Purity of boundaries of open complex varieties

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Abstract

We study the boundary of an open smooth complex algebraic variety U. We ask when the cohomology of the geometric boundary $Z = X \setminus U$ in a smooth compactification X is pure with respect to the mixed Hodge structure. Knowing the dimension of singularity locus of some singular compactification we give a bound for k above which the cohomology $H^k(Z)$ is pure. The main ingredient of the proof is purity of the intersection cohomology sheaf.

Key words:Mixed Hodge theory of singular varieties, intersection cohomology, resolution of singularities

MSC classification: 32S35, 55N33, 14E15, 32S20

1 Introduction

Let U be a smooth complex algebraic variety which is not compact. We study cohomological properties of U which are invariant with respect to modifications of the interior of U. In other words we investigate cohomological properties of the boundary. The boundary itself can have at least two meanings. First of all from the topological point of view we may treat an open smooth variety as the interior of a compact manifold with boundary. In this case the boundary would mean an odd dimensional real manifold. We call it the link at infinity. On the other hand from the geometric point of view we may compactify our variety in the category of algebraic varieties. The boundary is then a subvariety of the compactification. In addition we may require that the compactification is smooth. The condition that the boundary is a normal crossing divisor is irrelevant for us, although it is hidden in the construction of the mixed Hodge structure. The link at the infinity can be identified with the link of the boundary in the compactification. Regardless from the differences we show that the topological and geometric boundaries have a lot in common when the mixed Hodge structure is concerned.

To some extend we try to avoid specific methods of Hodge theory having in mind possible application (or rather open questions) for real algebraic geometry, as well as some questions about torsion for cohomology of complex varieties. The sections §2-§4 are valid in that generality. Nevertheless the results of §5 cannot be generalized and they hold only for rational cohomology of algebraic varieties. The strong functoriality of weight filtration implies that lower weight subspaces of topological and geometric boundary coincide, see Proposition 3. In similar situations this phenomenon was already described in [10, Prop. 7.1] and [2, Prop. 5.1].

The proof of the main result of §6 uses even stronger techniques. The purity of intersection cohomology sheaf [3] imposes some conditions on the link of the geometric boundary. We prove Theorem 11 which can be shortened to the following statement:

Theorem 1 *Suppose U is a complex smooth algebraic variety. Assume that U admits a singular compactification Y. Suppose that the singularities of the pair $(Y, Y \setminus U)$ is of dimension s. Then for any smooth compactification X the boundary $X \setminus U$ has pure cohomology $H^k(X \setminus U)$ for $k \geq \dim(U) + s$.*

The Theorem 1 for U admitting one-point compactification already appeared in [4, Th. 2.1.11]. A vast generalization was given in [13]. The present version gives a better bound for purity, although the situation considered here is less general.

One can treat Theorem 1 as a contractibility criterion. For a subvariety $Z \subset X$: if $H^k(Z)$ is not pure for $k \geq \dim(U)+s$, then the pair (X, Z) cannot be contracted to a pair (Y, W) with singularities of the dimension smaller or equal to s. Although the Theorem 1 resembles the Grauert criterion, it is of different nature. In the Grauert criterion the intersection form on Z depends on the embedding $Z \subset X$, whereas here the mixed Hodge structure of Z does not. Of course it is needles to say that our criterion is not sufficient for existence of a contraction.

In contrast to the previous paper [13] we try to present the subject as elementary as possible. We have avoided to use mixed Hodge modules [11] taking for granted that the cohomology with coefficients in a complex of sheaves of geometric origin has a natural mixed Hodge structure. By "geometric origin" we mean "obtained by the standard sheaf-theoretic operations". We assume that the varieties are defined over reals or over the complex numbers and we use classical topology. In our arguments we will apply resolution of singularities although a part of results depends only on the formal properties of mixed Hodge modules or Weil sheaves.

2 Topological boundary: the link at the infinity

Let us begin with description of some invariants of open manifolds which can be defined just using topology and basic properties resolution of singularities. Working with $\mathbb{Z}/2$-cohomology we can apply our construction also for real algebraic manifolds. For complex manifolds we can use any coefficients, not necessarily $\mathbb{Q}$.

