HECKE ALGEBRA ACTION ON TWISTED MOTIVIC CHERN CLASSES AND K-THEORETIC STABLE ENVELOPES

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ABSTRACT. Let G be a linear semisimple algebraic group and B its Borel subgroup. Let $\mathbb{T} \subset B$ be the maximal torus. We study the inductive construction of Bott-Samelson varieties to obtain recursive formulas for the twisted motivic Chern classes of Schubert cells in G/B. To this end we introduce two families of operators acting on the equivariant K-theory $K_{\mathbb{T}}(G/B)[y]$, the right and left Demazure-Lusztig operators depending on a parameter. The twisted motivic Chern classes coincide (up to normalization) with the K-theoretic stable envelopes. Our results imply wall-crossing formulas for a change of weight chamber and slope parameters. The right and left operators generate a twisted double Hecke algebra. We show that in the type A this algebra acts on the Laurent polynomials. This action is a natural lift of the action on $K_{\mathbb{T}}(G/B)[y]$ with respect to the Kirwan map. We show that the left and right twisted Demazure-Lusztig operators provide a recursion for twisted motivic Chern classes of matrix Schubert varieties.

1. Introduction

Schubert varieties and their cohomological invariants are important objects of enumerative geometry. These varieties are usually singular, yet they admit a well-studied resolution of singularities called Bott-Samelson resolution. In many cases the inductive construction of Bott-Samelson varieties gives rise to recursive formulas for various cohomological invariants. These formulas allow to compute the class of a Schubert cell from classes of smaller cells.

The study of inductive properties of various characteristic classes of Schubert varieties is widely presented in the literature. Starting from formulas for fundamental classes in cohomology [BGG73] or in K-theory [LS82, KK90] the recursion based on word length became a standard feature of cohomological study of homogenous varieties. Among further important contributions we mention [Bri97, Knu03] for fundamental classes, [AM16, MNS22] for c_{SM} classes, [AMSS19, MNS22, MS22] for motivic Chern classes, [AMSS22] for Hirzebruch-Todd classes, [SZZ20, SZZ21] for stable envelopes, [RW20, KRW20] for elliptic classes and [MNS22] for classes in the quantum cohomology. Most of the mentioned results are nicely reviewed in [MNS22]. In [RW22a] the study of such recurrences allowed to observe an instance of mirror symmetry. In this paper we study the inductive properties of the twisted motivic Chern class.

The motivic Chern class mC was defined in [BSY10]. It generalizes several other classes such as Chern-Schwartz-MacPherson c_{SM} class [Mac74], Baum-Fulton-MacPherson Todd class [BFM75], or L-class (see [SY07] for a broad survey). It may be thought of as a relative, K-theoretic version of the Hirzebruch-Todd genus [Hir56], which is also defined for singular varieties. The equivariant versions are due to [Ohm06, Web16, FRW21, AMSS19]. In [KW22] the twisted version of the motivic Chern class was defined. It takes into account a chosen fractional line bundle. The motivic Chern class (and its twisted

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version) of a locally closed subvariety $X \subset M$ can be explicitly computed in terms of a resolution of singularities of the closure of X.

We consider semisimple, simply connected algebraic group G with a chosen Borel subgroup B and a maximal torus $\mathbb{T}$. The rational Picard group $\mathrm{Pic}(G/B) \otimes \mathbb{Q}$ is isomorphic to the space of rational characters $\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$. We study the twisted motivic Chern class $\mathrm{mC}^{\mathbb{T}}(w, \lambda) = \mathrm{mC}^{\mathbb{T}}(X_w, \partial X_w; \mathcal{L}(\lambda))$ of the Schubert variety X_w for a Weyl group element w and a fractional character λ (see section 5.3 for a definition). To compute such classes we use the Bott-Samelson resolution. It assigns to a reduced word decomposition $\underline{w}$ of a Weyl group element w a resolution of singularities of the Schubert variety X_w

$$p_{\underline{w}} : Z_{\underline{w}} \to X_w \subset G/B\,.$$

Let s be a simple reflection, α_s the corresponding simple root and P_s the corresponding minimal parabolic subgroup. Suppose that w is a Weyl group element such that ws is longer than w. The Bott-Samelson variety $Z_{\underline{ws}}$ can be constructed inductively as a fiber product

$$\begin{tikzcd} Z_{\underline{ws}} \arrow[rr, "p_{\underline{ws}}"] \arrow[d] & & G/B \arrow[d] \\ Z_{\underline{w}} \arrow[r, "p_{\underline{w}}"] & G/B \arrow[r] & G/P_s \end{tikzcd}$$

This construction induces a recursive formula for the twisted class. We define the twisted Demazure-Lusztig operator

$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} = \frac{1 + y\mathcal{L}_s^*}{1 - \mathcal{L}_s} \cdot s^{\mathrm{R}} - \frac{(1+y) \cdot \mathcal{L}_s^{\lceil -\langle \lambda, \alpha_s^\vee \rangle \rceil}}{1 - \mathcal{L}_s} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\,,$$

where $\mathcal{L}_s$ is the relative tangent bundle of the projection $G/B \to G/P_s$ and s^{R} denotes the right Weyl group action of s (see section 3.2). Our first result is the following theorem.

Theorem (6.12)**.** *Let $w \in (G/B)^{\mathbb{T}} \simeq W$ be a fixed point and λ a general enough fractional character. Then*

$$(-y)^{\frac{1}{2}(l(w)+1-l(ws))} \cdot \mathrm{mC}^{\mathbb{T}}(ws, s\lambda) = \mathcal{T}^{\mathrm{R}}_{s,\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

We obtain also a left counterpart of the above recursive formula. We define the left twisted Demazure-Lusztig operator

$$\mathcal{T}^{\mathrm{L}}_{s,\lambda} := \frac{1 + y\alpha_s^{-1}}{1 - \alpha_s^{-1}} \cdot s^{\mathrm{L}} - \frac{(1+y) \cdot \alpha_s^{-\lceil \langle \lambda, \alpha_s^\vee \rangle \rceil}}{1 - \alpha_s^{-1}} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\,.$$

We use the recursive formula 6.12 to obtain the following theorem.

Theorem (8.1)**.** *Let $w \in (G/B)^{\mathbb{T}} \simeq W$ be a fixed point and λ a general enough fractional character. Then*

$$(-y)^{\frac{1}{2}(l(w)+1-l(sw))}\, \mathrm{mC}^{\mathbb{T}}(sw, \lambda) = \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

One of the important features of the twisted motivic Chern class is its connection with the K-theoretic stable envelope. Stable envelopes are characteristic classes defined initially for symplectic resolutions in three types: cohomological [MO19], K-theoretic [Oko17, OS22] and elliptic [AO21]. Their definition is still evolving see e.g. [Oko21] for a recent progress. In this paper we consider only the K-theoretic stable envelopes. They depend on a fractional line bundle called slope.

Stable envelopes for the cotangent variety of a homogeneous space were extensively studied see e.g [Su17, RTV15, RTV19, RSVZ19, SZ20]. They are tightly connected to the characteristic classes mentioned earlier. See [FR18, RV18, AMSS17] for a comparison with the c_{SM} class in cohomology. In the K-theory, a comparison between the stable envelope for a small anti-ample slope and the motivic Chern class was carried out in [AMSS19, FRW21, Kon22]. The twisted class of [KW22] is a generalization of the motivic Chern class which agrees with the stable envelope for a general enough slope. It allows to define the stable envelope in terms of a resolution of singularities of the Schubert variety. Our recursive formulas may be restated in the language of stable envelopes.

Corollary (6.13, 8.11). *Let $w \in (G/B)^{\mathbb{T}} \simeq W$ be a fixed point. Consider a general enough fractional character λ. Then*

$$q^{1/2} \cdot \mathrm{stab}^{s\lambda}_{\mathfrak{C}_+}(ws) = \mathcal{T}^{\mathrm{R},q}_{s,\lambda}(\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w))\,,$$
$$q^{1/2} \cdot \mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(sw) = \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w))\,.$$

where $\mathcal{T}^{\mathrm{R},q}_{s,\lambda}$ and $\mathcal{T}^{\mathrm{L},q}_{s,w\lambda}$ denote operators $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathcal{T}^{\mathrm{L}}_{s,w\lambda}$ after substitution $y = -q$, respectively.

Corollary (6.14). *Let $w \in (G/B)^{\mathbb{T}} \simeq W$ be a fixed point. Suppose that ws is longer than w. Consider an arbitrary fractional character λ and a small anti-ample character λ^-. For a small enough positive real number ε we have*

$$q^{1/2} \cdot \mathrm{stab}^{s\lambda+\varepsilon\lambda^-}_{\mathfrak{C}_+}(ws) = \mathcal{T}^{\mathrm{R},q}_{s,\lambda}(\mathrm{stab}^{\lambda+\varepsilon\lambda^-}_{\mathfrak{C}_+}(w))\,.$$

Best known recursive formulas computing the stable envelope for G/B concern only specific values of the slope parameter (see e.g. [SZZ20, proposition 3.3 and theorem 3.5] and [SZZ21, theorem 5.4]). Our formulas generalize these results and work for an arbitrary slope.

One of the important notions in the theory of K-theoretic stable envelopes are R-matrices (also called wall-crossing formulas). They describe the behavior of the stable envelope for elementary modifications of parameters (weight chamber and slope). They were studied in e.g. [OS22, SZZ21, GN17, Smi21]. In [SZZ21] the slope R-matrix was fully computed for generalized flag varieties.

Our recursive formulas allow for an alternative, geometric approach to the wall-crossing problem based on the study of Bott-Samelson resolution. The right recursion leads to the wall-crossing formula related to the change of slope (theorem 9.3), which is equivalent to the results of [SZZ21]. The left induction implies the wall-crossing formula related to the change of weight chamber (theorem 8.12).

Recursive formulas in the K-theory surprisingly rely on simple algebraic operators involving characters and the classes of line bundles. It turns out that one can build an algebra generated by natural lifts of operators $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathcal{T}^{\mathrm{L}}_{s,\lambda}$, denoted by $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$, or their variants depending on a scalar $\mathfrak{T}^{\mathrm{R}}_{s,a}$ and $\mathfrak{T}^{\mathrm{L}}_{s,a}$. That algebra acts on the representation ring $R(\mathbb{T}^2)$ extended by the formal variable y. Practically this means, that to check identities in this algebra it is enough to perform calculus involving Laurent polynomials in two sets of variables and y. This algebra formally resembles the construction of Hecke algebra of Ginzburg-Kapranov-Vasserot, but there is a difference: the algebra of [GKV97] acts on $R(\mathbb{T} \times \mathbb{T}^\vee)$, where $\mathbb{T}^\vee$ is the dual torus. On the other hand we have a dependence on λ, which means that our operators can be treated as functions on $\mathfrak{t}^*$. It is remarkable that the braid relations hold for the lifted operators. Braid relations can be deduced from the properties of the motivic Chern classes or simply checked by hand performing elementary transformation of rational functions. Our twisted Hecke algebra is presented in full generality in the last section. Before, starting from section 10 we analyze the A_n

case. The braid relations have the form (satisfied by left and right operators)

$$\mathscr{T}_{i,s_{i+1}s_i\lambda} \circ \mathscr{T}_{i+1,s_i\lambda} \circ \mathscr{T}_{i,\lambda} = \mathscr{T}_{i+1,s_is_{i+1}\lambda} \circ \mathscr{T}_{i,s_{i+1}\lambda} \circ \mathscr{T}_{i+1,\lambda}$$

or

$$\mathfrak{T}_{i,a} \circ \mathfrak{T}_{i+1,a+b} \circ \mathfrak{T}_{i,b} = \mathfrak{T}_{i+1,b} \circ \mathfrak{T}_{i,a+b} \circ \mathfrak{T}_{i+1,a}\,,$$

see lemma 11.3 and propositions 11.7, 11.9 The quadratic relations hold: if $\langle\lambda,\alpha_i^\vee\rangle \notin \mathbb{Z}$, $a \notin \mathbb{Z}$

$$\mathscr{T}^{\mathrm{L}}_{i,s_i\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i,\lambda} = -y\,\mathrm{id}\,,$$
$$\mathfrak{T}^{\mathrm{L}}_{i,-a} \circ \mathfrak{T}^{\mathrm{L}}_{i,a} = -y\,\mathrm{id}\,.$$

For integer values of the parameter a the relation is different, we have

$$\mathfrak{T}^{\mathrm{L}}_{i,1-a} \circ \mathfrak{T}^{\mathrm{L}}_{i,a} = -y\,\mathrm{id}\,,$$

see lemma 11.1. Dependence on the parameter is not a surprise. Felder in [Fel95], see also [EV98], stated braid relations in their incarnation of Yang-Baxter equation for R-matrices

$$R^{(12)}(z)R^{(13)}(z+w)R^{(23)}(w) = R^{(23)}(w)R^{(13)}(z+w)R^{(12)}(z)\,.$$

A natural question arises: How the operators $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ are related to geometry? What do they compute? We give an answer in the A_n case. We prove that after a suitable normalization the operators $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ provide recursive formulas for the twisted motivic Chern classes of matrix Schubert varieties $\mathcal{X}_w$. We consider only the maximal rank Schubert varieties in $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)$, i.e. $B\times B$-orbits of the permutation matrices. We prove

Theorem (12.6)**.** *The left and right recursions hold*

$$\mathscr{T}^{\mathrm{L}}_{i,w\lambda}\big(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{X}_w,\partial\mathcal{X}_w;D_{w,\lambda})\big) = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{X}_{s_iw},\partial\mathcal{X}_{s_iw};D_{s_iw,\lambda}) \quad \textit{if} \;\; l(s_iw)>l(w)\,,$$
$$\mathscr{T}^{\mathrm{R}}_{i,\lambda}\big(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{X}_w,\partial\mathcal{X}_w;D_{w,\lambda})\big) = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{X}_{ws_i},\partial\mathcal{X}_{ws_i};D_{ws_i,s_i\lambda}) \quad \textit{if} \;\; l(ws_i)>l(w)\,.$$

Here $\mathscr{B}$ is a certain fixed rational function (see the formula (19)) and $D_{w,\lambda}$ is a distinguished divisor contained in the boundary of the matrix Schubert variety (definition 15). The factor $\mathscr{B}$ is responsible for the motivic Chern class of the fiber of the projection $\mathrm{GL}_n \to \mathrm{GL}_n/B$, and it is present in a similar formula for Rimányi-Tarasov-Varchenko weight function, [RTV15, section 6.1]. For the proof we apply two resolutions of matrix Schubert varieties: left resolution and right resolution, giving rise to left and right Demazure-Lusztig operators.

The geometry hidden behind the operators $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ for a general semisimple group is not discussed. We plan to present it in another paper, not enlarging the present exposition.

Contents

2. Notations

All considered varieties are complex and quasi-projective. We consider an algebraic torus $\mathbb{T} \simeq (\mathbb{C}^*)^{\mathrm{rk}\,\mathbb{T}}$.

2.1. **Line bundles and divisors.** Let X be a quasiprojective $\mathbb{T}$-variety and $\mathcal{L} \to X$ an equivariant line bundle i.e. a line bundle together with a linearization. For any fixed point $x \in X^{\mathbb{T}}$, the fiber $\mathcal{L}_{|x}$ is a representation of the torus $\mathbb{T}$.

Suppose that X is a smooth $\mathbb{T}$-variety, $x \in X^{\mathbb{T}}$ a fixed point and D a $\mathbb{T}$-invariant codimension one subvariety. The line bundle $\mathcal{O}_X(D)$ (treated as a subsheaf of meromorphic functions) has the natural linearization such that

- $\mathcal{O}_X(D)_{|x}$ is trivial when $x \notin D$,
- $\mathcal{O}_X(D)_{|x}$ is the normal weight to D at x when x is a smooth point of D.

Suppose now that $D = \sum_{i=1}^{n} a_i D_i$ is an arbitrary $\mathbb{T}$-invariant divisor on X. The isomorphism

$$\mathcal{O}_X(D) \simeq \bigotimes_{i=1}^{n} \mathcal{O}_X(D_i)^{a_i}$$

induces the natural linearization of the bundle $\mathcal{O}_X(D)$.

2.2. **Round-up divisor.**

Definition 2.1 ([Laz04, definition 9.1.2])**.** Let $D = \sum q_i D_i$ be a $\mathbb{Q}$-divisor. The round-up divisor $\lceil D \rceil$ is given by

$$\lceil D \rceil = \sum \lceil q_i \rceil D_i \,.$$

In general rounding-up does not commute with pullbacks (cf. [Laz04, remark 9.1.4]). The following proposition shows commutation in a special case.

Proposition 2.2. *Let X be a smooth variety and $\pi : Y \to X$ a smooth morphism (for example a $\mathbb{P}^1$-bundle). Let D be a $\mathbb{Q}$-divisor on X. Then*

$$\lceil \pi^* D \rceil = \pi^* \lceil D \rceil \,.$$

The proof is straightforward.

