NOTES ON MIXED HODGE STRUCTURES
Srni – January 2005

Andrzej Weber
Instytut Matematyki, Uniwersytet Warszawski

This is a short introduction to mixed Hodge structures. The notes contain basic definitions and statements. There are no proofs. The motivic nature of the theory is exposed. In the end some applications to equivariant cohomology are given

Topology of algebraic varieties was studied from the very beginning of algebraic topology. Usual topological invariants often carry additional structures. For example: the cohomology of a complex smooth projective variety is equipped with a pure Hodge structure. Every cohomology class can be decomposed into (p,q)–types. The decomposition has well known symmetries which allow to form the famous Hodge diamond.
If we deal with singular or open varieties the situation is much more complicated, but on the other hand we obtain an interesting and reach structure. The structure in question is derived from the fact that each variety can be built up from smooth projective ones by means of set-theoretic operations and blow-ups. The result is a filtration in cohomology which does not depend on the way the variety was constructed. This filtration is highly functorial. It is called the **weight filtration**. Each piece of the associated gradation looks like the cohomology of a projective variety, i.e. it carries a pure Hodge structure. This way we arrive to the notion of the **mixed Hodge structure**, in short MHS. This theory was constructed by Pierre Deligne, who managed to transpose his results arising from Weil conjectures into the category of complex varieties. Now the theory of mixed Hodge structure is an indispensable tool for studying singular varieties.
After giving some simple but instructive examples I will describe the linear algebra which is hidden under the name of mixed Hodge structure. Then, I'll construct MHS for smooth open varieties using the logarithmic complex. To deal with singularities I'll construct MHS for simplicial varieties. I'll also give applications for equivariant cohomology when a linear algebraic group acts on an algebraic variety.

Our approach is explained by the diagram:

$$\begin{array}{ccccc} X & \longmapsto & M(X) & \longmapsto & H^*(X) \\ \\ \text{smooth projective variety} & \longmapsto & \text{pure motive} & \longmapsto & \text{pure Hodge structure} \\ \\ \text{open/singular variety} & \longmapsto & \text{mixed motive} & \longmapsto & \text{mixed Hodge structure} \end{array}$$

The reader can find a sufficient introduction to Chow motives e.g. in [M]. In fact the nature of motives is irrelevant. Only the interactions between them are important. Pure motives are organized into a category and the "interactions" are just the morphisms. We define the morphisms to be the correspondences. A correspondence from X to Y is an algebraic cycle in the product $X \times Y$. To compose morphisms we have to be able to intersect algebraic cycle and therefore we have to divide them by a relation which allows to develop intersection theory. We will not explain further the subject of motives, but we keep in mind that the homological constructions we describe have a geometric background.

In this exposition we do not mention many important issues such as variation of Hodge structures or their degeneration (see e.g. [St], [Cl]). In general we refer to the book of C. Voisin [Vo]. Also the vast theory of mixed Hodge modules [Sa] is not presented, although it seems unavoidable for subtle analysis of singularities.

Plan:

1. Introduction

motivation, simple examples of cohomology of complex algebraic varieties.

2. Linear algebra

pure Hodge structures, category of mixed Hodge structures, mixed Hodge complexes, degeneration of spectral sequences.

3. Open smooth varieties

logarithmic complex, Poincaré residues, construction of MHS, cohomology with compact supports.

4. Singular varieties

simplicial varieties, hypercovering of a singular variety, construction of MHS. virtual Poincaré polynomials.

5. Actions of algebraic groups

simplicial varieties arising from a group action, formality of equivariant cohomology, degeneration of Eilenberg-Moore spectral sequence.

1. Introduction

1.1 *Motivation.*

Let X be an algebraic variety defined over $\mathbb{F}_q$ and ϕ be the induced Frobenius automorphism acting on the étale cohomology $H^*_{\acute{e}t}(X \otimes \overline{\mathbb{F}}_q; \mathbb{Q}_\ell)$.

(1.1.1) **Corollaries from Weil conjectures:**

– all eigenvalues of ϕ are algebraic numbers of the module $q^{\frac{i}{2}}$ (with respect to any embedding $\mathbb{Q}_\ell \longrightarrow \mathbb{C}$),

– if X is smooth and projective then all eigenvalues of ϕ on $H^n_{\acute{e}t}(X \otimes \overline{\mathbb{F}}_q; \mathbb{Q}_\ell)$ have the module equal to $q^{\frac{n}{2}}$,

– let V_i be the subspace of $H^*_{\acute{e}t}(X \otimes \overline{\mathbb{F}}_q; \mathbb{Q}_\ell)$ corresponding to the eigenvalues of the module equal to $q^{\frac{i}{2}}$. Then the *weight filtration* $W_p = \bigoplus_{i \leq p} V_i$ can be computed in a geometric way. Moreover, it is strictly functorial.

(1.1.2) **Goal:** Construct a weight filtration for complex algebraic varieties and prove its functoriality. The Hodge structure on the associated graded group should play the rigidifying role of the Frobenius automorphism.

1.2 *Examples of cohomology of smooth open varieties.*

In this section we consider cohomology with arbitrary coefficients, but later the coefficients will be rational or complex numbers.

(1.2.1) Let X be a smooth variety, $\overline{X}$ its smooth compactification. Suppose, that $D = \overline{X} \setminus X$ is smooth, $c = \operatorname{codim} D$. Then we have Gysin exact sequence

$$\longrightarrow H^{n-2c}(D) \xrightarrow{j_!} H^n(\overline{X}) \xrightarrow{i^*} H^n(X) \xrightarrow{res} H^{n-2c+1}(D) \longrightarrow ,$$

where res is the residue map. The weight filtration of $H^n(X)$ looks as follows:

$$W_{n-1} = 0, \qquad W_n = im(i^*) = ker(res), \qquad W_{n+1} = H^n(X) .$$

(1.2.2) In general the bottom term of the weight filtration of a smooth variety X can be defined as follows

$$W_n H^n(X) = im(i^* : H^n(\overline{X}) \to H^n(X)) ,$$

where $i : X \to \overline{X}$ is any smooth compactification of X.