The first invariant we propose to consider is the cohomology of the link at infinity:

$$H^*(L_\infty U) := \varinjlim_{K\subset U} H^*(U \setminus K)\,,$$

where K runs through compact sets contained in U. The group $H^*(L_\infty U)$ is exactly the cohomology of the link L_Z, the link of the boundary set $Z = X \setminus U$, where X is a compactification of U. (For various approaches to the link of a subvariety, see [5].) This cohomology group is of finite dimension. It can be expressed in terms of sheaf operations on X:

$$H^*(L_\infty U) = H^*(L_Z) = H^*(Z; i^* Rj_* \mathbb{Q}_U)\,,$$

where $j : U \hookrightarrow X$ and $i : X \setminus U \hookrightarrow X$ are the inclusions.

3 Geometric boundary: image of boundary cycles

Another invariant considered by us is the image of boundary cycles

$$IB^*(U) = \mathrm{im}(H^*(Z) \to H^{*+1}_c(U)) = \ker(H^{*+1}_c(U) \to H^{*+1}(X))\,,$$

where X is a **smooth** compactification of U and $Z = X \setminus U$. The maps come from the long exact sequence of the pair (X, Z).

To show the independence of X we start with a purely topological lemma.

Lemma 2 *Suppose we have a map of real smooth oriented closed manifolds*

$$f : X_1 \to X_2$$

which is isomorphism of some open subsets

$$f_{|U_1} : U_1 = f^{-1}(U_2) \xrightarrow{\simeq} U_2\,.$$

Denote by

$$IB^*_i = \ker(H^{*+1}_c(U_i) \to H^{*+1}(X_i))$$

the kernels of the natural maps for $i = 1, 2$. Then f induces an isomorphism

$$f^* : IB^*_2 \to IB^*_1\,.$$

Proof. The map $f^* : H^*(X_2) \to H^*(X_1)$ is injective since it is a map of degree one of compact manifolds. The map f induces the transformation

$$\begin{array}{ccccc} IB_2^k & \hookrightarrow & H_c^{k+1}(U_2) & \to & H^{k+1}(X_2) \\ \downarrow & & \simeq \downarrow & & {\scriptstyle mono} \downarrow \\ IB_1^k & \hookrightarrow & H_c^{k+1}(U_1) & \to & H^{k+1}(X_1) \end{array}$$

It follows that $IB_2^k \to IB_1^k$ is an isomorphism. □

To prove the independence of $IB^*(U)$ on the compactification it remains to say that any two smooth compactifications are dominated by a third one.

4 Basic exact sequences

We will need three exact sequences to relate the described invariants. These exact sequences may be constructed topologically, but it is important to know that they come from distinguished triangles in the derived category of sheaves. It will follow, that for complex varieties the maps of the described exact sequences preserve the mixed Hodge structure.

We start with the sequence relating the cohomology of U and the cohomology of its link at the infinity. Let X be any compactification and $Z = X \setminus U$. It is possible to find a neighbourhood N of Z which retracts to Z and the boundary ∂N is homeomorphic to the link of Z. Considering the pair $(X \setminus N, \partial N)$ we arrive to the long exact sequence

$$\to H^k(U) \to H^k(L_Z) \xrightarrow{\delta} H_c^{k+1}(U) \to H^{k+1}(U) \to \; . \tag{1}$$

This exact sequence may be in fact obtained from the fundamental distinguished triangle (in the category of mixed Hodge modules on X)

$$\begin{array}{ccccc} i_* i^! G & & \longrightarrow & & G \\ & {\scriptstyle [+1]} \nwarrow & & \swarrow & \\ & & Rj_* j^* G & & \end{array} \tag{2}$$

where $G = j_! \mathbb{Q}_U$. By duality we obtain the triangle

$$\begin{array}{ccccc} i_! i^* Rj_* \mathbb{Q}_U & & \longleftarrow & & Rj_* \mathbb{Q}_U \\ & {\scriptstyle [+1]} \searrow & & \nearrow & \\ & & j_! \mathbb{Q}_U & & \end{array}$$

since $j_! j^! Rj_* \mathbb{Q}_U \simeq j_! \mathbb{Q}_U$. Applying the cohomology we obtain the sequence (1).