2.3. **Equivariant K-theory.** For a quasiprojective $\mathbb{T}$-variety X we consider the equivariant K-theory of coherent sheaves $G^{\mathbb{T}}(X)$ and the equivariant K-theory of vector bundles $K_{\mathbb{T}}(X)$. For a smooth $\mathbb{T}$-variety X these two notions coincide i.e. we have a canonical isomorphism

$$K_{\mathbb{T}}(X) \simeq G^{\mathbb{T}}(X) \,,$$

induced by taking the sheaf of sections.
The equivariant K-theory of a point is isomorphic to the ring of Laurent polynomials

$$K_{\mathbb{T}}(pt) \simeq \mathbb{Z}[\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)] \simeq \mathbb{Z}[t_1^{\pm}, \dots, t_{\mathrm{rk}\,\mathbb{T}}^{\pm}] \,.$$

Let $S \subset K_{\mathbb{T}}(pt)$ be the multiplicative system consisting of all nonzero elements. By the localized K-theory of X we denote the ring $S^{-1}K_{\mathbb{T}}(X)$. The localization theorem [Tho92, theorem 2.1] implies that the restriction map induces an isomorphism

$$S^{-1}K_{\mathbb{T}}(X) \simeq S^{-1}K_{\mathbb{T}}(X^{\mathbb{T}}) \,.$$

Suppose that a $\mathbb{T}$-variety X is smooth and projective. Suppose moreover that the fixed point set $X^{\mathbb{T}}$ is finite. Then the equivariant K-theory of X is a free $K_{\mathbb{T}}(pt)$-module[1] In this case we have an inclusion of rings

$$K_{\mathbb{T}}(X) \subset S^{-1}K_{\mathbb{T}}(X) \simeq S^{-1}K_{\mathbb{T}}(X^{\mathbb{T}})$$

and

$$K_{\mathbb{T}}(X^{\mathbb{T}}) \simeq \bigoplus_{x \in X^{\mathbb{T}}} \mathbb{Z}[t_1^{\pm}, \dots, t_{\mathrm{rk}\,\mathbb{T}}^{\pm}] \,.$$

[1]The proof is fairly easy induction on Białynicki-Birula skeleta, as in [FRW21, section 9]. For the structure of the K-theory of homogeneous spaces see [KK90] or [Uma13].

For an equivariant complex vector bundle E over a $\mathbb{T}$-variety X we define λ_y class as

$$\lambda_y(E) := \sum_{k=0}^{\mathrm{rk}(E)} y^k[\Lambda^k E] \in K_{\mathbb{T}}(X)[y]\,.$$

The K-theoretic Euler class of E is defined as

$$eu(E) := \lambda_{-1}(E^*) = \sum_{k=0}^{\mathrm{rk}(E)} (-1)^k[\Lambda^k E^*] \in K_{\mathbb{T}}(X)\,.$$

The Lefschetz-Riemann-Roch formula (LRR for short) allows to compute push-forwards using restriction to the fixed point set. Let us remind it here.

Theorem 2.3 ([Tho92, theorem 3.5] and [CG10, theorem 5.11.7])**.** *Let X and Y be smooth $\mathbb{T}$-varieties. Let $p : X \to Y$ be an equivariant proper morphism. Consider a component of the fixed point set $F \subset Y^{\mathbb{T}}$. Let I be the set of components of the fixed point set $p^{-1}(F)^{\mathbb{T}}$. For $\alpha \in S^{-1}K_{\mathbb{T}}(X)$ we have*

$$\frac{(p_*\alpha)_{|F}}{eu(\nu(F \subset Y))} = \sum_{F' \in I} (p_{F'})_* \frac{\alpha_{|F'}}{eu(\nu(F' \subset X))} \in S^{-1}K_{\mathbb{T}}(F)\,,$$

where $\nu(F \subset Y)$ is the normal bundle and $p_{F'} \colon F' \to F$ is the restriction of p.

We will apply LRR theorem for Bott-Samelson resolution, where the fixed point sets are finite, hence the pushforward maps $(p_{F'})^*$ are isomorphism and in practice, identifying $F' \simeq F \simeq pt$, can be treated as the identities.

3. Generalized flag varieties

We summarize the facts from Lie theory and about Coxeter groups which we will further use.

3.1. **Notation and assumptions.**

- G is a connected, semisimple, simply connected complex Lie group with a chosen Borel subgroup B and a maximal torus $\mathbb{T}$, such that $\mathbb{T} \subset B \subset G$.
- W is the Weyl group of G. We use the identification $W \simeq (G/B)^{\mathbb{T}}$.
- For a Weyl group element $w \in W$ let X_w° be its B-orbit. We call this orbit Schubert cell of w. The Schubert variety X_w is the closure of the Schubert cell X_w° in G/B. Denote by ∂X_w the boundary
$$\partial X_w = X_w \setminus X_w^\circ\,.$$
- ι_w denotes the inclusion $\iota_w \colon X_w \hookrightarrow G/B$.
- $\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ is the group of characters.
- $\mathfrak{t}$ is the real part of Lie algebra of the torus $\mathbb{T}$, i.e.
$$\mathfrak{t}^* \simeq \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}\,.$$
Weyl group acts on $\mathfrak{t}^*$ from the left.
- We fix a W-invariant inner product $\langle -, - \rangle$ on $\mathfrak{t}^*$.
- For a root $\alpha \in \mathfrak{t}^*$ of G let
$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$
be the dual root. If a root appears as an element of $\mathfrak{t}^*$ then we use the additive notaion. If a root appears as an element of the representation ring $R(\mathbb{T})$, then the multiplicative notation is applied. For example the dual of the line representation

given by the root α is denoted α^{-1}, which in some sources would be denoted by $e^{-\alpha}$.

- We call a reflection $s \in W$ simple if it corresponds to a simple root.
- For $w \in W$ let $l(w)$ be its length. It is equal to the dimension of the Schubert variety of w, i.e.
$$\dim(X_w) = l(w)\,.$$
- For a simple reflection $s \in W$ we consider the corresponding minimal parabolic subgroup P_s containing B and a lift of s in $N\mathbb{T}$. Let
$$\pi_s : G/B \to G/P_s$$
be the projection
- For a character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \simeq \mathrm{Hom}(B, \mathbb{C}^*)$ let $\mathcal{L}(\lambda) \in \mathrm{Pic}(G/B)$ be a line bundle of the form
$$\mathcal{L}(\lambda) \simeq G \times_B \mathbb{C}_{-\lambda}\,.$$
- The bundle $\mathcal{L}(\lambda)$ has a natural linearization such that for a fixed point $w \in (G/B)^{\mathbb{T}} \simeq W$ we have
$$\mathcal{L}(\lambda)_{|w} = -w\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)\,.$$
- For a $\mathbb{Q}$-character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ let $\mathcal{L}(\lambda) \in \mathrm{Pic}(G/B) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the corresponding fractional line bundle. This assignment induces an isomorphism
$$\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q} \simeq \mathrm{Pic}(G/B) \otimes \mathbb{Q}\,.$$
- We say that $\lambda \in \mathfrak{t}^*$ is general enough if $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for all roots α.
- We say that $\lambda \in \mathfrak{t}^*$ is anti-ample if $\langle \lambda, \alpha^\vee \rangle < 0$ for all simple roots α.

3.2. **Weyl group actions.** The Weyl group acts on the maximal torus $\mathbb{T}$ by conjugation. This induces a left Weyl group action on the characters $\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. The action extends naturally to an action on fractional characters and the equivariant K-theory of a point
$$K_{\mathbb{T}}(pt) \simeq \mathbb{Z}[\mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)]\,.$$
For a generalized flag variety G/B we have two actions (left and right) of the Weyl group on the equivariant K-theory $K_{\mathbb{T}}(G/B)$. The left action of $W = N\mathbb{T}/\mathbb{T}$ is well defined on $(G/B)^{\mathbb{T}}$. It does not preserve the torus action. It is invariant when twisted by the torus automorphism given by W action. On the other hand the right action is torus equivariant, but it is not holomorphic. It is obtained by the identification $G/B \simeq K/T$, where $K \subset G$ is a maximal compact group and $T \subset K$ is a maximal compact torus. We can assume that $T \subset \mathbb{T}$. The right action is the natural action of $W = NT/T$. See [MNS22, section 3.1] or [Knu03] for a definition and detailed discussion of these actions. For an element $w \in W$ we denote the left action of w by w^{L} and the right one by w^{R}. On restrictions to the fixed point set the actions are given by the following formulas.

Proposition 3.1. *For $w, \sigma \in W$ and $x \in K_{\mathbb{T}}(G/B)$ we have*
$$w^{\mathrm{R}}(x)_{|\sigma} = x_{|\sigma w}\,,$$
$$w^{\mathrm{L}}(x)_{|\sigma} = w(x_{|w^{-1}\sigma})\,.$$

Remark 3.2. The action of W on $K_{\mathbb{T}}(G/B)$ via w^{R} satisfies $(uw)^{\mathrm{R}} = u^{\mathrm{R}} w^{\mathrm{R}}$ i.e. is an action from the left, despite its name.

3.3. **Reflections.** The Weyl group can be identified with the group generated by the reflections in the simple roots of the Lie algebra. We record the following proposition following directly from the properties of root systems (cf. [Hum90, proposition in section 1.2]).

Proposition 3.3. *Let α and β be roots and s_α, s_β the corresponding reflections. Let $w \in W$ be a Weyl group element. Then*

$$s_\beta = w s_\alpha w^{-1} \iff \beta = \pm w\alpha \in \mathfrak{t}^* .$$

We will need further properties of the Weyl group.

Proposition 3.4. *Let $s \in W$ be any reflection. There exists $w \in W$ such that wsw^{-1} is a simple reflection.*

Proof. Let α be the positive root corresponding to s. By proposition 3.3 it is enough to show that there exists $w \in W$ such that $w\alpha$ is a simple root. This is a standard fact, see e.g. [Car05, proposition 5.12]. □

Proposition 3.5. *Suppose $q_1, q_2, q_3 \in W$ are reflections such that $q_3 = q_1q_2q_1$, q_1 is simple and $q_1 \neq q_2$. Let $w \in W$ be any Weyl group element. Then*

$$l(wq_1) < l(wq_1q_2) \iff l(w) < l(wq_3) .$$

Proof. First let us note that $l(w) < l(wq_i)$ if and only if the weight of the tangent space of the one dimensional $\mathbb{T}$-orbit joining w with wq_i at w is a negative root. Let α_2, α_3 be positive roots corresponding to reflections q_2 and q_3, respectively. The reflection q_1 is simple and $q_1 \neq q_2$, therefore $q_1\alpha_2$ is also a positive root (see e.g. [Car05, lemma 5.9]). Thus $q_1\alpha_2 = \alpha_3$ (cf. proposition 3.3). The tangent weight at w to the one-dimensional $\mathbb{T}$-orbit connecting wq_3 and w is equal to $-w\alpha_3$. The tangent weight at wq_1 to the one-dimensional $\mathbb{T}$-orbit connecting wq_1q_2 and wq_1 is equal to $-wq_1\alpha_2$. These two weights are equal which proves the lemma. □

Proposition 3.6. *Let s be a simple reflection, α_s the corresponding simple root and $w \in W$ a Weyl group element.*

- *Suppose that $l(w) < l(ws)$ then $w\alpha_s$ is a positive root.*
- *Suppose that $l(w) < l(sw)$ then $w^{-1}\alpha_s$ is a positive root.*

Proof. The first part follows from the fact that $-w\alpha_s$ is the tangent weight at w of the orbit connecting w and ws. The second part follows from the first since $l(w) < l(sw)$ if and only if $l(w^{-1}) < l(w^{-1}s)$. □

4. Bott-Samelson resolution

The construction of Demazure-Lusztig operators are based on the geometry of Bott-Samelson resolution. We briefly summarize what we need below.

4.1. **Inductive construction.** Let $\underline{w}$ be a reduced word decomposition of $w \in W$. We denote by

$$p_{\underline{w}} : Z_{\underline{w}} \to X_w \subset G/B$$

the corresponding Bott–Samelson resolution of singularities (see e.g. [BK05, section 2.2], [AMSS19, section 2] or [RW20, section 3]). Let

$$\partial Z_{\underline{w}} = p_{\underline{w}}^{-1}(\partial X_w), \qquad Z^\circ_{\underline{w}} = Z_{\underline{w}} \setminus \partial Z_{\underline{w}} .$$

It is a standard fact that

$$p_{\underline{w}} : (Z_{\underline{w}}, \partial Z_{\underline{w}}) \to (X_w, \partial X_w)\,.$$

is a simple normal crossing resolution of singularities. The boundary $\partial Z_{\underline{w}}$ consists of $l(w)$ components corresponding to omitting a single letter in $\underline{w}$. Denote these components by $\partial_j Z_{\underline{w}}$ for $j \in \{1, 2, \ldots, l(w)\}$.

The Bott–Samelson resolution may be constructed inductively. Consider a simple reflection s. Suppose that $l(ws) > l(w)$. Then we have a pullback diagram

(1)
$$\begin{tikzcd}
Z_{\underline{ws}} \arrow[rr, "p_{\underline{ws}}"] \arrow[d, "\pi_{\underline{ws}}"] & & X_{ws} \arrow[r, "\iota_{ws}"] & G/B \arrow[d, "\pi_s"] \\
Z_{\underline{w}} \arrow[r, "p_{\underline{w}}"] & X_w \arrow[r, "\iota_w"] & G/B \arrow[r, "\pi_s"] & G/P_s
\end{tikzcd}$$

The map $\pi_{\underline{ws}} : Z_{\underline{ws}} \to Z_{\underline{w}}$ has a $\mathbb{T}$-invariant section $i : Z_{\underline{w}} \to Z_{\underline{ws}}$. We have an equality

$$\partial Z_{\underline{ws}} = \pi_{\underline{ws}}^{-1}(\partial Z_{\underline{w}}) \cup i(Z_{\underline{w}})\,.$$

One may use the above construction to inductively describe the fixed point set $Z_{\underline{w}}^{\mathbb{T}}$. It can be identified with the set of binary sequences of length $l(w)$. For the empty word identification is obvious, the variety $Z_\varnothing$ is a single point. Suppose that $l(ws) > l(w)$. Let ε be a binary sequence of length $l(w)$. By the inductive assumption it corresponds to a fixed point in $Z_{\underline{w}}^{\mathbb{T}}$. The preimage $\pi_{\underline{ws}}^{-1}(\varepsilon)$ is a projective line $\mathbb{P}^1$ equipped with a nontrivial action of $\mathbb{T}$. The set $\pi_{\underline{ws}}^{-1}(\varepsilon)^{\mathbb{T}}$ consists of two points. Identify these fixed points with $(\varepsilon, 0)$ and $(\varepsilon, 1)$. We assume that $(\varepsilon, 0) = i(\varepsilon)$ is the point that lies in the image of section i. For a detailed discussion of the fixed point set and an example see [RW20, section 3.2].

4.2. **Pull-backs of line bundles.** Consider a fractional character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$. We are interested in the pullback $p_{\underline{w}}^* \mathcal{L}(\lambda)$ of the corresponding line bundle to the Bott–Samelson variety. Let us recall several classical results.

Proposition 4.1 (Kempf lemma [Kem76, section 2, lemma 3])**.** *Let $\underline{w}$ be a reduced word decomposition of $w \in W$. Suppose that s is the reflection corresponding to a simple root α_s. Assume that $l(ws) > l(w)$. Then, we have an isomorphism*

$$p_{\underline{ws}}^* \mathcal{L}(s\lambda) \simeq \pi_{\underline{ws}}^* p_{\underline{w}}^* \mathcal{L}(\lambda) \otimes \mathcal{O}_{Z_{\underline{ws}}}(-\langle \lambda, \alpha_s^\vee \rangle \cdot i(Z_{\underline{w}})) \in \mathrm{Pic}(Z_{\underline{ws}})\,.$$

Corollary 4.2. *Kempf lemma holds equivariantly. Consider the natural $\mathbb{T}$-linearization of bundle $\mathcal{L}(\lambda)$ (i.e. such that the weight at* id $\in G/B$ *is equal to $-\lambda$) and the natural linearization of $\mathcal{O}_{Z_{\underline{ws}}}(i(Z_{\underline{w}}))$. We have an isomorphism*

$$p_{\underline{ws}}^* \mathcal{L}(s\lambda) \simeq \pi_{\underline{ws}}^* p_{\underline{w}}^* \mathcal{L}(\lambda) \otimes \mathcal{O}_{Z_{\underline{ws}}}(-\langle \lambda, \alpha_s^\vee \rangle \cdot i(Z_{\underline{w}})) \in \mathrm{Pic}^{\mathbb{T}}(Z_{\underline{ws}})\,.$$

Proof. We only need to prove that $\mathbb{T}$-linearizations of both sides agree. Due to [Bri15, proposition 2.10] it is enough to check that the weights at some chosen fixed point agree. On the maximal fixed point $p_{\underline{ws}}^{-1}(ws)$ both bundles have weight $-w\lambda$. □

Proposition 4.3 (Chevalley formula [Dem74, Paragraph 4.3], see also [KRW20, proposition 4.1] for this formulation)**.** *Let $\underline{w} = s_{i_1}, s_{i_2}, \ldots, s_{i_{l(w)}}$ be a reduced word decomposition of $w \in W$. Let α_j be the simple root corresponding to the reflection s_{i_j}. Consider elements*

$$\gamma_j = s_{i_{l(w)}} s_{i_{l(w)-1}} \ldots s_{i_{j+1}} \cdot \alpha_j \in \mathfrak{t}^*\,.$$

There is an isomorphism

$$p_{\underline{w}}^*\mathcal{L}(\lambda) \simeq \mathcal{O}_{Z_{\underline{w}}}\Big(\sum_{j=1}^{l(w)} \langle \lambda, \gamma_j^\vee \rangle \cdot \partial_j Z_{\underline{w}}\Big) \in \operatorname{Pic}(Z_{\underline{w}})\,.$$

Proposition 4.4. *The coefficients $\langle \lambda, \gamma_j^\vee \rangle$ from Chevalley formula belong to the set*

$$\{\langle \lambda, \alpha^\vee \rangle | \alpha \text{ is a root, } l(w) > l(ws_\alpha)\}\,,$$

where s_α is the reflection corresponding to the root α.