(1.2.3) **Proposition:** *The subgroup $im(i^* : H^n(\overline{X}) \to H^n(X)) \subset H^n(X)$ does not depend on the compactification. The proposition holds for arbitrary coefficients of cohomology.*

Proof: For any two compactifications $\overline{X}_1$ and $\overline{X}_2$ construct another one: let $\overline{X}_3$ be a resolution of the closure of the diagonal embedding $X \hookrightarrow \overline{X}_1 \times \overline{X}_2$. The projection $p_i : \overline{X}_3 \to \overline{X}_i$ for $i = 1, 2$ induces a surjection $(p_i)_! : H^n(\overline{X}_3) \to H^n(\overline{X}_i)$ since it is a map of closed manifold of degree 1. The restriction map $H^n(\overline{X}_3) \to H^n(X)$ factors through $(p_i)_!$. Therefore $im(H^n(\overline{X}_i) \to H^n(X)) = im(H^n(\overline{X}_3) \to H^n(X))$.

(1.2.4) We define another filtration in cohomology: *Hodge filtration.* Let $\Omega^\bullet$ denote the complex of holomorphic differential forms. It contains a subcomplex $\Omega^{\bullet \geq p}$. We identify $H^n(X; \Omega^\bullet)$ with $H^n(X; \mathbb{C})$. Let

$$F^p = im(H^n(X; \Omega^{\bullet \geq p}) \to H^n(X; \Omega^\bullet)) \subset H^n(X; \mathbb{C}) .$$

Hodge filtration is decreasing. If X is compact then

$$F^p = \bigoplus_{r \geq p} H^{r,n-r}(X) .$$

Moreover

$$F^p \cap \overline{F^{n-p}} = H^{p,n-p}(X) \qquad \text{and} \qquad H^n(X; \mathbb{C}) = F^p \oplus \overline{F^{n-p+1}} .$$

We say that the filtration F^p is a pure Hodge structure of the weight n.

(1.2.5) **Proposition:** *In the example 1.2.1 Hodge filtration induces a pure Hodge structure of the weight n on Gr^W_n and of the weight $n+1$ on Gr^W_{n+1}.*

We have to choose a suitable subcomplex of $\Omega^\bullet$ and its resolution. Then the assertion follows from 2.1.8. We obtain a **mixed Hodge structure** in $H^*(X)$.

1.3 *Examples of cohomology of compact varieties.*

Let X be a compact variety with the smooth singular locus Z. Let $\mu : \widetilde{X} \to X$ be a resolution of singularities. Suppose $E = \mu^{-1}Z$ is smooth. We have Mayer-Vietoris exact sequence

$$\longrightarrow H^{n-1}(E) \xrightarrow{\delta} H^n(X) \longrightarrow H^n(\widetilde{X}) \oplus H^n(Z) \longrightarrow H^n(E) \xrightarrow{\delta} .$$

The weight filtration of $H^n(X)$ looks as follows:

$$W_{n-2} = 0, \qquad W_{n-1} = im(\delta), \qquad W_n = H^n(X)\,.$$

Since $\mu^*_{|E} : H^n(Z) \to H^n(E)$ is injective we have

$$W_{n-1} = ker(H^n(X) \to H^n(\widetilde{X}))\,.$$

Also in this case one constructs a Hodge filtration in $H^n(X;\mathbb{C})$.

(1.3.1) In general the term W_{n-1} of the weight filtration of a compact variety X can be defined as follows

$$W_{n-1}H^n(X) = ker(\mu^* : H^n(X) \to H^n(\widetilde{X}))\,,$$

where $\mu : \widetilde{X} \to X$ is any resolution of singularities of X.

(1.3.2) **Proposition:** *The subgroup $ker(\mu^* : H^n(X) \to H^n(\widetilde{X})) \subset H^n(X)$ does not depend on the resolution of singularities. The proposition holds for arbitrary coefficients of cohomology.*

Proof is analogous to 1.2.3.

(1.3.3) One can show that for a compact algebraic variety the group $W_{n-1}H^n(X) \otimes \mathbb{Q}$ is a topological invariant of X. In fact it is the kernel of the map to intersection cohomology of Goresky-MacPherson [GoMP], see [We].

2. Linear algebra

2.1 *Pure Hodge structures.*

(2.1.1) **Definition:** *A pure Hodge structure of the weight n is a finite dimensional $\mathbb{Q}$-vector space $V_\mathbb{Q}$ and a decreasing filtration F^p of $V_\mathbb{C} = V_\mathbb{Q} \otimes \mathbb{C}$ such that*

$$V_\mathbb{C} = F^p \oplus \overline{F^{n-p+1}}\,.$$

If $p+q=n$ we define $V^{p,q} = F^p \cap \overline{F^q}$.

(2.1.2) An alternative but equivalent definition of the Hodge structure is a decomposition $V_\mathbb{C} = \bigoplus_{p+q=n} V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$. Then we define $F^p = \bigoplus_{r \geq p} V^{r,n-r}$.

(2.1.3) Example: The cohomology of a projective variety $H^n(X)$ with the Hodge filtration constructed in 1.2.4 is a pure Hodge structure of the weight n.

(2.1.4) Additionally one may consider an integral Hodge structure, i.e. a lattice $V_\mathbb{Z}$ in $V_\mathbb{Q}$.

(2.1.5) For arbitrary $i \in \mathbb{Z}$ we define $\mathbb{Q}(i)$ by: $V_\mathbb{Q} = \mathbb{Q}$, $F^{-i} = V_\mathbb{C}$, $F^{-i+1} = 0$. It is a pure Hodge structure of the weight $-2i$. With respect to the natural tensor product of Hodge structures we have $\mathbb{Q}(i) \otimes \mathbb{Q}(j) = \mathbb{Q}(i+j)$. The structure $\mathbb{Q}(1)$ is of a special importance: it is called the Tate object.