We also need an exact sequence relating $H^k(Z)$ and $H^k(L_Z)$. Topologically we have a retraction $N \to Z$. The exact sequence for the manifold with boundary $(N, \partial N)$

$$\to H^k(N) \to H^k(\partial N) \to H^{k+1}(N, \partial N) \to H^{k+1}(N) \to$$

becomes

$$\to H^k(Z) \to H^k(L_Z) \to H^{k+1}(X,U) \to H^{k+1}(Z) \to \ . \qquad (3)$$

The sheaf theoretic definition is given below. Let us restrict the triangle (2) with $G = \mathbb{Q}_X$ to Z. We have $i^* i_* i^! \mathbb{Q}_X = i^! \mathbb{Q}_X$ and we obtain the triangle

$$\begin{array}{ccc} i^!\mathbb{Q}_X & \longrightarrow & \mathbb{Q}_Z \\ {\scriptstyle [+1]}\nwarrow & & \swarrow \\ & i^* Rj_* \mathbb{Q}_U . & \end{array}$$

The associated sequence of cohomology is just (3). It plays the fundamental role in our further consideration.

Of course the third exact sequence used by us is the sequence of the pair (X,Z)

$$\to H^k(X) \to H^k(Z) \to H^{k+1}_c(U) \to H^{k+1}(X) \to \ . \qquad (4)$$

To relate the groups $H^*(L_\infty(U)) = H^*(L_Z)$ and $IB^*(U) = \mathrm{im}(H^k(Z) \to H^{k+1}_c(U))$ we apply the map of exact sequences (1) and (4) induced by the inclusion

$$(X \setminus int(N), \partial N) \subset (X,N)\,.$$

We obtain the commutative diagram

$$\begin{array}{ccc} H^k(Z) & \to & H^{k+1}_c(U) \\ \downarrow & & \| \\ H^k(L_Z) & \to & H^{k+1}_c(U) \end{array} \quad .$$

We see that

$$IB^k(U) = \mathrm{im}(H^k(Z) \to H^{k+1}_c(U)) \subset \mathrm{im}(H^k(L_\infty(U)) \to H^{k+1}_c(U))\,.$$

In general the inclusion is proper.

5 Mixed Hodge structure

From now on we consider only complex algebraic varieties and rational cohomology.

The considered invariants $H^k(L_\infty U)$ and $IB^k(U)$ are equipped with a mixed Hodge structures. The first one is given by the sheaf-theoretic description:

$$H^*(L_\infty U) = H^*(Z; i^* Rj_* \mathbb{Q}_U)\,.$$

The second one, $IB^*(U)$, has a structure induced from $H^*(Z)$. In the situation of Lemma 2 the map $IB^k_2 \to IB^k_1$ preserves quotient mixed Hodge structures and since it is an isomorphism of vector spaces it must be also an isomorphism of all weight subspaces. In fact by the definition of the mixed Hodge structure we have

$$IB^k(U) = W_k H_c^{k+1}(U)\,.$$

For us the most interesting part is the weight subspace W_{k-1}. Using basic properties of the mixed Hodge structure we will give three description of that weight space.

Proposition 3 *Let X be a smooth compactification of U and $Z = X \setminus U$. Then the following groups are isomorphic:*

1. $W_{k-1}H^k(L_\infty U)\,,$
2. $W_{k-1}H_c^{k+1}(U)\,,$
3. $W_{k-1}H^k(Z)\,.$

Note that in the statement of the theorem we do not assume that Z is a smooth divisor with a normal crossing. As a corollary from Proposition 3 we have

Corollary 4 *Let X be a smooth compactification of U and $Z = X \setminus U$. The cohomology $H^k(Z)$ is pure of weight k if and only if $H^k(L_Z)$ is of weight $\geq k$.*

Also we note (compare [10, Prop. 7.1]):

Corollary 5 *The impure part of cohomology of the boundary set $W_{k-1}H^k(Z)$ does not depend on the smooth compactification.*

Remark 6 Note that the group $W_{k-1}H^k(Z)$ is a topological invariant of Z, since by [12] it is the kernel of the canonical map to the intersection cohomology $H^k(Z) \to IH^k(Z)$. Also by the construction of the mixed Hodge structure we have

$$W_{k-1}H^k(Z) = ker(g^* : H^k(Z) \to H^k(\widetilde{Z}))\,,$$

where $g : \widetilde{Z} \to Z$ is any dominating proper map from a smooth variety, possibly of bigger dimension.