Proof. Let s_{γ_j} be the reflection corresponding to the root γ_j. We need to prove that

$$l(w) > l(ws_{\gamma_j})\,.$$

Let $w_{>j} = s_{i_{j+1}} s_{i_{j+2}} \dots s_{i_{l(w)}} \in W$. We have $\gamma_j = w_{>j}^{-1} \cdot \alpha_j$. Proposition 3.3 implies that

$$s_{\gamma_j} = w_{>j}^{-1} s_{i_j} w_{>j} \in W\,.$$

It follows that ws_{γ_j} is represented by the word $\underline{w}$ with s_{i_j} omitted. Therefore $l(w) > l(ws_{\gamma_j})$. ☐

Remark 4.5. In proposition 4.4 we do not assume that roots are positive. Of course the set of coefficients may be much smaller, e.g. all appearing coefficients are nonnegative if λ is dominant.

5. Twisted motivic Chern class

The K-theoretic characteristic classes of singular varieties are our main protagonists. Below we recall the twisted version of motivic Chern classes.

5.1. **Motivic Chern class.** The motivic Chern class was defined in [BSY10] (see also [SY07] for a survey). Its equivariant version is due to [AMSS19, FRW21]. Here we remind only the definition of $\mathbb{T}$-equivariant motivic Chern class. Consult [AMSS19, FRW21] for a detailed account.

Definition 5.1 (after [FRW21, section 2.3])**.** The motivic Chern class assigns to every $\mathbb{T}$-equivariant map of quasi-projective $\mathbb{T}$-varieties $f: Z \to Y$ an element

$$\mathrm{mC}^{\mathbb{T}}(f) = \mathrm{mC}^{\mathbb{T}}(Z \xrightarrow{f} Y) \in G^{\mathbb{T}}(Y)[y]$$

such that the following properties are satisfied

1. Additivity: Let Z be a $\mathbb{T}$-variety and $U \subset Z$ an invariant open subvariety. Then

$$\mathrm{mC}^{\mathbb{T}}(Z \xrightarrow{f} Y) = \mathrm{mC}^{\mathbb{T}}(U \xrightarrow{f_{|U}} Y) + \mathrm{mC}^{\mathbb{T}}(Z \setminus U \xrightarrow{f_{|Z\setminus U}} Y)\,.$$

2. Functoriality: For an equivariant proper map $g: Y \to Y'$ we have

$$\mathrm{mC}^{\mathbb{T}}(Z \xrightarrow{g\circ f} Y') = g_*\,\mathrm{mC}^{\mathbb{T}}(Z \xrightarrow{f} Y)\,.$$

3. Normalization: For a smooth $\mathbb{T}$-variety X we have

$$\mathrm{mC}^{\mathbb{T}}(\mathrm{id}_X) = \lambda_y(T^*X)\,.$$

The equivariant motivic Chern class is the unique assignment satisfying the above properties. For a smooth $\mathbb{T}$-variety X we may consider the class $\mathrm{mC}^{\mathbb{T}}(Z \to X)$ as an element of $K_{\mathbb{T}}(X)[y]$.

The above definition is meaningful also in the non-equivariant setting, i.e. for $\mathbb{T}$ equal to the trivial group (cf. [BSY10]). See [AMSS19, theorem 4.2] for an equivalent definition in terms of a natural transformation of functors.

5.2. **Twisted class.** The twisted motivic Chern class was defined in [KW22]. We repeat its definition here. Let $(Y, \partial Y)$ be a pair, consisting of an algebraic quasiprojective $\mathbb{T}$-variety Y and an invariant closed subvariety $\partial Y \subset Y$. We assume that the complement $Y^\circ = Y \setminus \partial Y$ is smooth. Let Δ be an invariant $\mathbb{Q}$-Cartier divisor on Y with support contained in the boundary $|\Delta| \subset \partial Y$.

Definition 5.2 ([KW22, definition 2.2])**.** The twisted motivic Chern class of a triple $(Y, \partial Y; \Delta)$ as above is defined by the formula

$$\mathrm{mC}^{\mathbb{T}}(Y, \partial Y; \Delta) = f_*\Big(\mathcal{O}_Z(\lceil f^*(\Delta)\rceil) \cdot \mathrm{mC}^{\mathbb{T}}(Z^\circ \subset Z)\Big)\,,$$

where $f : (Z, \partial Z) \to (Y, \partial Y)$ is a resolution of singularities such that $\partial Z = f^{-1}(\partial Y)$ is a simple normal crossing divisor and $Z^\circ = Z \setminus \partial Z$. We assume that $f_{|Z^\circ} : Z^\circ \to Y^\circ$ is an isomorphism.

It is proved in [KW22, corollary 4.6] that this class is well defined, i.e. it does not depend on a choice of a SNC resolution of singularities.

Remark 5.3. The twisted motivic Chern class may be obtained as a limit of the equivariant version of Borisov-Libgober elliptic class [BL03], whenever it is defined, see [KW22, Section 3] for a detailed discussion.

5.3. **Twisted classes of Schubert cells and stable envelopes.** The K-theoretic stable envelopes were defined in [Oko17, OS22]. They depend on a fractional line bundle called slope. It turns out that for homogeneous varieties the stable envelopes for a small anti-ample slope coincide (up to normalization) with the motivic Chern classes of Schubert varieties, see [AMSS19, theorem 8.5], [FRW21] and [Kon22, theorem 6.2]. The twisted class is a class depending on a fractional line bundle which generalizes the motivic Chern class and coincides with the stable envelopes for an arbitrary slope, see [KW22, theorem 7.1]. Consider a generalized flag variety G/B. Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ be a fractional character and $w \in W$ a fixed point.

Proposition 5.4 ([KW22, section 10])**.** *There exists a unique $\mathbb{T}$-invariant $\mathbb{Q}$-Cartier divisor $\Delta_{w,\lambda}$ on X_w such that*

(1) Divisor $\Delta_{w,\lambda}$ represents $\mathcal{L}(\lambda)_{|X_w}$.
(2) Support of $\Delta_{w,\lambda}$ is contained in the boundary ∂X_w.

Definition 5.5. The twisted motivic Chern class of a Schubert cell is defined as

$$\mathrm{mC}^{\mathbb{T}}(w, \lambda) := \iota_{w*}\, \mathrm{mC}^{\mathbb{T}}(X_w, \partial X_w; \Delta_{w,\lambda}) \in K_{\mathbb{T}}(G/B)[y]\,. \tag{2}$$

Remark 5.6. To compute the class $\mathrm{mC}^{\mathbb{T}}(w, \lambda)$ one may use the Bott-Samelson resolution. The divisor $p_{\underline{w}}^*\Delta_{w,\lambda}$ is the unique $\mathbb{T}$-invariant $\mathbb{Q}$-divisor representing $p_{\underline{w}}^*\mathcal{L}(\lambda)$ with support contained in $\partial Z_{\underline{w}}$. Its coefficients are computed by Chevalley formula (cf. propositions 4.2 and 4.4).

Remark 5.7 ([KW22, proposition 2.5]). The twisted motivic Chern class generalizes the motivic Chern class, i.e. for a small anti-ample λ we have

$$\mathrm{mC}^{\mathbb{T}}(w, \lambda) = \mathrm{mC}^{\mathbb{T}}(w, 0) = \mathrm{mC}^{\mathbb{T}}(X_w^\circ \subset G/B)\,.$$

Theorem 5.8 ([KW22, theorem 7.1 and remark 6.4])**.** *Suppose that a fractional character λ is general enough. Then the class*

$$q^{-\frac{1}{2}\dim X_w^+}\, \mathrm{mC}^{\mathbb{T}}_{-q}(w, \lambda) \in K_{\mathbb{T}}(G/B)[q^{1/2}, q^{-1/2}]$$

is equal to the stable envelope $\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w)\,.$

Remark 5.9. We write $\mathrm{mC}^{\mathbb{T}}_{-q}$ for the image of the class $\mathrm{mC}^{\mathbb{T}}$ in a map

$$\rho : K_{\mathbb{T}}(G/B)[y] \to K_{\mathbb{T}}(G/B)[q^{-1/2}, q^{1/2}]\,,$$

which sends y to $-q$.

Corollary 5.10. *Let λ be an arbitrary rational character. Let λ^- be a small anti-ample fractional character and ε a small enough positive rational number. Then the class*

$$q^{-\frac{1}{2}\dim X_w^+}\,\mathrm{mC}^{\mathbb{T}}_{-q}(w,\lambda) \in K_{\mathbb{T}}(G/B)[q^{1/2}, q^{-1/2}]$$

is equal to the stable envelope $\mathrm{stab}^{\lambda+\varepsilon\lambda^-}_{\mathfrak{C}_+}(w)$.

Remark 5.11. We do not include the definition of stable envelope here. For a definition adapted to generalized flag varieties see e.g. [AMSS19, SZZ20, KW22]

Remark 5.12. We write $\mathrm{stab}^{\lambda}_{\mathfrak{C}}(w)$ for the stable envelope of the fixed point $w \in W$ for weight chamber $\mathfrak{C}$, slope λ and the tangent polarization. We consider stable envelopes only for the tangent bundle TG/B as a polarization. This restriction does not reduce the generality of results. Stable envelopes corresponding to different polarizations are related by a shift of slope and renormalization, see [Oko17, Paragraph 9.1.12]. We usually consider stable envelopes for the positive weight chamber $\mathfrak{C}^+$.

6. Twisted Demazure-Lusztig operators

Many cohomological invariants of Schubert varieties can be computed inductively due to the inductive construction of resolution of singularities. By application of Demazure or Bernstein-Gelfand-Gelfand operators, starting from the class of 0-dimensional Schubert variety, the fundamental class can be computed, [BGG73]. The same is true for the motivic Chern classes in K-theory as proven in [AMSS19], where Demazure-Lusztig operators are employed. The twisted version demands a twisted version of operators.

6.1. **The definitions and main properties.** Let α_s be a simple root and $s \in W$ the corresponding reflection. Denote by $\mathcal{L}_s$ the relative tangent bundle of the projection $\pi_s : G/B \to G/P_s$. Let us recall the definition of Demazure-Lusztig operator.

Definition 6.1. The Demazure-Lusztig operator $\mathcal{T}^{\mathrm{R}}_s$ is defined by

$$\begin{aligned}\mathcal{T}^{\mathrm{R}}_s &= \lambda_y(\mathcal{L}^*_s)\cdot \pi_s^*\pi_{s*} - \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\\ &= \frac{1+y\mathcal{L}^*_s}{1-\mathcal{L}_s}\cdot s^{\mathrm{R}} - \frac{(1+y)}{1-\mathcal{L}_s}\cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\,.\end{aligned}$$

They appeared already in Lusztig's article [Lus85] on the K-theory of flag varieties. In fact, $K_{\mathbb{T}}(G/B)[q,q^{-1}]$ is a free, rank one module over the Hecke algebra, hence it admits both right and left Hecke action. The relation with characteristic classes was exposed in [AMSS19]. Because of the relation with Hirzebruch χ_y-genus we use the variable y which is equal to $-q$ from the Lusztig's work. The left Demazure-Lusztig $\mathcal{T}^{\mathrm{L}}_s$ operator is defined by (cf. [MNS22, Section 5.3])

$$\mathcal{T}^{\mathrm{L}}_s := \frac{1+y\alpha_s^{-1}}{1-\alpha_s^{-1}}\cdot s^{\mathrm{L}} - \frac{(1+y)}{1-\alpha_s^{-1}}\cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\,.$$

Note that here α_s^{-1} denotes the one-dimensional representation, the dual of α_s. We use here the multiplicative notations for characters.

Remark 6.2. For the flag varieties in type A the left Demazure-Lusztig operators appeared in form of the R-matrix relations connecting weight functions [RTV15, section 7.2], [RTV19, theorem 3.2]. In [FRW21] it was shown, that the trigonometric weight functions coincide with the motivic Chern classes of Schubert cells.

We define a twisted version of the Demazure-Lusztig operators operators.

Definition 6.3.

1) For $a \in \mathbb{R}$ we define the twisted Demazure-Lusztig operator

$$\begin{aligned}\mathcal{T}^{\mathrm{R}}_{s,a} :=& \frac{1+y\mathcal{L}^*_s}{1-\mathcal{L}_s} \cdot s^{\mathrm{R}} - \frac{(1+y)\cdot \mathcal{L}_s^{\lceil a \rceil}}{1-\mathcal{L}_s} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]} \\ =& \frac{1+y\mathcal{L}^*_s}{1-\mathcal{L}_s} \cdot s^{\mathrm{R}} + \frac{(1+y)\cdot \mathcal{L}_s^{\lceil a \rceil -1}}{1-\mathcal{L}^*_s} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\end{aligned}$$

2) Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a fractional character. We define the twisted Demazure-Lusztig operator $\mathcal{T}^{\mathrm{R}}_{s,\lambda} : K_{\mathbb{T}}(G/B)[y] \to K_{\mathbb{T}}(G/B)[y]$ by

$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} = \mathcal{T}^{\mathrm{R}}_{s,-\langle \lambda, \alpha_s^\vee \rangle} .$$

3) For $a \in \mathbb{R}$ we define the left twisted Demazure-Lusztig operator

$$\begin{aligned}\mathcal{T}^{\mathrm{L}}_{s,a} :=& \frac{1+y\alpha_s^{-1}}{1-\alpha_s^{-1}} \cdot s^{\mathrm{L}} - \frac{(1+y)\cdot \alpha_s^{-\lceil a \rceil}}{1-\alpha_s^{-1}} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]} \\ =& \frac{1+y\alpha_s^{-1}}{1-\alpha_s^{-1}} \cdot s^{\mathrm{L}} + \frac{(1+y)\cdot \alpha_s^{1-\lceil a \rceil}}{1-\alpha_s} \cdot \mathrm{id}_{K_{\mathbb{T}}(G/B)[y]} .\end{aligned}$$

4) Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a fractional character. We define the twisted left Demazure-Lusztig operator $\mathcal{T}^{\mathrm{L}}_{s,\lambda} : K_{\mathbb{T}}(G/B)[y] \to K_{\mathbb{T}}(G/B)[y]$ by

$$\mathcal{T}^{\mathrm{L}}_{s,\lambda} = \mathcal{T}^{\mathrm{L}}_{s,\langle \lambda, \alpha_s^\vee \rangle} .$$

Formally the definitions make sense after the inversion of the elements $1-\mathcal{L}$ and $1-\alpha$, but it will be clear later, that the operations lead to $K_{\mathbb{T}}(G/B)[y]$, see remark 6.5.

6.2. **Right Demazure-Lusztig operators.** We note several straightforward consequences of the above definition.

Proposition 6.4. *Let s be a simple reflection, λ a fractional character and $a \in \mathbb{R}$ a real number.*

(1) For $a \in (-1, 0]$ the twisted operators are equal to Demazure-Lusztig operators

$$\mathcal{T}^{\mathrm{R}}_{s} = \mathcal{T}^{\mathrm{R}}_{s,a} , \qquad \mathcal{T}^{\mathrm{L}}_{s} = \mathcal{T}^{\mathrm{L}}_{s,a}$$

(2) Operators $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ are morphisms of $K_{\mathbb{T}}(pt)[y]$-modules.