(2.1.6) A morphism of Hodge structures is a map $f_\mathbb{Q} : V_\mathbb{Q} \to V'_\mathbb{Q}$ such that $f_\mathbb{C}(F^p V_\mathbb{C}) \subset F^p(V'_\mathbb{C})$, where $f_\mathbb{C} = f_\mathbb{Q} \otimes \mathbb{C}$.

(2.1.7) **Proposition:** *A map of Hodge structures of the same weights strictly preserves the filtration, i.e.*

$$im(f_\mathbb{C}) \cap F^p V'_\mathbb{C} = f_\mathbb{C}(F^p V_\mathbb{C}) \,.$$

(2.1.8) **Proposition:** *Pure Hodge structures of a fixed weight form an abelian category. The kernel, cokernel and image have the induced pure Hodge structures.*

2.2 *Category of mixed Hodge structures.*

(2.2.1) **Definition:** *A mixed Hodge structure (MHS) consists of a finite dimensional $\mathbb{Q}$-vector space $V_\mathbb{Q}$, an increasing filtration W_ℓ of $V_\mathbb{Q}$ (weight filtration) and a decreasing filtration F^p of $V_\mathbb{C}$ (Hodge filtration). Moreover the induced Hodge filtration on*

$$Gr_\ell^W V_\mathbb{C} = (W_\ell V_\mathbb{Q}/W_{\ell-1}V_\mathbb{Q}) \otimes \mathbb{C} = W_\ell V_\mathbb{C}/W_{\ell-1}V_\mathbb{C}$$

is a pure Hodge structure of the weight ℓ. We assume that $W_\ell V_\mathbb{Q} = 0$ for ℓ small enough and $W_\ell V_\mathbb{Q} = V_\mathbb{Q}$ for ℓ big enough.

(2.2.2) We say that a mixed Hodge structure is of the weight $\leq w$ (or $\geq w$) if $Gr_\ell^W V_\mathbb{C} = 0$ for $\ell > w$ (resp. $\ell < w$).

(2.2.3) **Proposition:** *A morphism of MHSs (i.e. a map which preserves filtrations) preserves filtrations strictly.*

It follows that there are no nontrivial maps between the structure of the weight $\geq w+1$ and the structure of the weight $\leq w$.

(2.2.4) Let

$$I^{p,q} = (F^p \cap W_{p+q}) \cap (\overline{F^q} \cap W_{p+q} + \overline{F^{q-1}} \cap W_{p+q-2} + \overline{F^{q-2}} \cap W_{p+q-3} + \ldots)$$

An useful fact for the proof of the above Proposition is the following splitting:

(2.2.5) **Lemma:** [GrSc, Lemma 1.12] *We have* $W_\ell = \bigoplus_{p+q\leq\ell} I^{p,q}$ *and* $F^p = \bigoplus_{r\geq p}\bigoplus_q I^{r,q}$.

Therefore if $p+q=\ell$ the projection $W_\ell \to Gr_\ell^W$ sends $I^{p,q}$ isomorphicaly to $(Gr_\ell^W)^{p,q}$. Moreover $I^{p,q} \equiv \overline{I^{q,p}} \mod W_{\ell-2}$ (not only mod $W_{\ell-1}$).

(2.2.6) **Corollary:** *A map of mixed Hodge structures which is an isomorphism of the underlying vector spaces is an isomorphism of the mixed Hodge structures.*

(2.2.7) **Proposition:** *Mixed Hodge structures form an abelian monoidal category. The kernel, cokernel and image have the induced mixed Hodge structures.*

(2.2.8) In the example 1.2.1 for a smooth curve X and D which consists of $k+1$ points we have

$$Gr_1^W H^1(X) = H^1(\overline{X}), \qquad Gr_2^W H^1(X) \simeq \mathbb{Q}(-1)^{\oplus k}.$$

In particular $H^1(\mathbb{C}^*) = Gr_2^W H^1(\mathbb{C}^*) \simeq \mathbb{Q}(-1)$ is a pure Hodge structure of type $(1,1)$.

2.3 *Mixed Hodge complexes.*

(2.3.1) **Definition:** *A mixed Hodge complex consists of:*
– a complex of $\mathbb{Q}$-vector spaces $A^\bullet_{\mathbb{Q}}$ with a weight filtration W_ℓ,
– a complex of $\mathbb{C}$-vector spaces $A^\bullet_{\mathbb{C}}$ with a weight filtration also denoted by W_ℓ,
– a filtered quasiisomorphism $A^\bullet_{\mathbb{Q}} \otimes \mathbb{C} \simeq A^\bullet_{\mathbb{C}}$,
– Hodge filtration $F^p A^\bullet_{\mathbb{C}}$,
such that for all $\ell \in \mathbb{Z}$:
– the map induced by the differential

$$Gr_\ell^W d : Gr_\ell^W A^\bullet_{\mathbb{C}} \to Gr_\ell^W A^{\bullet+1}_{\mathbb{C}}$$

strictly preserves the Hodge filtration,
– the cohomology $H^n(Gr_\ell^W A^\bullet_{\mathbb{Q}})$ with the induced Hodge filtration is a pure Hodge structure of the weight $n+\ell$.

(2.3.2) We say that the cohomology of a mixed Hodge complex $A^\bullet_{\mathbb{Q}}$ is a pure if $H^n(A^\bullet_{\mathbb{Q}})$ is pure Hodge structure of the weight n.