The entire cohomology of the boundary of a smooth compactification is not an invariant of U. Of course when we blow up something at the boundary then the cohomology is modified, nevertheless the lower parts of weight filtration remains unchanged.

Remark 7 With help of the Decomposition Theorem of [3] we have better insight to what happens with the cohomology of the boundary. Let f be a map of pairs $(X_1, Z_1) \to (X_2, Z_2)$ which is an isomorphism outside Z_1. The push-forward of the constant sheaf on X_1 decomposes:

$$Rf_*\mathbb{Q}_{X_1} \simeq \mathbb{Q}_{X_2} \oplus \bigoplus_\alpha IC(V_\alpha; L_\alpha)\,.$$

The supports of the intersection sheaves $IC(V_\alpha; L_\alpha)$ are contained in Z_2, therefore

$$H^*(Z_1) = H^*(Z_2; (Rf_*\mathbb{Q}_{X_1})_{|Z_2}) \simeq H^*(Z_2) \oplus \bigoplus_\alpha IH^*(V_\alpha; L_\alpha)\,.$$

Again we see that the difference between $H^*(Z_1)$ and $H^*(Z_2)$ is pure since $\bigoplus_\alpha IH^*(V_\alpha; L_\alpha)$ is a summand of $H^*(X_1)$.

Remark 8 Using another powerful tool, namely the Weak Factorization Theorem [1], we can trace how the cohomology of the boundary may change. Each time when we blow up a smooth center S contained in the boundary the pure summand $\operatorname{coker}\big(H^*(S) \to H^*(\mathbb{P}N_{S/X})\big)$ contributes to the cohomology of the blown up boundary. Here $H^*(\mathbb{P}N_{S/X})$ is the projectivization of the normal bundle of S in X.

The proof of Proposition 3 is divided into Lemmas 9 and 10.

Lemma 9 *We have*

$$W_{k-1}H^k(Z) \simeq W_{k-1}H^{k+1}_c(U)\,.$$

Proof. We recall that $H^k(X)$ is of weight k and $H^{k+1}(X)$ is of weight $k+1$. Therefore the long exact sequence

$$\to H^k(X) \to H^k(Z) \xrightarrow{\delta} H^{k+1}_c(U) \to H^{k+1}(X) \to$$

induces an isomorphism of graded pieces for $\ell < k$

$$Gr^W_\ell H^k(Z) \simeq Gr^W_\ell H^{k+1}_c(U)\,.$$

It follows that the boundary map $W_{k-1}H^k(Z) \to W_{k-1}H^{k+1}_c(U)$ is an isomorphism. $\square$

Lemma 10 *We have*

$$W_{k-1}H^k(L_Z) \simeq W_{k-1}H^{k+1}_c(U)\,.$$

Proof. We consider the long exact sequence (1). Since U is smooth $W_{k-1}H^k(U) = 0$. Therefore for $\ell < k$

$$Gr^W_\ell H^k(L_Z) \simeq Gr^W_\ell H^{k+1}_c(U)\,.$$

Again the boundary map $W_{k-1}H^k(L_Z) \to W_{k-1}H^{k+1}_c(U)$ is an isomorphism. $\square$

6 Singular versus smooth compactifications

Let $W \subset Y$ be a pair of varieties. Assume that $Y \setminus W$ is smooth. By the singularity of the pair we mean the set of points at which W in Y analytically does not look like a submanifold (of any dimension) in a manifold. The singularity set consists of points at which W or Y is singular. Below we give the exact statement of our main result.

Theorem 11 *Let U be a smooth variety. Suppose that U admits a compactification Y and let $W = Y \setminus U$ be the boundary set. Denote by s the dimension of the singularities of the pair (Y, W). Let X be a smooth compactification of U and $Z = X \setminus U$. For $k \geq \dim(U) + \dim(W)$ we have:*
i) the cohomology of the link $H^k(L_Z)$ is of weight $\geq k+1$,
ii) the restriction map $H^k(Z) \to H^k(L_Z)$ vanishes.