(3) For a small enough positive real number $\varepsilon > 0$ we have

$$\mathcal{T}^{\mathrm{R}}_{s,a} = \mathcal{T}^{\mathrm{R}}_{s,a-\varepsilon} , \qquad \mathcal{T}^{\mathrm{L}}_{s,a} = \mathcal{T}^{\mathrm{L}}_{s,a-\varepsilon}$$

(4) Let $\lambda^- \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be an anti-ample fractional character. For a small enough positive real number $\varepsilon > 0$ we have

$$\mathcal{T}^{\mathrm{R}}_{s,a} = \mathcal{T}^{\mathrm{R}}_{s,\lambda-\varepsilon\lambda^-} , \qquad \mathcal{T}^{\mathrm{L}}_{s,a} = \mathcal{T}^{\mathrm{L}}_{s,\lambda+\varepsilon\lambda^-}$$

(5) Let λ be a fractional character. The twisted Demazure-Lusztig operator is equal to

$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} = \lambda_y(\mathcal{L}_s^*)\cdot\pi_s^*\pi_{s*} - \left(1 + \frac{(1+y)(1-\mathcal{L}_s^{\lceil -\langle\lambda,\alpha_s\rangle\rceil})}{1-\mathcal{L}_s}\right)\cdot \mathrm{id}$$

$$= \mathcal{T}^{\mathrm{R}}_s - \frac{(1+y)(1-\mathcal{L}_s^{\lceil -\langle\lambda,\alpha_s\rangle\rceil})}{1-\mathcal{L}_s}\cdot \mathrm{id}\,.$$

Remark 6.5. For every integer $n\in\mathbb{Z}$ the element $1-\mathcal{L}_s^n$ is divisible by $1-\mathcal{L}_s$ in the equivariant K-theory $K_{\mathbb{T}}(G/B)$. Therefore, the twisted Demazure-Lusztig operators define maps

$$\mathcal{T}^{\mathrm{R}}_{s,a} : K_{\mathbb{T}}(G/B)[y] \to K_{\mathbb{T}}(G/B)[y]\,.$$

Similarly $\mathcal{T}^{\mathrm{L}}_{s,a}$ acts on $K_{\mathbb{T}}(G/B)[y]$. This action does not preserve the $K_{\mathbb{T}}(pt)$-module structure. On the other hand, unlike $\mathcal{T}^{\mathrm{R}}_{s,a}$, it descenss to an action on $K_{\mathbb{T}}(G/P)[y]$ for any parabolic group $P\supset B$.

We prove quadratic relation for the twisted Demazure-Lusztig polynomials.

Proposition 6.6. *Let $s\in W$ be a simple reflection and $a\in\mathbb{R}$ a real number. Suppose that $a\notin\mathbb{Z}$. Then*

$$\mathcal{T}^{\mathrm{R}}_{s,a}\circ\mathcal{T}^{\mathrm{R}}_{s,-a} = -y\,\mathrm{id}_{K_{\mathbb{T}}(G/B)[y]}\,.$$

Proof. For two real numbers a,b and $\xi\in K_{\mathbb{T}}(G/B)[y]$ the composition $\mathcal{T}^{\mathrm{R}}_{s,a}\circ\mathcal{T}^{\mathrm{R}}_{s,b}$ evaluated at ξ is of the form

$$\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s}\left(\frac{1+y\mathcal{L}_s}{1-\mathcal{L}_s^*}\xi - \frac{(1+y)\mathcal{L}_s^{-\lceil b\rceil}}{1-\mathcal{L}_s^*}s^{\mathrm{R}}\xi\right) - \frac{(1+y)\mathcal{L}_s^{\lceil a\rceil}}{1-\mathcal{L}_s}\left(\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s}s^{\mathrm{R}}\xi - \frac{(1+y)\mathcal{L}_s^{\lceil b\rceil}}{1-\mathcal{L}_s}\xi\right)\,.$$

Suppose that $b=-a$ and the number a is not an integer, then

$$-\lceil b\rceil = -\lceil -a\rceil = \lfloor a\rfloor = \lceil a\rceil - 1\,.$$

In the expression above the coefficient of $s^{\mathrm{R}}\xi$ is equal to

$$-\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s}\cdot\frac{(1+y)\mathcal{L}_s^{\lceil a\rceil-1}}{1-\mathcal{L}_s^*} - \frac{(1+y)\mathcal{L}_s^{\lceil a\rceil}}{1-\mathcal{L}_s}\cdot\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s} = 0\,.$$

The coefficient of ξ is equal to

$$\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s}\cdot\frac{1+y\mathcal{L}_s}{1-\mathcal{L}_s^*} + \frac{(1+y)\mathcal{L}_s^{\lceil a\rceil}}{1-\mathcal{L}_s}\cdot\frac{(1+y)\mathcal{L}_s^{1-\lceil a\rceil}}{1-\mathcal{L}_s}\,.$$

Simplifying we obtain $-y$. □

Corollary 6.7. *Suppose that $a\in\mathbb{Z}$. Then*

$$\mathcal{T}^{\mathrm{R}}_{s,a}\circ\mathcal{T}^{\mathrm{R}}_{s,-a} = -y - (1+y)\mathcal{L}_s^a\cdot\mathcal{T}^{\mathrm{R}}_{s,-a}\,.$$

For $a=0$ we obtain the standard relation

$$(\mathcal{T}^{\mathrm{R}}_s + y)(\mathcal{T}^{\mathrm{R}}_s + 1) = 0\,.$$

Proof. Let $\varepsilon>0$ be a small enough positive real number. We have

$$\mathcal{T}^{\mathrm{R}}_{s,-a} = \mathcal{T}^{\mathrm{R}}_{s,-a-\varepsilon}\,,\qquad \mathcal{T}^{\mathrm{R}}_{s,a} = \mathcal{T}^{\mathrm{R}}_{s,a+\varepsilon} - (1+y)\mathcal{L}_s^a\,.$$

Therefore

$$\mathcal{T}^{\mathrm{R}}_{s,a}\circ\mathcal{T}^{\mathrm{R}}_{s,-a} = (\mathcal{T}^{\mathrm{R}}_{s,a+\varepsilon} - (1+y)\mathcal{L}_s^a)\circ\mathcal{T}^{\mathrm{R}}_{s,-a-\varepsilon} = \mathcal{T}^{\mathrm{R}}_{s,a+\varepsilon}\circ\mathcal{T}^{\mathrm{R}}_{s,-a-\varepsilon} - (1+y)\mathcal{L}_s^a\cdot\mathcal{T}^{\mathrm{R}}_{s,-a}$$

The result follows from proposition 6.6. □

Corollary 6.8. *Let $s \in W$ be a simple reflection and $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ a fractional character.*

(1) Suppose that $\langle \lambda, \alpha_s^\vee \rangle \notin \mathbb{Z}$, then
$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s,s\lambda} = -y\,\mathrm{id}\,.$$
(2) Suppose that $\langle \lambda, \alpha_s^\vee \rangle \in \mathbb{Z}$. Then
$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s,s\lambda} = -y \cdot \mathrm{id} - (1+y)\mathcal{L}_s^{-\langle \lambda, \alpha_s \rangle} \cdot \mathcal{T}^{\mathrm{R}}_{s,s\lambda}\,.$$

Further properties of the twisted Demazure-Lusztig operators are treated in section 7.

6.3. **Bott-Samelson recursion.** In this section we prove our first inductive formula. Let α_s be a simple root and $s \in W$ the corresponding reflection. Denote by $\mathcal{L}_s$ the relative tangent bundle of the projection $\pi_s : G/B \to G/P_s$.

Theorem 6.9. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a fractional character and $w \in W \simeq (G/B)^{\mathbb{T}}$ a fixed point such that $l(ws) > l(w)$. Then*
$$\mathrm{mC}^{\mathbb{T}}(ws, s\lambda) = \mathcal{T}^{\mathrm{R}}_{s,\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

Remark 6.10. Theorem 6.9 is the limit case of the induction for the elliptic classes of [RW20]. We prefer to stay entirely in the framework of K-theory and analyze the Bott-Samelson construction from the point of view of twisted motivic Chern classes.

Before proving the above theorem we note several corollaries.

Corollary 6.11. *For $\lambda = 0$ we recover* [AMSS19, corollary 5.2].

Corollary 6.12. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a general enough fractional character, i.e. such that $\langle \lambda, \alpha_s \rangle$ is not an integer. For a fixed point $w \in W$ such that $l(w) > l(ws)$ we have*
$$(-y) \cdot \mathrm{mC}^{\mathbb{T}}(ws, s\lambda) = \mathcal{T}^{\mathrm{R}}_{s,\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$
Therefore, for an arbitrary $w \in W$ we have
$$(-y)^{\frac{1}{2}(l(w)-l(ws)+1)} \cdot \mathrm{mC}^{\mathbb{T}}(ws, s\lambda) = \mathcal{T}^{\mathrm{R}}_{s,\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

Proof. It follows from theorem 6.9 and corollary 6.8. □

We recall that the condition *general enough* means here and elsewhere that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for any root α. The above corollary may be restated in the language of stable envelopes.

Corollary 6.13. *Let $w \in W$ be an arbitrary fixed point and λ a general enough fractional character. Then*
$$q^{1/2} \cdot \mathrm{stab}^{s\lambda}_{\mathfrak{C}_+}(ws) = \mathcal{T}^{\mathrm{R},q}_{s,\lambda}(\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w))\,,$$
where $\mathcal{T}^{\mathrm{R},q}_{s,\lambda}$ denotes the operation $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ after substitution $y = -q$.

Proof. It follows directly from theorem 5.8 and corollary 6.12. □

We may obtain a new interesting recursion for stable envelopes choosing non-generic value of λ.

Corollary 6.14. *Let $w \in W$ be a fixed point such that $l(ws) > l(w)$.Let λ be an arbitrary character and $\lambda^- \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ a small anti-ample character. For a small enough positive real number $\varepsilon > 0$ we have*
$$q^{1/2} \cdot \mathrm{stab}^{s\lambda+\varepsilon\lambda^-}_{\mathfrak{C}_+}(ws) = \mathcal{T}^{\mathrm{R},q}_{s,\lambda}(\mathrm{stab}^{\lambda+\varepsilon\lambda^-}_{\mathfrak{C}_+}(w))\,,$$
where $\mathcal{T}^{\mathrm{R},q}_{s,\lambda}$ denotes operator $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ after substitution $y = -q$.

Proof. It follows directly from corollary 5.10 and theorem 6.9. □

Remark 6.15. Corollaries 6.14 and 6.13 induce different recursive formulas, e.g. we may obtain [SZZ20, theorem 3.5] taking $\lambda = 0$ in corollary 6.14 and [SZZ21, theorem 5.4] from corollary 6.13 (see corollary 7.12).

The rest of this section is devoted to the proof of theorem 6.9. We split the proof into several lemmas. For an arbitrary element $\sigma \in W$ with a reduced word decomposition $\underline{\sigma}$ we define

$$\mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\sigma,\lambda) := \frac{\mathrm{mC}^{\mathbb{T}}(\sigma,\lambda)}{eu(TG/B)} \in S^{-1}K_{\mathbb{T}}(G/B)[y]\,,$$
$$\mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{\sigma},\lambda) := \frac{\mathrm{mC}^{\mathbb{T}}(Z^{\circ}_{\underline{\sigma}} \subset Z_{\underline{\sigma}}) \cdot \mathcal{O}_{Z_{\underline{\sigma}}}(\lceil p^*_{\underline{\sigma}}\Delta_{\sigma,\lambda}\rceil)}{eu(TZ_{\underline{\sigma}})} \in S^{-1}K_{\mathbb{T}}(Z_{\underline{\sigma}})[y]\,.$$

Consider the situation as in theorem 6.9. Fix a reduced word decomposition $\underline{w}$ of w. It induces a reduced word decomposition $\underline{ws}$ of ws.

Lemma 6.16. *Consider the situation as in theorem 6.9. Let ε be a binary sequence corresponding to a fixed point in $Z_{\underline{w}}$. Let $\widehat{\varepsilon}$ be a binary sequence of the form $\widehat{\varepsilon} = (\varepsilon,\delta)$, where $\delta \in \{0,1\}$. We have following equalities in the localized K-theory of a point $S^{-1}K^{\mathbb{T}}(pt)[y]$.*

(1)
$$eu(TZ_{\underline{ws}})_{|\widehat{\varepsilon}} = eu(TZ_{\underline{w}})_{|\varepsilon} \cdot (1-\mathcal{L}^*_s)_{|\widehat{\varepsilon}}\,,$$

(2)
$$\mathcal{O}_{Z_{\underline{ws}}}(\lceil p^*_{\underline{ws}}\Delta_{ws,s\lambda}\rceil)_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{Z_{\underline{w}}}(\lceil p^*_{\underline{w}}\Delta_{w,\lambda}\rceil)_{|\varepsilon} \cdot \mathcal{L}_{s\,|\widehat{\varepsilon}}^{\lceil -\langle\lambda,\alpha_s\rangle\rceil} & \text{when } \delta = 0\,,\\ \mathcal{O}_{Z_{\underline{w}}}(\lceil p^*_{\underline{w}}\Delta_{w,s\lambda}\rceil)_{|\varepsilon} & \text{when } \delta = 1\,,\end{cases}$$

(3)
$$\mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{ws},s\lambda)_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{w},\lambda)_{|\varepsilon} \cdot \left(\frac{(1+y)\cdot\mathcal{L}_s^{\lceil -\langle\lambda,\alpha_s\rangle\rceil -1}}{1-\mathcal{L}^*_s}\right)_{|\widehat{\varepsilon}} & \text{when } \delta = 0\,,\\ \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{w},\lambda)_{|\varepsilon} \cdot \left(\frac{1+y\mathcal{L}^*_s}{1-\mathcal{L}^*_s}\right)_{|\widehat{\varepsilon}} & \text{when } \delta = 1\,.\end{cases}$$

(4) In particular
$$\mathrm{mC}^{\mathbb{T}}(Z^{\circ}_{\underline{ws}} \subset Z_{\underline{ws}})_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}}(Z^{\circ}_{\underline{w}} \subset Z_{\underline{w}})_{|\varepsilon} \cdot (1+y)\cdot \mathcal{L}^*_{s|\widehat{\varepsilon}} & \text{when } \delta = 0\,,\\ \mathrm{mC}^{\mathbb{T}}(Z^{\circ}_{\underline{w}} \subset Z_{\underline{w}})_{|\varepsilon} \cdot (1+y\mathcal{L}^*_s)_{|\widehat{\varepsilon}} & \text{when } \delta = 1\,.\end{cases}$$

Proof. **1)** The bundle $p^*_{\underline{ws}}\mathcal{L}_s$ is the relative tangent bundle of the projection $\pi_{\underline{ws}}$. Therefore we have a short exact sequence
$$0 \to p^*_{\underline{ws}}\mathcal{L}_s \to TZ_{\underline{ws}} \to \pi^* TZ_{\underline{w}} \to 0\,.$$
The claim follows from the multiplicative properties of Euler class.

2) Denote the divisor $i(Z_{\underline{w}}) \subset Z_{\underline{ws}}$ by D. The bundle $\mathcal{O}_{Z_{\underline{ws}}}(D)_{|D}$ is isomorphic to $\mathcal{L}_{s|D}$. The point $\widehat{\varepsilon}$ belongs to D if and only if $\delta = 0$. Therefore

$$\mathcal{O}_{Z_{\underline{ws}}}(D)_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{L}_{s|\widehat{\varepsilon}} & \text{when } \delta = 0\\ 1 & \text{when } \delta = 1\end{cases} \tag{3}$$

Kempf lemma (proposition 4.1) implies that
$$p^*_{\underline{ws}}\Delta_{ws,s\lambda} = \pi^*_{\underline{ws}} p^*_{\underline{w}}\Delta_{w,\lambda} + \langle -\lambda,\alpha_s\rangle \cdot D\,.$$

We use this equality to obtain

$$\begin{aligned}\mathcal{O}_{Z_{\underline{ws}}}(\lceil p^*_{\underline{ws}}\Delta_{ws,s\lambda}\rceil)_{|\widehat{\varepsilon}} &= \mathcal{O}_{Z_{\underline{ws}}}(\lceil \pi^*_{\underline{ws}} p^*_{\underline{w}}\Delta_{w,\lambda} + \langle -\lambda, \alpha_s\rangle \cdot D\rceil)_{|\widehat{\varepsilon}} \\ &= \mathcal{O}_{Z_{\underline{ws}}}(\pi^*_{\underline{ws}}\lceil p^*_{\underline{w}}\Delta_{w,\lambda}\rceil)_{|\widehat{\varepsilon}} \cdot \mathcal{O}_{Z_{\underline{ws}}}(\lceil\langle -\lambda, \alpha_s\rangle\rceil \cdot D)_{|\widehat{\varepsilon}} \\ &= \mathcal{O}_{Z_{\underline{w}}}(\lceil p^*_{\underline{w}}\Delta_{w,\lambda}\rceil)_{|\varepsilon} \cdot \mathcal{O}_{Z_{\underline{ws}}}(D)^{\lceil\langle -\lambda,\alpha_s\rangle\rceil}_{|\widehat{\varepsilon}}\end{aligned}$$

where the second equality follows from proposition 2.2. The claim follows from formula (3).