(2.3.3) We describe a mixed Hodge complex $A^\bullet_{\mathbb{C}}$ computing the cohomology of $X = \mathbb{C}^*$. Let $\mathcal{A}^\bullet(\mathbb{C}^*)$ be the complex of all smooth differential forms on $\mathbb{C}^*$.
– $A^\bullet_{\mathbb{C}} \subset \mathcal{A}^\bullet(\mathbb{C}^*)$ is generated by $\frac{1}{2\pi\mathbf{i}}\frac{dz}{z}$ over $\mathcal{A}^\bullet(\mathbb{P}^1)$,
– weight filtration: $W_0 A^\bullet_{\mathbb{C}} = \mathcal{A}^\bullet(\mathbb{P}^1)$ and $W_1 A^\bullet_{\mathbb{C}} = A^\bullet_{\mathbb{C}}$,
– Hodge filtration is obtained by intersecting with the usual filtration by Hodge types $F^p\mathcal{A}^n(\mathbb{C}^*) = \oplus_{r\geq p}\mathcal{A}^{r,q}(\mathbb{C}^*)$,
– the integral structure is given by embedding the $A^\bullet_{\mathbb{C}}$ into the complex of the cochains $\mathrm{Hom}(C_*^{\mathrm{smooth}}(X);\mathbb{C}) = \mathrm{Hom}(C_*^{\mathrm{smooth}}(X);\mathbb{Q}) \otimes \mathbb{C}$. (It is much easier to introduce a real structure in $A^\bullet_{\mathbb{C}}$ distinguishing the real-valued forms.)
Then

$$Gr_0^W A^\bullet_{\mathbb{C}} = \mathcal{A}^\bullet(\mathbb{P}^1), \qquad Gr_1^W A^\bullet_{\mathbb{Q}} \simeq \mathbb{Q}(-1)^{\oplus 2}.$$

The short exact sequence of complexes

$$0 \longrightarrow Gr_0^W A^\bullet_{\mathbb{Q}} \longrightarrow A^\bullet_{\mathbb{Q}} \longrightarrow Gr_1^W A^\bullet_{\mathbb{Q}} \longrightarrow 0$$

induces a short exact sequences of pure Hodge structures of the weight 2

$$0 \longrightarrow H^1(X) \longrightarrow \mathbb{Q}(-1)^{\oplus 2} \xrightarrow{\delta} H^2(\mathbb{P}^1) = \mathbb{Q}(-1) \longrightarrow 0.$$

2.4 *Degeneration of spectral sequences.*

The following theorem is of key importance:

(2.4.1) **Theorem:** [D2, Scholie 8.1.8] *Let* $A^\bullet{}_{\mathbb{Q}}$ *be a mixed Hodge complex. Then:*
– *the spectral sequence associated with the weight filtration degenerates on* ${}_WE_2$,
– *the spectral sequence associated with the Hodge filtration degenerates on* ${}_FE_1$,
– *the shifted weight filtration* $W_\ell H^n(A^\bullet{}_{\mathbb{Q}})$ *which is the image* $H^n(W_{\ell-n}A^\bullet{}_{\mathbb{Q}})$ *together with the induced Hodge filtration is a mixed Hodge structure.*

3. Open smooth varieties

3.1 *Logarithmic complex.*

We will generalize the example 2.3.3 Let X be a smooth complex algebraic variety which can be embedded in a projective space. We find a smooth projective compactification $\overline{X}$. We assume that $D = \overline{X} \setminus X$ is a smooth divisor with normal crossings i.e. the irreducible components of $D = \bigcup_{i=1}^m D_i$ are smooth and locally D is given by the equation $z_1 z_2 \dots z_k = 0$ in a certain coordinate system. Let $\mathcal{A}^\bullet{}_{\mathbb{Q}}(\overline{X}, \log\langle D\rangle) \subset \mathcal{A}^\bullet{}_{\mathbb{Q}}(X)$ be the subcomplex consisting of differential forms which locally can be written as a sum

$$\sum_I \tfrac{1}{(2\pi\,\mathbf{i})^j} \frac{dz_{i_1}}{z_{i_1}} \wedge \frac{dz_{i_2}}{z_{i_2}} \wedge \dots \wedge \frac{dz_{i_j}}{z_{i_j}} \wedge \omega$$

with $I = \{i_1, i_2, \dots, i_j\} \subset \{1, 2, \dots, k\}$ and $\omega \in \mathcal{A}^\bullet(\overline{X})$. The weight filtration W_ℓ is given by $j \le \ell$. The Hodge filtration is given as in 2.3.3.

(3.1.1) **Proposition:** *The embedding* $\mathcal{A}^\bullet{}_{\mathbb{Q}}(\overline{X}, \log\langle D\rangle) \subset \mathcal{A}^\bullet{}_{\mathbb{Q}}(X)$ *is a quasiisomorphism, i.e. induces an isomorphism of cohomology.*

Proof. It is enough to check the isomorphism locally and deduce the global isomorphism by the local-to-global spectral sequence.

3.2 *Poincaré residues.*

(3.2.1) For a multindex $I = \{i_1, i_2, \dots, i_\ell\}$ define $X_I = D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_\ell}$ and $X^\ell = \coprod_{|I|=\ell} X_I$ with $X^0 = \overline{X}$. For $\ell = 0, 1, \dots m$ we have the residue map

$$res_\ell : H^*\big(W_\ell \mathcal{A}^\bullet{}_{\mathbb{Q}}(\overline{X}, \log\langle D\rangle)\big) \to H^{*-\ell}\big(\mathcal{A}^\bullet{}_{\mathbb{Q}}(X^\ell)\big)$$

obtained by extracting the polar part of the differential form along X^ℓ:

$$\tfrac{1}{(2\pi\,\mathbf{i})^\ell} \frac{dz_{i_1}}{z_{i_1}} \wedge \frac{dz_{i_2}}{z_{i_2}} \wedge \dots \wedge \frac{dz_{i_\ell}}{z_{i_j}} \wedge \omega \longmapsto \omega_{|X_I}\,.$$

(3.2.2) **Proposition:** *The map* res_ℓ *descends to an isomorphism*

$$H^*\big(Gr^W_\ell \mathcal{A}^\bullet{}_{\mathbb{Q}}(\overline{X}, \log\langle D\rangle)\big) \simeq H^{*-\ell}\big(\mathcal{A}^{\bullet-\ell}{}_{\mathbb{Q}}(X^\ell)\big)\,.$$

3.3 *Construction of MHS.*

(3.3.1) **Theorem:** *The complex* $\mathcal{A}^\bullet{}_{\mathbb{Q}}(\overline{X}, \log\langle D\rangle)$ *together with the described filtrations is a mixed Hodge complex.*

(3.3.2) **Theorem:** *The resulting mixed Hodge structure in* $H^*(X)$ *does not depend on the compactification.*

Proof. As in 1.2.3 any two compactifications can be compared with a help of a third one. We obtain maps of the mixed Hodge structures which are isomorphisms of the underlying vector spaces. Therefore they are isomorphisms.