For $k \geq \dim(U) + s$ we have:
iii) the cohomology of the boundary $H^k(Z)$ is pure of weight k, that is

$$W_{k-1}H^k(Z) = 0$$

iv) the cohomology of the link $H^k(L_Z)$ is of weight $\geq k$.

Note that by Proposition 3 the claim iv) does not depend on the choice of the smooth compactification X.

Let $n = \dim(U)$. By Poincaré duality we have

$$H^k(Z)^* = H^{2n-k}(X, U)(n)\,,$$

$$H^k(L_Z)^* = H^{2n-1-k}(L_Z)(n)\,,$$

where (n) denotes the Tate twist shifting the weights by $2n$. The dual version of the Theorem 11 is the following:

Theorem 12 *With the assumption of Theorem 11:*

For $k \leq \dim(U) - \dim(W)$ we have
i') the cohomology of the link $H^{k-1}(L_Z)$ is of weight $\leq k-1$,
ii') the boundary map $H^{k-1}(L_Z) \to H^k(X, U)$ vanishes.

For $k \leq \dim(U) - s$ we have
iii') the cohomology $H^k(X, U)$ is pure of weight k, that is

$$W_k H^k(X, U) = H^k(X, U)$$

iv') the cohomology of the link $H^{k-1}(L_Z)$ is of weight $\leq k$.

To distinguish two copies of U in X and in Y we will use the letter V for the copy of U in Y. The identification map $U \to V$ is denoted by f:

$$\begin{array}{ccccc} Z = X \setminus U & \subset & X = \overline{U} & \supset & U \\ & & & \simeq & \downarrow f \\ W = Y \setminus V & \subset & Y = \overline{V} & \supset & V \end{array}$$

Remark 13 In our setup, we can apply completion and resolution of singularities. Therefore X can be replaced by a dominating smooth variety for which the map f extends to the boundary.

Some information about the weights of cohomology of the link and the boundary can be deduced when we have a proper map $f : U \to V$ and a compactification of V. A statement which generalizes i) and ii) in terms of a defect of semismallness [4] is formulated in [13]. The direct generalization of iii) and iv) would involve precise information about the singularities of the perverse cohomology sheaves ${}^p\mathcal{H}^k Rf_*\mathbb{Q}_U$.

The Theorem 11 can be localized around a topological component of $X \setminus U$. Precisely, consider the set of ends, i.e. $U_\infty = \pi_0(X \setminus U)$. This set does not depend on the choice of X provided that X is normal. A map of algebraic varieties which is proper induces a map of their ends. To deduce purity of the cohomology of a part of the boundary of U it is enough to have information about a singular completion of the corresponding end.

7 Proofs

Before the proof of Theorem 11 let us recall the key property of the link of a subvariety

Theorem 14 ([5]) *Let Y be a variety and let W be a compact subvariety. Let us assume that $Y \setminus W$ is smooth. Then $H^k(L_W)$ is of weight $\leq k$ for the degrees $k < \dim(Y) - \dim(W)$.*

Theorem 14 immediately follows from the purity of the intersection sheaf [6, 3] since the stalk cohomology $\mathcal{H}^k(IC_Y)$ is isomorphic to $\mathcal{H}^k(Rj_*\mathbb{Q}_V)$ for $k < \dim(Y) - \dim(W)$ and $H^*(L_W) = H^*(W; (Rj_*\mathbb{Q}_V)_{|W})$.

Remark 15 In [2, §6] the Decomposition Theorem of [3] was used to give estimates for the dimension of intersection cohomology of the link by means of resolution. But it seems that the purity of the intersection sheaf was not used directly.

Proof of (12.i'-ii').