4) Suppose that $\delta = 0$. Then $\widehat{\varepsilon} \in D$. The divisor $\partial Z_{\underline{ws}}$ is SNC. Therefore (cf. [KW22, lemma 9.7])

$$\begin{aligned}\mathrm{mC}^{\mathbb{T}}(Z^\circ_{\underline{ws}} \subset Z_{\underline{ws}})_{|D} &= (1+y)\cdot \mathcal{O}_{Z_{\underline{ws}}}(-D)_{|D} \cdot \mathrm{mC}^{\mathbb{T}}(D^\circ \subset D) \\ &= (1+y)\cdot \mathcal{L}^*_{s|D} \cdot \mathrm{mC}^{\mathbb{T}}(Z^\circ_{\underline{w}} \subset Z_{\underline{w}})\end{aligned}$$

Suppose that $\delta = 1$ then

$$\begin{aligned}\mathrm{mC}^{\mathbb{T}}(Z^\circ_{\underline{ws}} \subset Z_{\underline{ws}})_{|\widehat{\varepsilon}} &= \mathrm{mC}^{\mathbb{T}}(\pi^{-1}_{\underline{ws}}(Z^\circ_{\underline{w}}) \subset Z_{\underline{ws}})_{|\widehat{\varepsilon}} \\ &= \mathrm{mC}^{\mathbb{T}}(Z^\circ_{\underline{w}} \subset Z_{\underline{w}})_{|\varepsilon} \cdot \lambda_y(\mathcal{L}^*_s)_{|\widehat{\varepsilon}}\,,\end{aligned}$$

where the first equality comes from localness of the motivic Chern class [FRW21, section 2.3] and the second from the Verdier-Rieman-Roch formula [AMSS19, theorem 4.2 (4)].

3) This point follows from **1–2)** and **4)**. □

Lemma 6.17. *Consider the situation as in theorem 6.9. For $\sigma \in (G/B)^{\mathbb{T}}$ we have*

$$\mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(ws, s\lambda)_{|\sigma} = \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(w,\lambda)_{|\sigma} \cdot \frac{(1+y)\cdot \mathcal{L}_{s|\sigma}^{\lceil -\langle\lambda,\alpha_s\rangle\rceil - 1}}{1-\mathcal{L}^*_{s|\sigma}} + \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(w,\lambda)_{|\sigma s} \cdot \frac{1+y\mathcal{L}^*_{s|\sigma}}{1-\mathcal{L}^*_{s|\sigma}}\,.$$

Proof. By definition

$$\mathrm{mC}^{\mathbb{T}}(ws, s\lambda) = p_{\underline{ws}*}\,\mathrm{mC}^{\mathbb{T}}(Z^\circ_{\underline{ws}} \subset Z_{\underline{ws}}) \cdot \mathcal{O}_{Z_{\underline{ws}}}(\lceil p^*_{\underline{ws}}\Delta_{ws,\lambda}\rceil)\,.$$

The LRR formula (theorem 2.3) implies that

$$\mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(ws, s\lambda)_{|\sigma} = \sum_{\widehat{\varepsilon}\in (Z_{\underline{ws}})^{\mathbb{T}},\ p_{\underline{ws}}(\varepsilon')=\sigma} \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{ws}, s\lambda)_{|\varepsilon'}\,. \tag{4}$$

We have a set decomposition

$$\{\widehat{\varepsilon} \in Z^{\mathbb{T}}_{\underline{ws}} | p_{\underline{ws}}(\widehat{\varepsilon}) = \sigma\} = \{(\varepsilon, 0) | \varepsilon \in Z^{\mathbb{T}}_{\underline{w}},\, p_{\underline{w}}(\varepsilon) = \sigma\} \sqcup \{(\varepsilon,1) | \varepsilon \in Z^{\mathbb{T}}_{\underline{w}},\, p_{\underline{w}}(\varepsilon) = \sigma s\}\,.$$

This allows to split the sum (4) into two sums. Lemma 6.16 (4) and the LRR formula implies that the first part is equal to

$$\sum_{\varepsilon\in(Z_{\underline{w}})^{\mathbb{T}},\ p_{\underline{w}}(\varepsilon)=\sigma} \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{w},\lambda)_{|\varepsilon} \cdot \frac{(1+y)\cdot \mathcal{L}_{s|\widehat{\varepsilon}}^{\lceil -\langle\lambda,\alpha_s\rangle\rceil - 1}}{1-\mathcal{L}^*_{s|\widehat{\varepsilon}}} = \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(w,\lambda)_{|\sigma} \cdot \frac{(1+y)\cdot \mathcal{L}_{s|\sigma}^{\lceil -\langle\lambda,\alpha_s\rangle\rceil - 1}}{1-\mathcal{L}^*_{s|\sigma}}\,,$$

and the second part to

$$\sum_{\varepsilon\in(Z_{\underline{w}})^{\mathbb{T}},\ p_{\underline{w}}(\varepsilon)=\sigma s} \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(\underline{w},\lambda)_{|\varepsilon} \cdot \frac{1+y\mathcal{L}^*_{s|\widehat{\varepsilon}}}{1-\mathcal{L}^*_{s|\widehat{\varepsilon}}} = \mathrm{mC}^{\mathbb{T}}_{\mathrm{loc}}(w,\lambda)_{|\sigma s} \cdot \frac{1+y\mathcal{L}^*_{s|\sigma}}{1-\mathcal{L}^*_{s|\sigma}}\,.$$

This proves the lemma. □

Proof of theorem 6.9. Let $\sigma \in (G/B)^{\mathbb{T}}$ be an arbitrary fixed point. We multiply the equation from lemma 6.17 by the Euler class $eu(T_\sigma G/B)$. After simplification we obtain

$$\mathrm{mC}^{\mathbb{T}}(ws, s\lambda)_{|\sigma} = \frac{1+y\mathcal{L}^*_{s|\sigma}}{1-\mathcal{L}_{s|\sigma}} \cdot \mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|\sigma s} - \frac{(1+y)\cdot \mathcal{L}_{s|\sigma}^{\lceil -\langle\lambda,\alpha_s\rangle\rceil}}{1-\mathcal{L}_{s|\sigma}} \cdot \mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|\sigma}\,.$$

Theorem 6.9 follows from the localization isomorphism. □

7. Twisted Hecke algebra

We describe the algebra of operations $\mathcal{T}^{\mathrm{R}}_{s,a} \in \mathrm{End}(K_{\mathbb{T}}(G/B)[y])$. In this section we show that one can define the operators $\mathcal{T}^{\mathrm{R}}_{w,a}$ for any $w \in W$ in a way that the resulting algebra is a deformation of the Hecke algebra and it acts in the expected way on the twisted Schubert classes. Further on, in section 15, purely algebraically, we lift this action to the endomorphisms of the representation ring $R(\mathbb{T}\times\mathbb{T})[y]$. From that point of view both operators $\mathcal{T}^{\mathrm{R}}_{s,a}$ and $\mathcal{T}^{\mathrm{L}}_{s,a}$ play equal roles.

Proposition 7.1. *Let s be a simple reflection and $a \in \mathbb{R}$ a real number. The operator $\mathcal{T}^{\mathrm{R}}_{s,a}$ commute with the left Weyl group action.*

Proof. The proof is analogous to [MNS22, proposition 5.9 (a)]. The left and right Weyl group actions commute (see [MNS22, proposition 5.3 (c)]). Moreover, the line bundle $\mathcal{L}_s$ is G-equivariant, so it is preserved by the left Weyl group action. □

Proposition 7.2. *Left and right twisted Demazure-Lusztig operators commute. Let $s_1, s_2 \in W$ be simple reflections and $a, b \in \mathbb{R}$ real numbers. Then*

$$\mathcal{T}^{\mathrm{R}}_{s_1,a} \circ \mathcal{T}^{\mathrm{L}}_{s_2,b} = \mathcal{T}^{\mathrm{L}}_{s_2,b} \circ \mathcal{T}^{\mathrm{R}}_{s_1,a}\,.$$

Proof. The twisted Demazure-Lusztig operator $\mathcal{T}^{\mathrm{R}}_{s_1,a}$ commutes with both the left Weyl group action and multiplication by a character from $S^{-1}K_{\mathbb{T}}(pt)[y]$, see proposition 7.1. □

Corollary 7.3. *Let $s_1, s_2 \in W$ be simple reflections and $\lambda_1, \lambda_2 \in \mathrm{Hom}(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}$ fractional characters. Then*

$$\mathcal{T}^{\mathrm{R}}_{s_1,\lambda_1} \circ \mathcal{T}^{\mathrm{L}}_{s_2,\lambda_2} = \mathcal{T}^{\mathrm{L}}_{s_2,\lambda_2} \circ \mathcal{T}^{\mathrm{R}}_{s_1,\lambda_1}\,.$$

The twisted operators satisfy certain braid relations with parameters described in lemma 11.3 which allows to define operator $\mathcal{T}^{\mathrm{R}}_{w,\lambda}$ for an arbitrary element $w \in W$.

Definition 7.4. Let $w \in W$ be a Weyl group element and $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_l})$ its reduced word decomposition. Let $w_{>k} \in W$ be composition of the last $l-k$ letters of $\underline{w}$, i.e.

$$w_{>k} = s_{i_{k+1}} s_{i_{k+2}} \dots s_{i_l} \in W\,.$$

For a fractional character λ we define the twisted Demazure-Lusztig operator $\mathcal{T}^{\mathrm{R}}_{\underline{w},\lambda}$ by

$$\mathcal{T}^{\mathrm{R}}_{\underline{w},\lambda} := \mathcal{T}^{\mathrm{R}}_{s_{i_1}, w_{>1}\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_{i_2}, w_{>2}\lambda} \circ \dots \circ \mathcal{T}^{\mathrm{R}}_{s_{i_{l-1}}, s_{i_l}\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_{i_l},\lambda}\,.$$

Proposition 7.5. *Let λ be a fractional character and $w \in W$ a Weyl group element. Let $\underline{w}$ and $\underline{w}'$ be two reduced word representations of w. Then*

$$\mathcal{T}^{\mathrm{R}}_{\underline{w},\lambda} = \mathcal{T}^{\mathrm{R}}_{\underline{w}',\lambda}\,.$$

Proof. It is enough to check that the equality holds on some basis of $S^{-1}K_{\mathbb{T}}(G/B)[y]$ over $S^{-1}K_{\mathbb{T}}(pt)[y]$. The elements $\{\sigma^{\mathrm{L}}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id},\lambda))\}_{\sigma\in W}$ form such a basis. Thus, it is enough to check that

$$\mathcal{T}^{\mathrm{R}}_{\underline{w},\lambda}(\sigma^{\mathrm{L}}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id},\lambda))) = \mathcal{T}^{\mathrm{R}}_{\underline{w}',\lambda}(\sigma^{\mathrm{L}}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id},\lambda)))\,.$$

Proposition 7.1 and multiple use of theorem 6.9 prove that both sides are equal to

$$\sigma^{\mathrm{L}}(\mathrm{mC}^{\mathbb{T}}(w^{-1}, w\lambda))\,.$$

□

The proof above uses the fact that the motivic Chern classes for a fixed λ span $S^{-1}K_{\mathbb{T}}(G/B)[y]$. An alternative proof is provided by using directly the braid relation. Eventually the result follows from the calculus of operations acting on Laurent polynomials, see section 11.2.

Example 7.6. Let $G = \mathrm{SL}_3 \subset \mathrm{GL}_3$, $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{t}^* \simeq \mathbb{R}^3 \subset \mathfrak{gl}_3$. For the word $\underline{w} = s_2 s_1 s_2$ we have:

$$\mathcal{T}^{\mathrm{R}}_{s_1s_2s_1,\lambda} = \mathcal{T}^{\mathrm{R}}_{s_1,s_2s_1\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_2,s_1\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_1,\lambda} = \mathcal{T}^{\mathrm{R}}_{s_1,(\lambda_2,\lambda_3,\lambda_1)} \circ \mathcal{T}^{\mathrm{R}}_{s_2,(\lambda_2,\lambda_1,\lambda_3)} \circ \mathcal{T}^{\mathrm{R}}_{s_1,(\lambda_1,\lambda_2,\lambda_3)}\,.$$

The corresponding parameters $\lceil\langle -w^{-1}_{<j}\lambda, \alpha_{s_{i_j}}\rangle\rceil$ in definition 6.3 (4) are equal to:

$$\lceil \lambda_3 - \lambda_2 \rceil, \qquad \lceil \lambda_3 - \lambda_1 \rceil, \qquad \lceil \lambda_2 - \lambda_1 \rceil\,.$$

For the word $\underline{w} = s_2 s_1 s_2$:

$$\mathcal{T}^{\mathrm{R}}_{s_2s_1s_2,\lambda} = \mathcal{T}^{\mathrm{R}}_{s_2,s_1s_2\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_1,s_2\lambda} \circ \mathcal{T}^{\mathrm{R}}_{s_2,\lambda} = \mathcal{T}^{\mathrm{R}}_{s_2,(\lambda_3,\lambda_1,\lambda_2)} \circ \mathcal{T}^{\mathrm{R}}_{s_1,(\lambda_1,\lambda_3,\lambda_2)} \circ \mathcal{T}^{\mathrm{R}}_{s_2,(\lambda_1,\lambda_2,\lambda_3)}\,.$$

The corresponding parameters $\lceil\langle -w^{-1}_{<j}\lambda, \alpha_{s_{i_j}}\rangle\rceil$ are equal to:

$$\lceil \lambda_2 - \lambda_1 \rceil, \qquad \lceil \lambda_3 - \lambda_1 \rceil, \qquad \lceil \lambda_3 - \lambda_2 \rceil\,.$$

We obtain relation

$$\mathcal{T}^{\mathrm{R}}_{s_1,\lambda_3-\lambda_2} \circ \mathcal{T}^{\mathrm{R}}_{s_2,\lambda_3-\lambda_1} \circ \mathcal{T}^{\mathrm{R}}_{s_1,\lambda_2-\lambda_1} = \mathcal{T}^{\mathrm{R}}_{s_2,\lambda_2-\lambda_1} \circ \mathcal{T}^{\mathrm{R}}_{s_1,\lambda_3-\lambda_1} \circ \mathcal{T}^{\mathrm{R}}_{s_2,\lambda_3-\lambda_2}\,.$$

Definition 7.7. Let λ be a fractional character and $w \in W$ a Weyl group element. We define

$$\mathcal{T}^{\mathrm{R}}_{w,\lambda} := \mathcal{T}^{\mathrm{R}}_{\underline{w},\lambda}\,,$$

For any reduced word representation of $\underline{w}$ of w.

Proposition 7.8. *Let $w, w' \in W$ be Weyl group elements and $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ a fractional character.*

(1) Suppose that $l(w') + l(w) = l(w'w)$, then

$$\mathcal{T}^{\mathrm{R}}_{w^{-1},\lambda}(\mathrm{mC}^{\mathbb{T}}(w', \lambda)) = \mathrm{mC}^{\mathbb{T}}(w'w, w^{-1}\lambda)\,.$$

(2) Suppose that λ is general enough, then

$$\mathcal{T}^{\mathrm{R}}_{w^{-1},\lambda}(\mathrm{mC}^{\mathbb{T}}(w', \lambda)) = (-y)^{\frac{1}{2}(l(w')+l(w)-l(w'w))}\,\mathrm{mC}^{\mathbb{T}}(w'w, w^{-1}\lambda)\,.$$

Proof. The first part follows from theorem 6.9. The second is a consequence of corollary 6.12. □

Remark 7.9. „General enough” means that $\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}$ for any root α.

Corollary 7.10. *Let $w, w' \in W$ be Weyl group elements and $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ a general enough fractional character. Then the corresponding stable envelopes are related by the recursive relation*

$$\mathcal{T}^{\mathrm{R},q}_{w^{-1},\lambda}(\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w')) = q^{l(w)/2} \cdot \mathrm{stab}^{w^{-1}\lambda}_{\mathfrak{C}_+}(w'w)\,.$$

The above results imply [SZZ21, theorem 5.4]. To see this we need the following proposition.

Proposition 7.11. *Let $s \in W$ be a simple reflection and $w \in W$ a Weyl group element such that $l(sw) > l(w)$. Let $\lambda^- \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a small anti-ample slope. Then*

$$\mathcal{T}^{\mathrm{R}}_{s,sw\lambda^-} = \mathcal{T}^{\mathrm{R}}_s \,.$$

Proof. It is enough to show that

$$\lceil -\langle sw\lambda^-, \alpha_s^\vee\rangle \rceil = \lceil \langle w\lambda^-, \alpha_s^\vee\rangle \rceil = 0\,.$$

The slope λ^- is small enough, therefore we only need to prove that $\langle w\lambda^-, \alpha_s^\vee\rangle$ is negative. We have

$$\langle w\lambda^-, \alpha_s^\vee\rangle = \langle \lambda^-, w^{-1}\alpha_s^\vee\rangle < 0\,,$$

where the last inequality follows from proposition 3.6 implying that $w^{-1}\alpha_s^\vee$ is positive. □

Corollary 7.12 ([SZZ21, theorem 5.4])**.** *Let $w, w' \in W$ be Weyl group elements and $\lambda^- \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ a small anti-ample slope. Then*

$$\mathcal{T}^{\mathrm{R},q}_{w^{-1}}(\mathrm{stab}^{w\lambda^-}_{\mathfrak{C}_+}(w')) = q^{l(w)/2} \cdot \mathrm{stab}^{\lambda^-}_{\mathfrak{C}_+}(w'w)\,.$$

Proof. Set $\lambda = w\lambda^-$. By the corollary 7.10

$$\mathcal{T}^{\mathrm{R},q}_{w^{-1},w\lambda^-}(\mathrm{stab}^{w\lambda^-}_{\mathfrak{C}_+}(w')) = q^{l(w)/2} \cdot \mathrm{stab}^{\lambda^-}_{\mathfrak{C}_+}(w'w)\,.$$

It is enough to prove that $\mathcal{T}^{\mathrm{R}}_{w^{-1},w\lambda^-} = \mathcal{T}^{\mathrm{R}}_{w^{-1}}$. Fix any reduced word decomposition $\underline{w} = (s_{i_1}, s_{i_2}, \dots, s_{i_m})$ of w. By definition of the twisted operators it is enough to prove that for any k we have

$$\mathcal{T}^{\mathrm{R}}_{s_{i_k},(w_{<k})^{-1}w\lambda^-} = \mathcal{T}^{\mathrm{R}}_{s_{i_k},w_{\geq k}\lambda^-} = \mathcal{T}^{\mathrm{R}}_{s_{i_k},s_{i_k}w_{>k}\lambda^-} = \mathcal{T}^{\mathrm{R}}_{s_{i_k}}\,,$$

where $w_{<k}$ is the element corresponding to the first $k-1$ letters of the word $\underline{w}$ and $w_{\geq k} = s_{i_k}w_{>k} = w_{<k}^{-1}w$. The last equation follows from proposition 7.11. □

Remark 7.13. In [SZZ20, 7.12] a different notation is used. Namely our $\mathrm{stab}^{w\lambda^-}_{\mathfrak{C}_+}(w')$ denotes $\mathrm{stab}^{+,w\nabla_-}_{w'}$.