(3.3.3) The spectral sequence associated to the weight filtration has the first table

$$ {}_W E_1^{-pq} = H^{q-2p}(X^p)\,. $$

$$
\begin{array}{ccccccccccc}
0 & \to & H^0(X^2) & \to & H^2(X^1) & \to & H^4(X^0) & \to & 0 & {\scriptstyle 4} \\
 & & 0 & \to & H^1(X^1) & \to & H^3(X^0) & \to & 0 & {\scriptstyle 3} \\
 & & 0 & \to & H^0(X^1) & \to & H^2(X^0) & \to & 0 & {\scriptstyle 2} \\
 & & & & 0 & \to & H^1(X^0) & \to & 0 & {\scriptstyle 1} \\
 & & & & 0 & \to & H^0(X^0) & \to & 0 & {\scriptstyle 0} \\
{\scriptstyle -3} & & {\scriptstyle -2} & & {\scriptstyle -1} & & {\scriptstyle 0} & & {\scriptstyle 1} &
\end{array}
$$

The differential d_1 is the alternating sum of Gysin maps $H^*(X^I) \to H^{*+2}(X^{I\setminus\{i\}})$. Since we care about the Hodge structure we should rather write $H^*(X^I) \to H^{*+2}(X^{I\setminus\{i\}})(1)$, where $V(1)$ denotes the Tate twist $V \otimes \mathbb{Q}(1)$. According to 2.4.1 the spectral sequence ${}_W E$ degenerates on ${}_W E_2$. Therefore

$$ Gr_\ell^W H^n(X) = H^{n-\ell}\big(\dots \to \underbrace{H^{n-4}(X^2)(-2)}_{-2} \to \underbrace{H^{n-2}(X^1)(-1)}_{-1} \to \underbrace{H^n(X^0)}_{0} \to 0\,\big)\,. $$

3.4 *Cohomology with compact supports.*

It turns out that for many purposes the cohomology with compact supports is more convenient. Let $d = \dim X$.

(3.4.1) Since $H^n_c(X) = \big(H^{2d-n}(X)\big)^*$ we have the dual spectral sequence

$$ {}_W E_1^{pq} = H^q(X^p)\,. $$

$$
\begin{array}{cccccccccccc}
{\scriptstyle 2d} & 0 & \to & H^{2d}(X^0) & \to & 0 & & & & & & \\
{\scriptstyle 2d-1} & 0 & \to & H^{2d-1}(X^0) & \to & 0 & & & & & & \\
{\scriptstyle 2d-2} & 0 & \to & H^{2d-2}(X^0) & \to & H^{2d-2}(X^1) & \to & 0 & & & & \\
{\scriptstyle 2d-3} & 0 & \to & H^{2d-3}(X^0) & \to & H^{2d-3}(X^1) & \to & 0 & & & & \\
{\scriptstyle 2d-4} & 0 & \to & H^{2d-4}(X^0) & \to & H^{2d-4}(X^1) & \to & H^{2d-4}(X^2) & \to & 0 & & \\
 & & & \dots & & \dots & & \dots & & \dots & & \\
{\scriptstyle 0} & 0 & \to & H^0(X^0) & \to & H^0(X^1) & \to & H^0(X^2) & \to & H^0(X^3)) & \to & \\
 & {\scriptstyle -1} & & {\scriptstyle 0} & & {\scriptstyle 1} & & 2 & & 3 & &
\end{array}
$$

This time the differential d_1 comes from the usual restrictions of forms. The spectral sequence degenerates on ${}_W E_2$. Therefore

$Gr_\ell^W H^n_c(X) = H^{n-\ell}\big(0 \to \underbrace{H^\ell(X^0)}_{0} \to \underbrace{H^\ell(X^1)}_{1} \to \underbrace{H^\ell(X^2)}_{2} \to \dots\big)$. Loosely speaking this complex may be understood as an inclusion–exclusion formula.

(3.4.2) The above complex of pure Hodge structures is just a reflection of a more fundamental feature. Gillet and Soulé defined in [GiSo] a *weight complex* $W^\bullet(X)$ which is a complex of motives. Here $W^\ell(X)$ is equal to the motive $M(X^\ell)$. The homotopy class of this complex does not depend on the compactification $\overline{X}$. From it one can construct a weight filtration in integral cohomology of X.

4. Singular varieties

4.1 *Simplicial varieties.*

(4.1.1) For many purposes it is worth to consider simplicial varieties:

$$X_\bullet = \left\{ X_0 \leftleftarrows X_1 \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; X_2 \;\cdots \right\},$$

i.e. a sequence of varieties X_n with maps $\partial_i : X_n \to X_{n-1}$ for $i = 0, 1, \ldots, n$ satisfying the usual simplicial identities modeled on the restrictions to faces in the standard simplex. (The degeneracy maps are irrelevant for our purposes.)

(4.1.2) With a simplicial variety one associates a topological space: its geometric realization:

$$|X_\bullet| = \Big(\coprod_n (\Delta_n \times X_n)\Big)/\sim \qquad (p, \partial_i x) \sim (\delta^i p, x)\,,$$

where $\delta^i : \Delta^{n-1} \to \Delta^n$ is the standard inclusion of the i-th face into the standard simplex. If the simplicial variety is augumented, (i.e. there is given an additional map $\epsilon : X_0 \to X$) we obtain a map from the realization $|\epsilon| : |X_\bullet| \to X$

4.2 *Hypercovering of a singular variety.*

(4.2.1) **Definition:** *We say that an augumented simplicial variety $\epsilon : X_\bullet \to X$ is a hypercovering of X if the map $|\epsilon|$ has contractible fibers.*

(4.2.2) By the resolution of singularities each variety has a simplicial smooth hypercovering. If X is compact, then we may find a hypercovering consisting of smooth compact varieties. Moreover, two hypercoverings can be compared with a third one through simplicial maps.