By Remark 13 we assume that the map f extends to X. The extended map (denoted by the same letter) induces a map of sheaves $i'^* Rj'_* \mathbb{Q}_V \to Rf_* i^* Rj_* \mathbb{Q}_U$. Therefore the mixed Hodge structures of the isomorphic groups $H^*(L_W)$ and $H^*(L_Z)$ coincide. By Theorem 14 and the assumption on the dimension of W the cohomology $H^{k-1}(L_W)$ is of weight $\leq k-1$. The claim 12.ii' follows from the long exact sequence (3): the boundary map

$$H^{k-1}(L_Z) \to H^k(X, U)$$

vanishes because the first term is of weight $\leq k-1$ and the second term is of weight $\geq k$.

Proof of (12.i-ii) follows by duality.

Proof of (12.iii-iv) If $W = Sing(Y)$ then $s = \dim(W)$ and the statement i) is even stronger then required. The Proposition 3 implies ii).

Suppose now $Sing(Y) \subsetneq W$. We may assume that f extends to a map $X \to Y$ and also we may assume that the map f is a resolution of singularities of the pair (Y, W). Let $\widetilde{W} \subset X$ be the proper transform of W. Denote by $E \subset X$ the exceptional set of f and let $F = E \cap \widetilde{W}$. Consider the Mayer-Vietoris exact sequence for $Z = E \cup \widetilde{W}$:

$$\to H^{k-1}(E) \oplus H^{k-1}(\widetilde{W}) \xrightarrow{\alpha} H^{k-1}(F) \xrightarrow{\delta} H^k(Z) \to H^k(E) \oplus H^k(\widetilde{W}) \to .$$

By (11.i) applied to the map $(X, E) \to (Y, f(E))$ the cohomology of the link $H^k(L_E)$ is of weight $\geq k+1$ for $k \geq \dim(X) + s$. Hence by Proposition 3 the cohomology $H^k(E)$ is pure for $k \geq \dim(X) + s$. Of course $H^k(\widetilde{W})$ is pure since we assume that $\widetilde{W}$ is smooth. To prove the purity of $H^k(Z)$ it remains to show that the map δ of the Mayer-Vietoris sequence is trivial.

By (11.ii) applied to $F \subset \widetilde{W}$ the map

$$H^{k-1}(F) \to H^{k-1}(L_F)$$

vanishes for $k-1 \geq \dim(W) + s$. By the exact sequence (3) for that pair the restriction map

$$H^{k-1}(\widetilde{W}, \widetilde{W} \setminus F) \to H^{k-1}(F)$$

is surjective. The above map factors through $H^{k-1}(\widetilde{W})$, therefore the map

$$H^{k-1}(\widetilde{W}) \to H^{k-1}(F)$$

is surjective. It follows that the restriction map α is surjective and the boundary map δ is trivial for $k \geq \dim(X) + s \geq \dim(W) + s + 1$. This completes the proof.

Proof of (12.iii'-iv') follows by duality. $\square$

Remark 16 If the singularity set is empty then $s = -\infty$ by convention. The claims (11.i-ii) hold for all degrees by trivial reasons.

The special case when W is a point (an isolated singularity resolution) was studied from the very beginning of the theory. In that case both maps $H^{n-1}(L_Z) \to H^n(X,U)$ and $H^n(X) \to H^n(L_Z)$ are trivial. The map $H^n(X,U) \to H^n(Z)$ is an isomorphism. After the identification $H^n(X,U) = H^n(Z)^*$ we obtain a nondegenerate intersection form which was studied for example in [7].

8 Questions about real algebraic varieties

The Hodge theory for real algebraic varieties and $\mathbb{Z}/2$ coefficients is not available. The approach of [8, 9] does not lead to a strongly functorial weight filtration. Nevertheless one defines impure cohomology of a singular compact variety X: it is the kernel of $H^*(X) \to H^*(\tilde{X})$, where $\tilde{X}$ is any resolution. We say that the cohomology of a real variety is pure if the kernel $H^*(X) \to H^*(\tilde{X})$ is trivial. The definition does not depend on $\tilde{X}$. One can ask the question about the generalization of the Theorem 11:

Question 17 *With the assumption of Theorem 3 for real algebraic varieties: What properties of* (Y, W) *would guarantee purity of* $H^*(Z)$ *in some range of degrees?*

The dimension of the singularity set is far to weak invariant. It is well known that any real algebraic set can be contracted to a point.

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