8. Left induction

So far we have concentrated on the right Demazure-Lusztig operator. Below we study the left operations purely algebraically, not giving any geometric interpretation. The geometric source of the algebraic formulas is explained (at least in the A_n-case) by analysing resolutions of matrix Schubert varieties, see section 13.

8.1. **Recursive formula for G/B.** In this section we prove a counterpart of theorem 6.9 concerning the left Weyl group action. The proof is similar to the proof [RW20, proposition 7.3], see also [MNS22].

Theorem 8.1. *Let $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes \mathbb{Q}$ be a fractional character and $w \in (G/B)^{\mathbb{T}} \simeq W$ a fixed point.*

(1) Suppose that $l(sw) > l(w)$, then

$$\mathrm{mC}^{\mathbb{T}}(sw, \lambda) = \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

(2) Suppose that λ is general enough, then

$$(-y)^{\frac{1}{2}(l(w)-l(sw)+1)}\,\mathrm{mC}^{\mathbb{T}}(sw, \lambda) = \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda))\,.$$

Proof. First we prove the theorem for $w = \mathrm{id}$. The Schubert variety of id is a single point. Thus, the class $\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, \lambda)$ does not depend on the character λ. It vanishes at all fixed points other than id and at the fixed point id we have

$$\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, \lambda)_{|\mathrm{id}} = eu(T_{\mathrm{id}}G/B)\,.$$

Restrictions of the relative tangent bundle $\mathcal{L}_s$ are given by $\mathcal{L}_{s|\mathrm{id}} = \alpha_s^{-1}$ and $\mathcal{L}_{s|s} = \alpha_s$. (We treat α_s as a character and we apply the multiplicative notation.) It follows that

$$\mathcal{T}^{\mathrm{L}}_{s,\lambda}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, \lambda)) = \mathcal{T}^{\mathrm{R}}_{s,s\lambda}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, \lambda)) = \mathcal{T}^{\mathrm{R}}_{s,s\lambda}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, s\lambda)) = \mathrm{mC}^{\mathbb{T}}(s, \lambda)\,. \tag{5}$$

This proves the theorem for $w = \mathrm{id}$.
Let us focus on the case $l(ws) > l(w)$. Then $l(ws) = l(w) + l(s)$. We have

$$\begin{aligned}\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w, \lambda)) &= \mathcal{T}^{\mathrm{L}}_{s,w\lambda} \circ \mathcal{T}^{\mathrm{R}}_{w^{-1},w\lambda}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, w\lambda)) \\ &= \mathcal{T}^{\mathrm{R}}_{w^{-1},w\lambda} \circ \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(\mathrm{id}, w\lambda)) \\ &= \mathcal{T}^{\mathrm{R}}_{w^{-1},w\lambda}(\mathrm{mC}^{\mathbb{T}}(s, w\lambda)) \\ &= \mathrm{mC}^{\mathbb{T}}(sw, \lambda)\,,\end{aligned}$$

where the first and the fourth equality follow from proposition 7.8 (1), the second from lemma 7.2 and the third from equation (5).
The proof of the other case is analogous. In the last equality we need to use proposition 7.8 (2) instead of proposition 7.8 (1). □

Remark 8.2. Alternatively, we may prove theorem 8.1 using another inductive construction of the Bott-Samelson varieties and reasoning similar to the proof of theorem 6.9. Let w be a Weyl group element such that sw is longer than w. Then the variety $Z_{\underline{sw}}$ admits a locally trivial morphism to the projective line $\mathbb{P}^1$ with fiber $Z_{\underline{w}}$. Compare the proof of proposition 14.2.

8.2. **Recursion for G/P.** The left induction is also valid for homogeneous varieties G/P where P is an arbitrary parabolic subgroup. Denote by π the projection $\pi : G/B \to G/P$. Let W_P be a subgroup of W corresponding to P. The fixed point set $(G/P)^{\mathbb{T}}$ is in bijection with the set of left cosets W/W_P. Let $W^P \subset W$ be a set of minimal length representatives. For a coset wW_P we consider the Schubert variety

$$X_{wW_P} = \overline{B\pi(w)} \subset G/P\,, \qquad X^{\circ}_{wW_P} = B\pi(w) \subset G/P\,, \qquad \partial X_{wW_P} = X_{wW_P} \setminus X^{\circ}_{wW_P}\,.$$

Denote by i_{wW_P} the inclusion $X_{wW_P} \to G/P$. The rational Picard group $\mathrm{Pic}_{\mathbb{Q}}(G/P)$ coincides with $\mathrm{Pic}_{\mathbb{Q}}(G/B)^{W_P}$. For a coset wW_P and a rational character $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(G/P)$ let $\Delta^P_{wW_P,\lambda}$ be the unique $\mathbb{T}$-invariant divisor with support contained in the boundary ∂X_{wW_P}, which represents $\mathcal{L}(\lambda)_{|X_{wW_P}}$. We use notation

$$\mathrm{mC}^{\mathbb{T}}(wW_P, \lambda) := i_{wW_P*}\,\mathrm{mC}^{\mathbb{T}}(X_{wW_P}, \partial X_{wW_P}; \Delta^P_{wW_P,\lambda}) \in K^{\mathbb{T}}(G/P)[y]\,.$$

Lemma 8.3. *Let $w \in W^P$ and $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(G/P)$. Then*

$$\mathrm{mC}^{\mathbb{T}}(wW_P, \lambda) = \pi_*\,\mathrm{mC}^{\mathbb{T}}(w, \lambda)\,.$$

Proof. Fix a reduced word decomposition $\underline{w}$ of w. The element w is a minimal length representative, therefore the map

$$\pi_{|X_w} \colon X_w \to X_{wW_P}$$

is birational. It is isomorphic on the Schubert cell and $\pi^{-1}_{|X_w}(\partial X_{wW_P}) = \partial X_w$. Thus

$$(Z_{\underline{w}}, \partial Z_{\underline{w}}) \xrightarrow{p_{\underline{w}}} (X_w, \partial X_w) \xrightarrow{\pi_{|X_w}} (X_{wW_P}, \partial X_{wW_P})$$

is a SNC resolution of singularities.
The divisor $\pi^*_{|X_w}\Delta^P_{wW_P,\lambda}$ represents $\mathcal{L}(\lambda)_{|X_w}$. It is $\mathbb{T}$-invariant and contained in the boundary of Schubert variety X_w. Therefore (cf. [KW22, section 10])

$$\pi^*_{|X_w}\Delta^P_{wW_P,\lambda} = \Delta_{w,\lambda}\,.$$

It follows that both considered classes are equal to the push-forward of

$$\mathrm{mC}^{\mathbb{T}}(Z_{\underline{w}}, \partial Z_{\underline{w}}; p^*_{\underline{w}}\Delta_{w,\lambda})\,.$$

□

Lemma 8.4 ([Deo87, lemma 2.1]). *Suppose that $w \in W^P$ and s is a simple root. There are three possibilities:*

(1) *$l(sw) < l(w)$, then $sw \in W^P$,*
(2) *$l(sw) > l(w)$ and $sw \in W^P$,*
(3) *$l(sw) > l(w)$ and $sw \notin W^P$, then there exists a simple reflection $\tilde{s} \in W_P$, such that $sw = w\tilde{s}$.*

Let us analyse the first possibility:

Proposition 8.5. *Consider $w \in W^P$. Let s be a simple reflection and $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(G/P)$ a general enough fractional character. Suppose that $l(sw) < l(w)$, then*

$$\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) = (-y)\,\mathrm{mC}^{\mathbb{T}}(swW_P,\lambda)\,.$$

Proof. The proof is analogous to proof of [MNS22, theorem 4.3]. We have

$$\begin{aligned}\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) &= \mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\pi_*\,\mathrm{mC}^{\mathbb{T}}(w,\lambda)) = \\ &= \pi_*\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w,\lambda)) = (-y)\cdot\pi_*\,\mathrm{mC}^{\mathbb{T}}(sw,\lambda) = (-y)\cdot\mathrm{mC}^{\mathbb{T}}(swW_P,\lambda)\,.\end{aligned}$$

The first equality follows from lemma 8.3. The second from commutation of π_* with the left Weyl group action [MNS22, proposition 5.3 (d)], and the third from theorem 8.1 (2). The last one follows from lemmas 8.3 and 8.4. □

For a generic slope λ we have $\mathcal{T}^{\mathrm{L}}_{s,w\lambda}\circ\mathcal{T}^{\mathrm{L}}_{s,sw\lambda} = (-y)id$. This is proven purely algebraically, as in proposition 6.6, see also section 11.1. Replacing w by sw and using the identity we obtain the corollary:

Corollary 8.6. *Consider $w \in W^P$. Let s be a simple reflection and $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(G/P)$ a general enough fractional character. Suppose that $l(sw) > l(w)$ and $sw \in W^P$, then*

$$\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) = \mathrm{mC}^{\mathbb{T}}(swW_P,\lambda)\,.$$

It remains to describe the result of the action of the left Demazure-Lusztig operator in the case when $swW_P = wW_P$.

Proposition 8.7. *Consider $w \in W^P$. Let s be a simple reflection and $\lambda \in \mathfrak{t}^W_{\mathbb{Q}} \simeq \mathrm{Pic}_{\mathbb{Q}}(G/P)$ a general enough fractional character. Suppose that $l(sw) > l(w)$ and $sw \notin W^P$ then*

$$\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) = -y\,\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)\,.$$

Proof. We have

$$\begin{aligned}\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) &= \pi_*\left(\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(w,\lambda))\right) = \pi_*\left(\mathrm{mC}^{\mathbb{T}}(sw,\lambda)\right) = \\ &= \pi_*\left(\mathrm{mC}^{\mathbb{T}}(w\tilde{s},\lambda)\right) = \pi_*\left(\mathcal{T}^{\mathrm{R}}_{\tilde{s},\tilde{s}\lambda}(\mathrm{mC}^{\mathbb{T}}(w,\tilde{s}\lambda))\right) = \pi_*\left(\mathcal{T}^{\mathrm{R}}_{\tilde{s},\lambda}(\mathrm{mC}^{\mathbb{T}}(w,\lambda))\right)\,,\end{aligned}$$

where $\tilde{s}$ is given by lemma 8.4.(3). By our assumptions $\tilde{s}\lambda = \lambda$, hence the last equality. It remains to show that

$$\pi_*\left(\mathcal{T}^{\mathrm{R}}_{\tilde{s},\lambda}(\mathrm{mC}^{\mathbb{T}}(w,\lambda))\right) = -y\pi_*\left(\mathrm{mC}^{\mathbb{T}}(w,\lambda)\right).$$

The statemant follows from the followin lemma. □

Lemma 8.8. *Let $\tilde{s}$ be a simple reflection, such that $\tilde{s}\lambda = \lambda$. Consider a parabolic subgroup $P \subset G$ such that $P_{\tilde{s}} \subset P$. Denote by π the projection $\pi : G/B \to G/P$. Then*

(1)
$$\mathcal{T}^{\mathrm{R}}_{\tilde{s},\lambda} = \mathcal{T}^{\mathrm{R}}_{\tilde{s}},$$

(2)
$$\pi_* \circ \mathcal{T}^{\mathrm{R}}_{\tilde{s}} = -y\,\pi_*.$$

Proof. The part (1) follows from proposition 6.4 (5), since $\langle\lambda, \alpha^\vee_{\tilde{s}}\rangle = 0$. For the proof of (2) we can assume, that $P = P_{\tilde{s}}$. Then for $\xi \in K_{\mathbb{T}}(G/B)[y]$

$$\pi_*\mathcal{T}^{\mathrm{R}}_{\tilde{s}}(\xi) = \pi_*((1 + y\mathcal{L}^*_{\tilde{s}}) \cdot \pi^*\pi_*(\xi)) - \xi) = \pi_*((1 + y\mathcal{L}^*_{\tilde{s}})\pi^*\pi_*(\xi)) - \pi_*(\xi).$$

We set $\eta = \pi_*(\xi)$ and apply the projection formula

$$\pi_*\mathcal{T}^{\mathrm{R}}_{\tilde{s}}(\xi) = \pi_*((1 + y\mathcal{L}^*_{\tilde{s}}) \cdot \pi^*\eta) - \eta = \pi_*(1 + y\mathcal{L}^*_{\tilde{s}}) \cdot \eta - \eta.$$

Since π is a $\mathbb{P}^1$-fibration $\pi_*(1 + y\mathcal{L}^*_{\tilde{s}}) = (1 - y)$. Hence

$$\pi_*\mathcal{T}^{\mathrm{R}}_{\tilde{s}}(\xi) = (1 - y)\eta - \eta = -y\,\pi_*(\xi).$$

□

We sum up the properties of the left action:

Theorem 8.9. *Let $w \in W^P$. Let s be a simple reflection and $\lambda \in \mathrm{Pic}_{\mathbb{Q}}(G/P)$ a general enough fractional character. Then*

$$\mathcal{T}^{\mathrm{L}}_{s,w\lambda}(\mathrm{mC}^{\mathbb{T}}(wW_P,\lambda)) = (-y)^{\dim X^P_w - \dim X^P_{sw} + 1}\,\mathrm{mC}^{\mathbb{T}}(swW_P,\lambda).$$

Remark 8.10. For $\lambda = 0$ theorem 8.1 implies [MNS22, theorem 7.6].

8.3. **Wall crossing for change of the coweight chamber.** Due to theorem 5.8 we obtained a recursive formula for twisted motivic Chern classes. After translation to stable envelopes it reads as follows:

Corollary 8.11. *Let $\lambda \in \mathrm{Hom}(\mathbb{T},\mathbb{C}^*) \otimes \mathbb{Q}$ be a general enough fractional character and $w \in (G/B)^{\mathbb{T}} \simeq W$ a fixed point. Then*

$$q^{1/2} \cdot \mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(sw) = \mathcal{T}^{\mathrm{L},q}_{s,w\lambda}(\mathrm{stab}^{\lambda}_{\mathfrak{C}_+}(w)), \tag{6}$$

where $\mathcal{T}^{\mathrm{L},q}_{s,w\lambda}$ denotes the operator $\mathcal{T}^{\mathrm{L}}_{s,w\lambda}$ after substitution $y = -q$.

Stable envelopes depend on a choice of a coweight chamber $\mathfrak{C}$. Up to this point we considered only the positive coweight chamber $\mathfrak{C}_+$. All other chambers are of the form $\sigma\mathfrak{C}_+$ for some $\sigma \in W$. It follows from the naturality of stable envelopes ([AMSS19, lemma 8.2 (a)]) that for Weyl group elements $\sigma, w \in W$, arbitrary slope λ and arbitrary coweight chamber $\mathfrak{C}$ we have

$$\sigma^{\mathrm{L}}(\mathrm{stab}^{\lambda}_{\mathfrak{C}}(w)) = \mathrm{stab}^{\lambda}_{\sigma\mathfrak{C}}(\sigma w). \tag{7}$$

For any choice of coweight chamber $\mathfrak{C}$ and slope λ classes $\{\mathrm{stab}^{\lambda}_{\mathfrak{C}}(w)\}_{w\in W}$ form a base of the localized K-theory

$$S^{-1}K_{\mathbb{T}}(G/B)[q^{1/2}, q^{-1/2}]$$

over the localized Laurent polynomials ring

$$S^{-1}K_{\mathbb{T}}(pt)[q^{1/2},q^{-1/2}]\,.$$

Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two coweight chambers. Wall-crossing formula (or R-matrix) describes the base change matrix from the basis $\{\mathrm{stab}^\lambda_{\mathfrak{C}_1}(w)\}_{w\in W}$ to $\{\mathrm{stab}^\lambda_{\mathfrak{C}_2}(w)\}_{w\in W}$ (see [Oko17, Paragraph 9.2.11]). Recursion (6) may be used to obtain such a formula.