(4.2.3) If $X = \bigcup_{i=1}^m Y_i$ consists of smooth components, such that each intersection $Y_I = Y_{i_0} \cap Y_{i_1} \cap \ldots \cap Y_{i_n}$ is smooth then $X_n = \coprod_{|I|=n+1} Y_I$ together with the inclusion maps is a smooth hypercovering of X.

4.3 *Construction of MHS.*

(4.3.1) For a smooth compact simplicial variety let us define a mixed Hodge complex

$$\mathcal{A}^n(X_\bullet) = \bigoplus_{s+t=n} \mathcal{A}^s(X_t)\,,$$

which is the total complex of the bicomplex with

$$d^I\omega = d\omega\,, \qquad d^{II}\omega = \sum_{i=0}^{t} (-1)^i \partial_i^* \omega$$

for $\omega \in \mathcal{A}^\bullet(X_t)$. The rational structure and the Hodge filtration is given as in the previous section. The weight filtration is given by $t \geq -\ell$. The cohomology of the complex $\mathcal{A}^n(X_\bullet)$ is equal to $H^*(|X_\bullet|;\mathbb{C})$.

(4.3.2) Let X be a compact variety and $X_\bullet$ its hypercovering. Let us define the mixed Hodge structure in $H^*(X) = H^*(|X_\bullet|)$ as the structure which is induced from the mixed Hodge complex $\mathcal{A}^n_{\mathbb{Q}}(X_\bullet)$. Since any two hypercoverings are comparable, the induced structure does not depend on it.

(4.3.3) For the variety described in 4.2.3 the weight spectral sequence has the shape of 3.4.1 with X^j replaced by X_j.

(4.3.4) If a variety is singular and not compact one constructs MHS combining the methods presented in §3 and §4

(4.3.5) Guillen and Navarro Aznar ([GN-A1]) gave more efficient way of defining the mixed Hodge structure on $H^*(X)$. It is based on the *cubic hyperresolution* which encodes the pull-back diagrams for resolutions of successive singular strata. The strategy is to replace X by a diagram of nonsingular varieties. First we resolve X and obtain the pull-back square $\begin{array}{ccc} E & \to & \widetilde{X} \\ \downarrow & & \downarrow \\ Z & \to & X \end{array}$, where Z is the singular locus of X. The procedure is repeated for Z and E: we resolve their singularities in a consistent way. Finally we obtain a diagram indexed by vertices of a cube of dimension $d+1$ where $d = \dim X$.

(4.3.6) Gillet and Soulé have constructed in [GiSo] a weight complex $W^\bullet(X)$ which can also be used to compute MHS. It is defined up to a homotopy. For example we may set $W^\ell(X) = M(X_\ell)$ with differential which is an alternating sum of ∂_i^*.

4.4 *Virtual Poincaré polynomials.*

A nice application of the mixed Hodge structure is a method of computing the usual Betti numbers of smooth compact varieties. For a smooth compact variety X define the Poincaré polynomial

$$P_X(t) = \sum_{n=0}^{2d} (-1)^n \dim(H^n(X)) t^n \in \mathbb{Z}[\,t\,]\,.$$

(4.4.1) **Theorem:** *There is an unique extension of the assignment $X \mapsto P_X(t)$ for all algebraic varieties such that whenever Y is a closed subset in X and U is the complement then*

$$P_X(t) = P_Y(t) + P_U(t)\,.$$

Moreover

$$P_{X_1\times X_2}(t) = P_{X_1}(t)\,P_{X_2}(t)\,.$$

Proof. For an arbitrary variety X define the virtual Poincaré polynomial $P_X(t)$ to be the weighted Euler characteristic

$$P_X(t) = \sum_{\ell,n=0}^{2d} (-1)^n \dim(Gr_\ell^W H_c^n(X)) t^\ell \in \mathbb{Z}[\,t\,]\,.$$

If X is smooth and compact, then the only summand with fixed ℓ is $n = \ell$. The additivity follows from the long exact sequence of the pair

$$\to H_c^n(U) \to H_c^n(X) \to H_c^n(Y) \to H_c^{n+1}(U) \to \;.$$

This is a sequence of mixed Hodge structures, therefore it induces the long exact sequence of graded groups

$$\to Gr_\ell^W H_c^n(U) \to Gr_\ell^W H_c^n(X) \to Gr_\ell^W H_c^n(Y) \to Gr_\ell^W H_c^{n+1}(U) \to \;.$$

The Euler characteristic is additive with respect to entries of the long exact sequences.

(4.4.2) Similarly one defines a virtual Hodge polynomial

$$\sum_{p,q,\ell=0}^{2d} (-1)^{p+q} \dim(Gr_\ell^W H_c^n(X)^{p,q}) u^p v^q \in \mathbb{Z}[u,v]\,.$$

It is additive and multiplicative as well.

(4.4.3) Application: suppose X is a smooth compact variety. Assume that it is decomposed into locally closed subvarieties $T_i \simeq (\mathbb{C}^*)^{n_i}$. For example X is a toric variety. Then $P_X(t) = \sum_i (t^2-1)^{n_i}$. The formula follows since $P_{\mathbb{C}^*}(t) = P_{\mathbb{P}^1}(t) - 2P_{pt}(t) = t^2 - 1$. Setting $t = \sqrt{q}$ the Poincaré polynomial computes the number of points of a toric variety over the field $\mathbb{F}_q$.

(4.4.4) In general any invariant κ defined for smooth compact algebraic varieties with values in a commutative group uniquely extends to an additive invariant defined for arbitrary varieties *if and only if* it satisfies the blow-up relation

$$\kappa(\widetilde{X}) - \kappa(E) = \kappa(X) - \kappa(Z)\,,$$

see [GN-A2] or [Bi]. Here $\widetilde{X}$ is the blow-up of X along Z with the exceptional divisor E.

