cell decomposition and the decomposition into opposite cells. The intersection of the open cell with the opposite cell is the complement of two divisors $D_1 = \{x_{31} = 0\}$ and $D_2 = \{x_{21}x_{32} - x_{31} = 0\}$ defined by vanishing of lower-left corner determinants.



The weight of the variable x_{ij} is $t_i - t_j$. The local Hirzebruch class of the open cell is equal to

$$td^{\mathbb{T}}_y(\mathbb{C}^3)_{|0} - td^{\mathbb{T}}_y(D_1 \hookrightarrow \mathbb{C}^3)_{|0} - td^{\mathbb{T}}_y(D_2 \hookrightarrow \mathbb{C}^3)_{|0} + td^{\mathbb{T}}_y(D_1 \cap D_2 \hookrightarrow \mathbb{C}^3)_{|0}\,.$$

The intersection of the divisors is not transverse, but it is equal to the intersection of two coordinate lines $x_{21} = 0$ and $x_{32} = 0$ in the plane $\{x_{31} = 0\}$. Therefore by the inclusion-exclusion formula $td^{\mathbb{T}}_y(X \hookrightarrow \mathbb{C}^3)_{|0}$ is equal to

$$\frac{1+y\frac{T_2}{T_1}}{1-\frac{T_2}{T_1}} \cdot \frac{1+y\frac{T_3}{T_1}}{1-\frac{T_3}{T_1}} \cdot \frac{1+y\frac{T_3}{T_2}}{1-\frac{T_3}{T_2}} - 2\frac{1+y\frac{T_2}{T_1}}{1-\frac{T_2}{T_1}} \cdot \frac{1+y\frac{T_3}{T_2}}{1-\frac{T_3}{T_2}} + \frac{1+y\frac{T_2}{T_1}}{1-\frac{T_2}{T_1}} + \frac{1+y\frac{T_3}{T_2}}{1-\frac{T_3}{T_2}} - 1\,,$$

where $T_i = e^{-t_i}$. After the substitution $S_{ij} = T_i/T_j - 1$ and using the relation $(1+S_{21})(1+S_{32}) = 1 + S_{31}$ the Hirzebruch class can be written in the form

$$\frac{\delta}{S_{21}S_{31}S_{32}}(1+S_{21})(1+S_{31})\big(S_{21}S_{32} + \delta S_{21}S_{32} + \delta^2(1+S_{31})\big)\,,$$

We ask: Does positivity always holds for Schubert cells in a homogeneous space G/P? After substitution $T_i = e^{\delta t_i}$ and letting $\delta \to 0$ we obtain Chern-Schwartz-MacPherson class. The positivity of the *nonequivariant* Chern-Schwartz-MacPherson class in the case of ordinary Grassmannians (conjectured in [AM09]) was shown by Huh [Huh13]. Proving nonequivariant result Huh takes an advantage of the action of the Borel B group preserving the cells. The proof is based on the observation that the bundle $T\widetilde{M}(-\log D)$ has a lot sections for a suitable B-invariant resolution of the Schubert cell. In general Huh considers the situation when a solvable group B acts on the pair (M, U), where U is a an open set contained in the nonsingular locus of M. Under the assumption that the pair (M, U) admits a B-equivariant resolution $(\widetilde{M}, \widetilde{U})$ on which B acts with finitely many orbits it is shown that the Chern-Schwartz-MacPherson class is effective. The equivariant version of this result would give positivity after specialization $\delta = 0$. Nevertheless the example of non-simplicial toric varieties shows that the assumption of Huh is not strong enough to imply positivity of the Hirzebruch class.

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