Theorem 8.12. *Let $\lambda\in\mathrm{Hom}(\mathbb{T},\mathbb{C}^*)\otimes\mathbb{Q}$ be a general enough fractional character and $w,\sigma\in W$ Weyl group elements. Let $s\in W$ be a simple reflection. Then*

$$\mathrm{stab}^\lambda_{\sigma s\mathfrak{C}_+}(w)=\frac{(1-q)\cdot\sigma\alpha_s^{\lfloor\langle w\lambda,\sigma\alpha_s\rangle\rfloor}}{1-q\sigma\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\sigma\mathfrak{C}_+}(\sigma s\sigma^{-1}w)+\frac{q^{1/2}(1-\sigma\alpha_s^{-1})}{1-q\sigma\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\sigma\mathfrak{C}_+}(w)\,.$$

Proof. Let $\tilde w=s\sigma^{-1}w$. Corollary 8.11 for $\tilde w$ implies that

$$q^{1/2}\cdot\mathrm{stab}^\lambda_{\mathfrak{C}_+}(s\tilde w)=\frac{1-q\alpha_s^{-1}}{1-\alpha_s^{-1}}\cdot s^{\mathrm{L}}(\mathrm{stab}^\lambda_{\mathfrak{C}_+}(\tilde w))-\frac{(1-q)\cdot\alpha_s^{-\lceil\langle\tilde w\lambda,\alpha_s\rangle\rceil}}{1-\alpha_s^{-1}}\cdot\mathrm{stab}^\lambda_{\mathfrak{C}_+}(\tilde w)\,,$$

which may be rewritten as

$$s^{\mathrm{L}}(\mathrm{stab}^\lambda_{\mathfrak{C}_+}(\tilde w))=\frac{(1-q)\cdot\alpha_s^{-\lceil\langle\tilde w\lambda,\alpha_s\rangle\rceil}}{1-q\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\mathfrak{C}_+}(\tilde w)+\frac{q^{1/2}(1-\alpha_s^{-1})}{1-q\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\mathfrak{C}_+}(s\tilde w)\,.$$

We apply the left Weyl group action of σ^{L} to this equation and use formula (7). Note that σ^{L} acts also on the coefficients $\alpha_s\in K_{\mathbb{T}}(pt)$. We obtain

$$\mathrm{stab}^\lambda_{\sigma s\mathfrak{C}_+}(\sigma s\tilde w)=\frac{(1-q)\cdot\sigma\alpha_s^{-\lceil\langle\tilde w\lambda,\alpha_s\rangle\rceil}}{1-q\sigma\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\sigma\mathfrak{C}_+}(\sigma\tilde w)+\frac{q^{1/2}(1-\sigma\alpha_s^{-1})}{1-q\sigma\alpha_s^{-1}}\,\mathrm{stab}^\lambda_{\sigma\mathfrak{C}_+}(\sigma s\tilde w)\,.$$

Substitution $\tilde w=s\sigma^{-1}w$ completes the proof. The exponent

$$-\lceil\langle s\sigma^{-1}w\lambda,\alpha_s\rangle\rceil=-\lceil-\langle\sigma^{-1}w\lambda,\alpha_s\rangle\rceil$$

is equal to $\lfloor\langle w\lambda,\sigma\alpha_s\rangle\rfloor$. $\square$

The above theorem describes the wall-crossing formula from the coweight chamber $\sigma\mathfrak{C}_+$ to $\sigma s\mathfrak{C}_+$. All chambers are of the form $\sigma'\mathfrak{C}_+$ for some $\sigma'\in W$. Therefore, repetitive use of the above theorem determines the wall-crossing formula between arbitrary coweight chambers.

9. Wall-crossing formula for a change of the slope

As an application of the action of the left and right Demazure-Lusztig operators we will give an easy proof of the wall-crossing formula for the slope.

For a root α and an integer $n\in\mathbb{Z}$ let

$$H_{\alpha,n}=\{\lambda\in\mathfrak{t}^*|\langle\lambda,\alpha^\vee\rangle=n\}\subset\mathrm{Hom}(\mathbb{T},\mathbb{C}^*)\otimes_{\mathbb{Z}}\mathbb{R}\simeq\mathrm{Pic}(G/B)\otimes_{\mathbb{Z}}\mathbb{R}\,.$$

Definition 9.1. Alcove is a connected component of the complement of hyperplanes $H_{\alpha,n}$.

Let λ_1 and λ_2 be slopes belonging to two adjacent alcoves. The slope R-matrix is the base change matrix from the basis $\{\mathrm{stab}^{\lambda_1}_{\mathfrak{C}_+}(w)\}_{w\in W}$ to $\{\mathrm{stab}^{\lambda_2}_{\mathfrak{C}_+}(w)\}_{w\in W}$ (see [OS22, Section 2.2]). In [SZZ21] the slope R-matrix for G/B was computed.

Theorem 9.2 ([SZZ21, theorems 4.1 and 5.1])**.** *Let α be a positive root and s the corresponding reflection. Suppose that alcoves ∇_1 and ∇_2 are adjacent and separated by the wall $H_{\alpha,0}$ and the functional $\langle -,\alpha^\vee\rangle$ is positive on ∇_2. For any fixed point $w\in (G/B)^{\mathbb{T}}\simeq W$ we have*

$$\mathrm{stab}^{\nabla_1}_{\mathfrak{C}_+}(w)=\begin{cases}\mathrm{stab}^{\nabla_2}_{\mathfrak{C}_+}(w)+(q^{1/2}-q^{-1/2})\cdot \mathrm{stab}^{\nabla_2}_{\mathfrak{C}_+}(ws) & \text{if } l(w)>l(ws)\,,\\ \mathrm{stab}^{\nabla_2}_{\mathfrak{C}_+}(w) & \text{if } l(ws)>l(w)\,.\end{cases}$$

We propose an alternative approach to this result based on the geometry of Bott-Samelson resolution. Theorem 5.8 states that stable envelopes can be computed using twisted motivic Chern classes. We will prove that

Theorem 9.3. *Let α be a positive root and s the corresponding reflection. Suppose that alcoves ∇_1 and ∇_2 are adjacent and separated by the wall $H_{\alpha,0}$ and the functional $\langle -,\alpha^\vee\rangle$ is positive on ∇_2. Let $w\in (G/B)^{\mathbb{T}}\simeq W$ be a fixed point. Then*

a) The twisted motivic Chern class is constant on alcoves, i.e. if slopes λ_1,λ_2 belong to the same alcove, then

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1)=\mathrm{mC}^{\mathbb{T}}(w,\lambda_2)\,.$$

b) Choose slopes $\lambda_1\in\nabla_1$ and $\lambda_2\in\nabla_2$. Suppose that $l(ws)>l(w)$ then

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1)=\mathrm{mC}^{\mathbb{T}}(w,\lambda_2)\,.$$

c) Choose slopes $\lambda_1\in\nabla_1$ and $\lambda_2\in\nabla_2$. Suppose that $l(ws)<l(w)$ then

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1)=\mathrm{mC}^{\mathbb{T}}(w,\lambda_2)-(-y)^{\frac{1}{2}(l(w)-l(ws)-1)}(1+y)\cdot \mathrm{mC}^{\mathbb{T}}(ws_\alpha,\lambda_2)\,.$$

Remark 9.4. The above theorem is equivalent to theorem 9.2.

Remark 9.5. In theorems 9.2 and 9.3 only the walls $H_{\alpha,0}$ are considered. The formulas for general walls $H_{\alpha,n}$ can be deduced from the periodicity of the twisted motivic Chern classes and the stable envelopes (cf. [AMSS19, lemma 8.2(a)] and [KW22, remark 2.4]). For an integral weight $\mu\in \mathrm{Hom}(\mathbb{T},\mathbb{C}^*)$ we have

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda+\mu)=\mathcal{L}(\mu)^{-1}_{|w}\mathcal{L}(\mu)\cdot \mathrm{mC}^{\mathbb{T}}(w,\lambda)\,,$$
$$\mathrm{stab}^{\nabla+\mu}_{\mathfrak{C}_+}(w)=\mathcal{L}(\mu)^{-1}_{|w}\mathcal{L}(\mu)\cdot \mathrm{stab}^{\nabla}_{\mathfrak{C}_+}(w)\,.$$

See [SZZ21, corollary 5.3 and discussion after lemma 3.7] for a detailed account.

Proof of parts a) and b) of theorem 9.3. Choose a reduced world decomposition $\underline{w}$ of w. By definition the twisted motivic Chern class $\mathrm{mC}^{\mathbb{T}}(w,\lambda)$ is determined by the pullback bundle $p_{\underline{w}}^*\mathcal{L}(\lambda)$. This bundle is described by the Chevalley formula. The conclusion follows from proposition 4.4. □

The rest of this section is devoted to the proof of part c) of theorem 9.3. Due to the part a) we may assume that $\lambda_2=s\lambda_1$.

Lemma 9.6. *Consider the situation described in the part c). Suppose that s is a simple reflection. Then the theorem holds, i.e.*

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1)=\mathrm{mC}^{\mathbb{T}}(w,s\lambda_1)-(1+y)\,\mathrm{mC}^{\mathbb{T}}(ws,s\lambda_1)\,.$$

Proof. Let $\mathcal{L}_s$ be the relative tangent bundle of the projection $G/B \to G/P_s$. Theorem 6.9 implies that

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1) = s^{\mathrm{R}}\left(\mathrm{mC}^{\mathbb{T}}(ws,s\lambda_1)\right)\cdot\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s} - \mathrm{mC}^{\mathbb{T}}(ws,s\lambda_1)\cdot\frac{1+y}{1-\mathcal{L}_s}\,, \tag{8}$$

$$\mathrm{mC}^{\mathbb{T}}(w,s\lambda_1) = s^{\mathrm{R}}\left(\mathrm{mC}^{\mathbb{T}}(ws,\lambda_1)\right)\cdot\frac{1+y\mathcal{L}_s^*}{1-\mathcal{L}_s} - \mathrm{mC}^{\mathbb{T}}(ws,\lambda_1)\cdot\frac{(1+y)\cdot\mathcal{L}_s}{1-\mathcal{L}_s}\,. \tag{9}$$

Moreover, part *b*) implies that $\mathrm{mC}^{\mathbb{T}}(ws,\lambda_1) = \mathrm{mC}^{\mathbb{T}}(ws,s\lambda_1)$. Subtracting equations (8)–(9) we obtain the desired equality. □

Lemma 9.7. *Suppose $r,\tilde{s}\in W$ are reflections such that $s = r\tilde{s}r$, r is simple and $r\neq\tilde{s}$. If part c) of theorem 9.3 holds for reflection $\tilde{s}$ then it also holds for s.*

Proof. Let

$$a = \frac{1}{2}(l(w)-l(wr)-1)\,,\quad b = \frac{1}{2}(l(wr)-l(wr\tilde{s})-1)\,,\quad c = \frac{1}{2}(l(wr\tilde{s})-l(ws)+1)\,.$$

Let α_r be the positive root defining the reflection r. We compute the difference using corollary 6.12

$$\mathrm{mC}^{\mathbb{T}}(w,\lambda_1) - \mathrm{mC}^{\mathbb{T}}(w,s\lambda_1) = (-y)^a\left(\mathcal{T}^{\mathrm{R}}_{r,r\lambda_1}\left(\mathrm{mC}^{\mathbb{T}}(wr,r\lambda_1)\right) - \mathcal{T}^{\mathrm{R}}_{r,rs\lambda_1}\left(\mathrm{mC}^{\mathbb{T}}(wr,rs\lambda_1)\right)\right)\,.$$

The exponents in the definition of $\mathcal{T}^{\mathrm{R}}_{r,rs\lambda_1}$ and $\mathcal{T}^{\mathrm{R}}_{r,r\lambda_1}$ (definition 6.3) are equal to $\lceil-\langle r\lambda_1,\alpha_r\rangle\rceil$ and $\lceil-\langle rs\lambda_1,\alpha_r\rangle\rceil$ correspondingly. We have

$$-\langle r\lambda_1,\alpha_r^\vee\rangle = \langle\lambda_1,\alpha_r^\vee\rangle\,,\qquad -\langle rs\lambda_1,\alpha_r^\vee\rangle = \langle s\lambda_1,\alpha_r^\vee\rangle\,.$$

Since λ_1 and $s\lambda_1$ are separated from each other only by the wall $H_{\alpha,0}$ and $\alpha\neq\alpha_r$, thus

$$\lceil\langle\lambda_1,\alpha_r^\vee\rangle\rceil = \lceil\langle s\lambda_1,\alpha_r^\vee\rangle\rceil\,,$$

hence

$$\lceil-\langle r\lambda_1,\alpha_r\rangle\rceil = \lceil-\langle rs\lambda_1,\alpha_r\rangle\rceil\,,\qquad\text{and}\qquad \mathcal{T}^{\mathrm{R}}_{r,r\lambda_1} = \mathcal{T}^{\mathrm{R}}_{r,rs\lambda_1}\,.$$

Therefore

$$\begin{aligned}\mathrm{mC}^{\mathbb{T}}(w,\lambda_1) - \mathrm{mC}^{\mathbb{T}}(w,s\lambda_1) =&(-y)^a\cdot\mathcal{T}^{\mathrm{R}}_{r,r\lambda_1}\left(\mathrm{mC}^{\mathbb{T}}(wr,r\lambda_1) - \mathrm{mC}^{\mathbb{T}}(wr,\tilde{s}r\lambda_1)\right)\\ =&(-1)\cdot(-y)^{a+b}\cdot(1+y)\cdot\mathcal{T}^{\mathrm{R}}_{r,r\lambda_1}\left(\mathrm{mC}^{\mathbb{T}}(wr\tilde{s},r\lambda_1)\right)\\ =&(-1)\cdot(-y)^{a+b+c}\cdot(1+y)\cdot\mathrm{mC}^{\mathbb{T}}(ws,\lambda_1)\,.\end{aligned}$$

The second equality follows from theorem 9.3 for $\tilde{s}$ and the third from corollary 6.12. Note that the assumption $l(wr) > l(wr\tilde{s})$ for c) is satisfied by proposition 3.5. □

Proof of the part c). Choose $\sigma\in W$ of minimal length such that $s_0 = \sigma^{-1}s\sigma$ is a simple reflection (cf. proposition 3.4). Choose a reduced word decomposition $\underline{\sigma}$ of σ. Let $\sigma_k\in W$ be the product of last k letters of $\underline{\sigma}$. Let

$$s_k = \sigma_k s_0(\sigma_k)^{-1}\,.$$

We prove the theorem by induction on k. Lemma 9.6 proves the inductive assumption for $k=0$. Lemma 9.7 proves the inductive step. □

10. Upgrade from GL_n/B to $\mathrm{End}(\mathbb{C}^n)$.

From now on (except the last section) we concentrate on the A_n-case. We show that the construction which we performed for the flag varieties can be lifted to the level of matrices.

10.1. **The Kirwan map.** In the study of the homogeneous spaces for SL_n it is convenient to consider $G = \mathrm{GL}_n$ instead of SL_n. Let B be the standard Borel subgroup of GL_n consisting of the upper-triangular matrices. Let $\mathbb{T} \subset \mathrm{GL}_n$ be the diagonal torus. Let $\alpha_1, \dots, \alpha_n \in \mathfrak{t}^*$ be the simple roots of SL_n, i.e.

$$\alpha_k = (0, 0, \dots, \underbrace{1, -1,}_{k,\ k+1} \dots, 0) \in \mathbb{R}^n \simeq \mathfrak{t}^* .$$

The root α_k corresponds to the simple reflection $s_k = (k, k+1)$.
For $i \in \{1, 2, \dots, n\}$ denote by t_i the characters of $\mathbb{T}$ associated with coordinates. The equivariant K-theory of the point

$$K_{\mathbb{T}}(pt) = \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_n^{\pm1}]$$

will be denoted by $\mathbb{Z}[\underline{t}^{\pm1}]$ for short. The equivariant K-theory of the flag manifold

$$\mathrm{GL}_n/B = \{V_0 \subset V_1 \subset \dots \subset V_n = \mathbb{C}^n \,:\, \dim V_i = i \text{ for } i \in \{0, 1, \dots, n\}\}$$

is generated over $\mathbb{Z}[\underline{t}^{\pm1}]$ by the classes of the tautological line bundles V_i/V_{i-1}. Let $\underline{x} = \{x_i\}_{i=1,2,\dots,n}$ be a set of variables. The surjection

$$\begin{aligned} K_{\mathbb{T}\times\mathbb{T}}(\mathrm{End}(\mathbb{C}^n)) \simeq \mathbb{Z}[\underline{t}^{\pm1}, \underline{x}^{\pm1}] &\to K_{\mathbb{T}}(\mathrm{GL}_n/B)\,, \\ x_i \ &\mapsto \ [V_i/V_{i-1}] \end{aligned}$$

has a geometric interpretation. Let $\mathbb{T}^2$ be the product of tori and let $\underline{t}$ and $\underline{x}$ be the sets of coordinate characters. The first copy of $\mathbb{T}^2$ acts on $\mathrm{End}(\mathbb{C}^n)$ by the left multiplication, the second copy acts by the right multiplication. The composition of the restriction map and the natural isomorphism

$$\mathbb{Z}[\underline{t}^{\pm1}, \underline{x}^{\pm1}] \simeq K_{\mathbb{T}^2}(\mathrm{End}(\mathbb{C}^n)) \longrightarrow K_{\mathbb{T}^2}(\mathrm{GL}_n) \simeq K_{\mathbb{T}}(\mathrm{GL}_n/B) \tag{10}$$

will be denoted by κ and called the Kirwan map. Let $L_{i,x} = \frac{x_{i+1}}{x_i}$ and $L_{i,t} = \frac{t_{i+1}}{t_i}$ for $i \in \{1, 2, \dots, n-1\}$. We have

$$\kappa(L_{i,t}) = \mathcal{L}(\alpha_i)\,, \qquad \kappa(L_{i,x}) = \alpha_i^{-1}\,. \tag{11}$$

The bundle $\mathcal{L}(\alpha_i)$ is the line bundle tangent to the fibers of the elementary contraction $G/B \to G/P_i$. It is induced from the representation $\mathbb{C}_{-\alpha_i}$, see section 3.1. For computation it is convenient to use the composition of κ with the restriction to the fixed point set. The composition is of the form

$$\overline{\kappa} : \mathbb{Z}[\underline{t}^{\pm1}, \underline{x}^{\pm1}, y] \to K_{\mathbb{T}}((\mathrm{GL}_n/B)^{\mathbb{T}}[y]) = \bigoplus_{\sigma\in S_n} \mathbb{Z}[\underline{t}^{\pm1}, y]\,,$$

$$f(\underline{t}, \underline{x}, y) \mapsto \bigoplus_{\sigma\in S_n} f(\underline{t}, \underline{x}, y)_{x_i := t_{\sigma(i)}}\,.$$

Here S_n denotes the group of permutations of n elements, the Weyl group of GL_n. For a detailed description of the equivariant K-theory of the flag variety see e.g. [Uma13].