(4.4.5) By the same rule one can prove the uniqueness (up to homotopy) of the weight complex $W^\bullet(X)$. It is enough to show that the complexes $0 \to M(\widetilde{X}) \to M(E) \to 0$ and $0 \to M(X) \to M(Z) \to 0$ are homotopy equivalent. This is so since by Manin [Ma] $M(\widetilde{X}) \simeq M(X) \oplus \bigoplus_{i=1}^{c-1} M(Z)(-i)$ and $M(\widetilde{E}) \simeq \bigoplus_{i=0}^{c-1} M(Z)(-i)$, where $c = \operatorname{codim} Z$, and (i) is the motivic version of the i-fold Tate twist considered in 3.3.3.

5. Actions of algebraic groups

5.1 *Simplicial varieties arising from a group action*

(5.1.1) Let G be an algebraic linear group. The classifying space of G as a simplicial variety has been considered by Deligne [D3, §6.1 and §9.1]:

$$BG_\bullet = \left\{ pt \;\overset{\longleftarrow}{\longleftarrow}\; G \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; G \times G \; \cdots \right\} .$$

The arrows are multiplications or forgetting the edge factors.

(5.1.2) The Borel construction for a G-variety X is the simplicial variety

$$(EG \times_G X)_\bullet = \left\{ X \;\overset{\longleftarrow}{\longleftarrow}\; X \times G \;\overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}}\; X \times G \times G \; \cdots \right\} .$$

The arrows are the action, multiplications or forgetting the last factor. Of course we have $(EG \times_G pt)_\bullet \simeq BG_\bullet$.

(5.1.3) The equivariant cohomology of a G-variety is defined as $H^*(|(EG \times_G X)_\bullet|)$. It is a module over $H^*(|BG_\bullet|)$.

(5.1.4) Another algebraic model of the classifying space and the Borel construction has been considered by Totaro in [To]. Instead of simplicial methods he approximates BG by algebraic varieties. For example the classifying space of the general linear group is approximated by Grassmannians: $BGL_n = \lim_{N\to\infty} Grass_n(\mathbb{C}^N)$. We will not distinguish between models of BG and we will drop simplicial decorations.

The mixed Hodge structures of $H^*(BG)$ and $H^*(G)$ are of the following types:

(5.1.5) **Theorem:** *The Hodge structure of $H^*(BG)$ is pure. Only the types (p,p) occur.*

For example, $H^*(B\mathbb{C}^*) \simeq H^*(\mathbb{P}^\infty)$ is a polynomial algebra on one pure generator of Hodge type $(1,1)$.

(5.1.6) Suppose that G is connected. Let $P^* = P^1 \oplus P^3 \oplus P^5 \oplus \ldots$ be the space of primitive elements of the Hopf algebra $H^*(G)$, so that $H^*(G)$ is the exterior algebra $\bigwedge P^*$. Note that primitive elements occur only in odd degrees.

(5.1.7) **Theorem:** *Let $k > 0$. Then*
– $W_k H^k(G) = 0$;
– *for k even* $W_{k+1} H^k(G) = 0$;
– *for k odd* $W_{k+1} H^k(G) = P^k$ *and it is of the type* $(\frac{k+1}{2}, \frac{k+1}{2})$.

One can filter the cohomology of G by "complexity": $C_a H^*(G) = \bigwedge^{\leq a} P^*$. Then $C_a H^*(G) \cap H^k(G) = W_{a+k} H^k(G)$.

5.2 *Formality of equivariant cohomology.*

Passing from cohomology to equivariant cohomology and back is possible with a help of spectral sequences. We assume that G is **connected**.

(5.2.1) For computing equivariant cohomology one has the Leray spectral sequences of the projection map $EG \times X \to BG$

$$E_2^{p,q} = H^p(BG_\bullet) \otimes H^q(X) \Rightarrow H_G^{p+q}(X)\,.$$

This spectral sequences preserves Hodge structure. If $H^*(X)$ is pure then the entry $E_2^{p,q}$ is pure of the weight $p+q$. Therefore all the differentials (starting from d_2) vanish.

(5.2.2) **Corollary:** *If X is smooth and compact then $H^*_G(X) \simeq H^*(BG) \otimes H^*(X)$ as a $H^*(BG)$-module.*

Then we say that the equivariant cohomology is formal.

(5.2.3) Example: Let $G = \mathbb{C}^*$ and let X be smooth and compact. Consider the approximation of $EG \times_G X$ through the varieties $Y_n = \mathbb{C}^{n+1} \times_G X$. The space Y_n is fibered over $\mathbb{P}^n$. It admits a decomposition $Y_n = \coprod_{i=0}^n X \times \mathbb{C}^i$ which is the inverse image of the decomposition of $\mathbb{P}^n$ into cells. Therefore by [Ka] the motive of Y_n decomposes as

$$M(Y_n) = \textstyle\bigoplus_{i=0}^n M(X)(-i)$$

Informally we write

$$M_G(X) = \textstyle\bigoplus_{i=0}^\infty M(X)(-i)$$

It follows that $H^*_G(X;\mathbb{Z}) = H^*(X;\mathbb{Z}) \otimes H^*(BG;\mathbb{Z})$. The same argument applies when $G = GL_n$.

5.3 *Degeneration of Eilenberg-Moore spectral sequence.*

(5.3.1) For passing from equivariant cohomology to the usual one we have the Eilenberg-Moore spectral sequence

$$E_2^{-p,q} = \mathrm{Tor}_p^{q,H^*(BG)}(H^*_G(X),\mathbb{Q}) \Rightarrow H^{q-p}(X)\,.$$

The torsion product is the left derived functor of $- \otimes -$ and it is denoted by $\mathrm{Tor}_p^{H^*(BG)}(-,-)$. It is positively graded. Additionally it has an internal grading. Its degree–q piece is denoted by $\mathrm{Tor}_p^{q,H^*(BG)}(-,-)$.

By [Ha] we have:

(5.3.2) **Theorem:** *The entries of the Eilenberg–Moore spectral sequence are equipped with mixed Hodge structures. The differentials are the maps of the MHS.*

The following results have been proved in [FrWe]:

(5.3.3) **Theorem:** *If the equivariant cohomology $H^*_G(X)$ is pure, then $E_2^{-p,q}$ is pure of the weight q. The spectral sequence degenerates. The cohomology of X is given additively by:*

$$H^n(X) = \bigoplus_{q-p=n} \mathrm{Tor}_p^{q,H^*(BG)}(H^*_G(X),\mathbb{Q})\,.$$

The sum of terms with $q \le \ell$ coincides with $W_\ell H^*(X)$.

This theorem follows from the fact that $Tor^A_*(M,N)$ can be computed from the bar-complex Bar_*. Each entry of the bar-complex $Bar_p = M\otimes \overline{A}^{\otimes p}\otimes N$ is pure provided that A, M and A are pure.

It turns out that in many cases the purity assumption is satisfied.

(5.3.4) **Theorem:** *If a variety X is smooth and consists of finitely many orbits of G, then $H^*_G(X)$ is pure. The equivariant cohomology of X is given additively by:*

$$H^*_G(X) = \bigoplus H^{*-2c}(BH)\,,$$

where the sum is taken over all orbits $\mathcal{O} = G/H \subset X$, and $c = \mathrm{codim}\ \mathcal{O}$.

The proof is an easy induction with respect to orbits. To be able to order them we assume that X is quasiprojective.

(5.3.5) If the variety is singular one can prove an analogous result for equivariant cohomology replaced by equivariant homology with closed supports of Edidin–Graham.

(5.3.6) The motivic or geometric meaning of the above theorem is not clear. One would have to argue that the weight complex of the cosimplicial variety $Y_n = (EG\times_G X)\times BG^n$ is equivalent to the weight complex of X.

Bibliography:

[Ar] D. Arapura *Building Mixed Hodge Structures. The arithmetic and geometry of algebraic cycles,* 13–32, CRM Proc. Lecture Notes, 24, Amer. Math. Soc., (ArXiv math.AG/9908036)

[Bi] F. Bittner *The universal Euler characteristic for varieties of characteristic zero.* Compos. Math. 140 (2004), no. 4, 1011–1032.

[BZ] J.-L. Brylinski, S. Zucker *An overview of recent advances in Hodge theory, Several complex variebles VI,* 39-142, Encyclopaedia Math. Sci., 69, Springer, Berlin, 1990.

[Cl] C. H. Clemens *Degeneration of Kähler manifolds.* Duke Math. J. 44 (1977), no. 2, 215–290.

[D1] P. Deligne *Théorie de Hodge I,* Actes Congr. Int. Math. 1970 Vol I, 425-430.

[D2] P. Deligne *Théorie de Hodge II,* Publ. Math. I.H.E.S. No. 40 (1972) 5-57.

[D3] P. Deligne *Théorie de Hodge III,* Publ. Math. I.H.E.S. No. 44 (1974) 5-77.

[Du] A. H. Durfee *A naive guide to mixed Hodge structures,* Proc. Symp. in Pure Math. Vol. 40/I (1983) 313-320.

[FW] M. Franz, A. Weber *Eilenberg–Moore spectral sequence and weights in cohomology.* to appear in Ann. Inst. Fourier.

[GM] S. I. Gelfand, Yu. I. Manin *Homological algebra.* Springer Verlag 1999, Chapter 6.

[GoMP] M. Goresky, R. MacPherson *Intersection homology II.* Invent. Math. Vol 72 (1983) 77–130.

[GiSo] H. Gillet, C. Soulé *Descent, motives and K-theory.* J. Reine Angew. Math. 478 (1996), 127–176.

[GrSc] P. Griffiths, W. Schmid *Recent developments in Hodge theory, a discussion of techniques and results.* (Proceedings of the international Colloquium on Discrete Subgroups of Lie Groups - Tata Institute, Bombay 1973.) Oxford Univ. Press, London (1975), 31-127.

[GN-A1] F. Guillen, V. Navarro Aznar, P. Pascual Gainza, F. Puerta *Hyperresolutions cubiques et descente cohomologique.* Lecture Notes in Mathematics, 1335. Springer-Verlag, Berlin, 1988.

[GN-A2] F. Guillen, V. Navarro Aznar *Un critère d'extension des foncteurs définis sur les schámas lisses.* Publ. Math. I.H.E.S. No. 95 (2002), 1–91.

[Ha] R. M. Hain *The de Rham homotopy theory of complex algebraic varieties. I* K-Theory, Vol 1 (1987) no 3, pp 271-324.

[Ka] N. A. Karpenko *Grothendieck Chow motives of Severi-Brauer varieties.* Algebra i Analiz 7 (1995), no. 4, 196–213; translation in St. Petersburg Math. J. 7 (1996), no. 4, 649–661

[Ma] Yu. I. Manin *Correspondences, motives and monoidal transformations.* Math. USSR Sbornik 6 (1968), 439-470.

[Sa] M. Saito *Introduction to mixed Hodge modules.* Astérisque 179-180 (1989) 145–162.

[St] J. Steenbrink *Limits of Hodge structures.* Invent. Math. 31 (1975/76), no. 3, 229–257.

[To] B. Totaro *The Chow ring of a classifying space* Algebraic K-theory. Symp. Pure Math. Vol 67, ed. Raskind, Wayne et al. Providence, RI: AMS Proc. (1999) 249–281.

[Vo] C. Voisin *Hodge theory and complex algebraic geometry. I & II.* Cambridge Studies in Advanced Mathematics, 76 & 77. Cambridge University Press, Cambridge, 2002 & 2003

[We] A. Weber *Pure homology of algebraic varieties.* Topology, Vol 43, no 3 (2004) pp 635-644.

Uniwersytet Warszawaki, Instytut Matematyki
ul. Banacha 2, 02-097 Warszawa
POLAND
e-mail: `aweber@mimuw.edu.pl`