10.2. **Matrix Schubert varieties.** The action of $B \times B$ on $\mathrm{End}(\mathbb{C}^n)$ is given by

$$(b_1, b_2) \cdot a = b_1 a b_2^{-1}\,.$$

We identify $\mathrm{End}(\mathbb{C}^n)$ with the space of $n \times n$ matrices and we identify permutations with their matrices. For $w \in S_n$ we denote by $\mathcal{O}_w$ the $B \times B$-orbit of w. Let $\mathcal{X}_w = \overline{\mathcal{O}_w}$ be its closure. It is a maximal rank matrix Schubert variety. In general, by the matrix Schubert variety we understand the closure of a $B \times B$ orbit in $\mathrm{End}(\mathbb{C}^n)$. We will discuss here only the maximal rank orbits. They are exactly the orbits of the permutation matrices. For more details see [MS05, chapter 15]. Let $pr : \mathrm{GL}_n \to \mathrm{GL}_n/B$ be the quotient map. Then

$$\mathcal{X}_w \cap \mathrm{GL}_n = pr^{-1}(X_w)\,.$$

10.3. **The lift of boundary divisors.** Let $s_k = (k, k+1)$ be a simple reflection and let w_0 be the longest permutation. The Schubert variety $X_{w_0 s_k} \subset \mathrm{GL}_n/B$ is of codimension one. By the Chevalley formula for a fractional character $\lambda \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^n$ the line bundle $\mathcal{L}(\lambda)$ can be written (nonequivariantly) in terms of Schubert divisors as

$$\mathcal{L}(\lambda) = \mathcal{O}_{\mathrm{GL}_n/B}\left(\sum_{i=k}^{n-1} (\lambda_k - \lambda_{k+1}) X_{w_0 s_k}\right)\,.$$

The analogous formula for the restriction of $\mathcal{L}(\lambda)$ to Schubert variety X_w is given e.g. in [Bri05, proposition 1.4.5]. We will repeat this construction in $\mathrm{End}(\mathbb{C}^n)$.

For a matrix $a = \{a_{i,j}\}_{1\le i,j\le n}$ and $1 \le k \le n$ let

$$m_k(a) = \det\left(\{a_{i,j}\}_{1\le i,j\le k}\right)$$

be the first principal k-minor. Define

$$M_{w,k} = \{a \in \mathcal{X}_w \mid m_k(w^{-1}a) = 0\}\,. \tag{12}$$

Remark 10.1. The Schubert divisor $X_{w_0 s_k} \subset \mathrm{GL}_n/B$ is pulled back from the Grassmannian $Gr_k(\mathbb{C}^n)$, hence it is the zero locus of a section of the bundle $\Lambda^k V_k$. The divisor $M_{w_0,k}$ restricted to GL_n is the inverse image of the Schubert divisor in $X_{w_0 s_k}$

$$M_{w_0,k} \cap \mathrm{GL}_n = pr^{-1}(X_{w_0 s_k})\,.$$

The divisor $M_{w,k}$ is the restriction to $\mathcal{X}_w$ of a left shift of $M_{w_0,k}$.

Proposition 10.2. *For $w \in S_n$ the divisors $M_{w,k}$ are contained in the boundary of $\mathcal{X}_w$, i.e. they do not intersect $\mathcal{O}_w$.*

Proof. First let us note that for $a \in \mathrm{End}(\mathbb{C}^n)$, $b \in B$ we have

$$m_k(ab) = 0 \iff m_k(a) = 0\,. \tag{13}$$

Also for $b \in B$, $w \in S_n$

$$m_k(w^{-1}bw) \neq 0\,. \tag{14}$$

We show that $m_k(w^{-1}a) \neq 0$ for $a = b_1 w b_2^{-1}$, $b_1, b_2 \in B$. Since

$$m_k(w^{-1}b_1 w b_2^{-1}) = 0 \iff m_k(w^{-1}b_1 w) = 0$$

and $b_1 \in B$ by (13–14) we obtain the claim. □

For a fixed sequence of rational numbers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ let

$$D_{w,\lambda} = \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) M_{w,k}\,. \tag{15}$$

We set for convenience $\lambda_{n+1} = 0$. In this way for $w = \mathrm{id}$

$$D_{\mathrm{id},\lambda} = \mathrm{div}\left(\prod_{i=1}^{n} a_{ii}^{\lambda_i}\right) \subset \mathcal{X}_{\mathrm{id}} = \overline{B}\,.$$

Here $\operatorname{div}(f)$ stands for the $\mathbb{Q}$-divisor of the formal expression with rational exponents. The divisor $D_{w,\lambda}$ is defined so that

$$(D_{w,\lambda})_{|\mathrm{GL}_n\cap\mathcal{X}_w} = pr^*(\Delta_{w,\lambda})\,. \tag{16}$$

See section 5.3 for the definition of $\Delta_{w,\lambda}$.

Example 10.3. For $n=2$, $S_2=\{\mathrm{id},s_1\}$

$$\begin{aligned} D_{\mathrm{id},\lambda} &= (\lambda_1-\lambda_2)\operatorname{div}(a_{11}) + \lambda_2 \operatorname{div}\det\begin{pmatrix} a_{11} & a_{12}\\ 0 & a_{22}\end{pmatrix}\\ &= \lambda_1\operatorname{div}(a_{11}) + \lambda_2\operatorname{div}(a_{22})\,,\\ D_{s_1,\lambda} &= (\lambda_1-\lambda_2)\operatorname{div}(a_{21}) + \lambda_2 \operatorname{div}\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\,. \end{aligned}$$

The boundary of the open orbit consists of the degenerate or upper-triangular matrices

$$\partial\mathcal{X}_{s_1} = \{a\in\operatorname{End}(\mathbb{C}^2)\ :\ \det(a)=0\ \vee\ a_{21}=0\}$$

Consider the map resolving the singularity of the boundary

$$\mu : P_1\times_B \overline{B} \to \operatorname{End}(\mathbb{C}^2)\,.$$

In one of the standard affine charts it is given by the formula

$$\mu(c,b_{11},b_{12},b_{22}) = \begin{pmatrix} 1 & 0\\ c & 1\end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12}\\ 0 & b_{22}\end{pmatrix} = \begin{pmatrix} b_{11} & b_{12}\\ c\,b_{11} & c\,b_{12}+b_{22}\end{pmatrix}\,.$$

Then

$$\begin{aligned}\mu^*(D_{s_1,\lambda}) &= \operatorname{div}\left((c\,b_{11})^{\lambda_1-\lambda_2}(b_{11}b_{22})^{\lambda_2}\right)\\ &= (\lambda_1-\lambda_2)\operatorname{div}(c) + \lambda_1\operatorname{div}(b_{11}) + \lambda_2\operatorname{div}(b_{22})\,.\end{aligned}$$

11. Universal algebra

We define an algebra acting on the ring of Laurent polynomials. We will show that the generating operations satisfy the same quadratic and braid relations, as those acting on the level of K-theory of GL_n/B. The described algebra can be compared with the algebra generated by Demazure-Lusztig operators which computes the motivic Chern classes of matrix Schubert varieties.

11.1. **The small algebra.** Let us define operations

$$\mathfrak{T}^{\mathrm{R}}_{i,a}\colon \mathbb{Z}[z_1^{\pm1},z_2^{\pm1},\dots,z_n^{\pm1},y] \longrightarrow \mathbb{Z}[z_1^{\pm1},z_2^{\pm1},\dots,z_n^{\pm1},y]$$

on Laurent polynomials in $z_1,z_2,\dots,z_n$. The operations depend on $i\in\{1,2,\dots,n-1\}$ and $a\in\mathbb{Q}$. For $f\in\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},\dots,z_n^{\pm1},y]$ let

$$L_i = \frac{z_{i+1}}{z_i}$$

and

$$\mathfrak{T}^{\mathrm{R}}_{i,a}(f) = -\frac{(1+y)L_i^{\lceil a\rceil}\cdot f - \left(1+y\,L_i^{-1}\right)\cdot s_if}{1-L_i}\,,$$

where

$$s_if(z_1,z_2,\dots,z_i,z_{i+1},\dots,z_n,y) = f(z_1,z_2,\dots,z_{i+1},z_i,\dots,z_n,y)\,.$$

If we set $z_i=z_{i+1}$ then the numerator is equal to 0, therefore, it is divisible by z_i-z_{i+1}, which is equivalent to divisibility in Laurent polynomials by $1-\frac{z_i}{z_{i+1}}$. Hence the operators $\mathfrak{T}^{\mathrm{R}}_{i,a}$ transform Laurent polynomials into Laurent polynomials. The operators $\mathfrak{T}^{\mathrm{R}}_{i,a}$ can be treated as variants of the isobaric divided differences, see e.g. [RS18, section 2.1]. Another way of writing the operators $\mathfrak{T}^{\mathrm{R}}_{i,a}$ is the following

$$\mathfrak{T}^{\mathrm{R}}_{i,a}(f)=\frac{\left(1+y\,L_i^{-1}\right)\cdot s_if}{1-L_i}+\frac{(1+y)L_i^{\lceil a\rceil-1}\cdot f}{1-L_i^{-1}}\,.$$

This form is adapted to the localization formula. A slight modification of the operator $\mathfrak{T}^{\mathrm{R}}_{i,a}$ will be useful

$$\begin{aligned}\mathfrak{T}^{\mathrm{L}}_{i,a}(f)&=-\frac{(1+y)L_i^{\lceil a\rceil}\cdot f-\left(1+y\,L_i\right)\cdot s_if}{1-L_i}\\&=\frac{\left(1+y\,L_i\right)\cdot s_if}{1-L_i}+\frac{(1+y)L_i^{\lceil a\rceil-1}\cdot f}{1-L_i^{-1}}\,.\end{aligned}$$

It is elementary to note that

$$\mathfrak{T}^{\mathrm{L}}_{i,a}((1+y\,L_i)f)=(1+y\,L_i)\;\mathfrak{T}^{\mathrm{R}}_{i,a}(f)\,,$$

i.e. $\mathfrak{T}^{\mathrm{L}}_{i,a}$ differs from $\mathfrak{T}^{\mathrm{R}}_{i,a}$ by conjugation with the multiplication by $(1+y\,L_i)$.

The properties described below hold for both $\mathfrak{T}^{\mathrm{R}}_{i,a}$ and $\mathfrak{T}^{\mathrm{L}}_{i,a}$, therefore we will simply use the notation $\mathfrak{T}_{i,a}$.

Lemma 11.1 (Quadratic relation)**.** *Suppose $a\notin\mathbb{Z}$, then*

$$\mathfrak{T}_{i,-a}\circ\mathfrak{T}_{i,a}=-y\,\mathrm{id}\,.$$

If $a\in\mathbb{Z}$, then

$$\mathfrak{T}_{i,1-a}\circ\mathfrak{T}_{i,a}=-y\,\mathrm{id}\,.$$

Proof. The proof is identical as in proposition 6.8. □

Remark 11.2. For a formula involving the composition $\mathfrak{T}_{i,b}\circ\mathfrak{T}_{i,a}$ for arbitrary $a,b\in\mathbb{Q}$ see theorem 15.1.

The operators $\mathfrak{T}_{i,a}$ satisfy the braid relation with parameters:

Lemma 11.3. *For $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Q}$ we have*

$$\mathfrak{T}_{i,\lambda_3-\lambda_2}\circ\mathfrak{T}_{i+1,\lambda_3-\lambda_1}\circ\mathfrak{T}_{i,\lambda_2-\lambda_1}=\mathfrak{T}_{i+1,\lambda_2-\lambda_1}\circ\mathfrak{T}_{i,\lambda_3-\lambda_1}\circ\mathfrak{T}_{i+1,\lambda_3-\lambda_2}$$

Remark 11.4. The vector $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ can be understood as the weight for the maximal torus of GL_3, then the above formula coincides with the formula from example 7.6.

Proof. The formula can be checked by hand (or rather using a computer algebra software). We can assume that $n=3$, $i=1$. Since in the formulas appears $\lceil\lambda_i-\lambda_j\rceil$ one is led to consider various cases depending on the fractional parts. Setting $a=\lambda_1-\lambda_2$ and $b=\lambda_2-\lambda_3$ it is enough to check two cases assuming for $a,b\in\mathbb{Z}$

$$\mathfrak{T}_{1,a}\circ\mathfrak{T}_{2,a+b}\circ\mathfrak{T}_{1,b}=\mathfrak{T}_{2,b}\circ\mathfrak{T}_{1,a+b}\circ\mathfrak{T}_{2,a}$$

and

$$\mathfrak{T}_{1,a}\circ\mathfrak{T}_{2,a+b+1}\circ\mathfrak{T}_{1,b}=\mathfrak{T}_{2,b}\circ\mathfrak{T}_{1,a+b+1}\circ\mathfrak{T}_{2,a}\,.$$

Verification is an elementary, although lengthy, calculation.

There is an alternative method of the proof, not demanding direct calculation, but relying on the fact that Demazure-Lusztig operators generate an action of the Hecke algebra on the K-theory of GL_n/B. It will be presented after theorem 11.11. □

11.2. **Operators with parameters in $\mathfrak{t}_n^* \subset \mathfrak{gl}_n^*$.**

Definition 11.5. For $\lambda \in \mathfrak{t}_n^*$ we define operator $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ acting on $\mathbb{Z}[\underline{x}^{\pm 1}, y]$

$$\mathscr{T}^{\mathrm{R}}_{i,\lambda} = \mathfrak{T}^{\mathrm{R}}_{i,-\langle \lambda, \alpha_i^\vee \rangle}\,,$$

after substitution $x_i = z_i$. We extend the action of the operator $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ to

$$\mathscr{T}^{\mathrm{R}}_{i,\lambda} \colon \mathbb{Z}[\underline{t}^{\pm 1}, \underline{x}^{\pm 1}, y] \to \mathbb{Z}[\underline{t}^{\pm 1}, \underline{x}^{\pm 1}, y]$$

linearly with respect to $\mathbb{Z}[\underline{t}^{\pm 1}]$. Explicitly for $f \in \mathbb{Z}[\underline{x}^{\pm 1}, \underline{t}^{\pm 1}, y]$ we have

$$\mathscr{T}^{\mathrm{R}}_{i,\lambda}(f) = \frac{(1+y)\left(\frac{x_i}{x_{i+1}}\right)^{1-\lceil \lambda_{i+1}-\lambda_i \rceil}}{1-\frac{x_i}{x_{i+1}}} f + \frac{\left(1+y\frac{x_i}{x_{i+1}}\right)}{1-\frac{x_{i+1}}{x_i}} s_i^x f\,, \tag{17}$$

where the reflection s_i^x acts on x-variables, switching x_i with x_{i+1}.

Remark 11.6. The assignment $\lambda \mapsto \mathscr{T}^{\mathrm{R}}_{i,\lambda}$ is constant on the alcoves in $\mathfrak{t}^*$.

Directly from lemmas 11.1 and 11.3 we obtain:

Proposition 11.7. *For a generic λ we have*

$$\mathscr{T}^{\mathrm{R}}_{i,\lambda} \circ \mathscr{T}^{\mathrm{R}}_{i,s_i\lambda} = -y\,\mathrm{id}\,.$$

and for arbitrary λ

$$\mathscr{T}^{\mathrm{R}}_{i,s_{i+1}s_i\lambda} \circ \mathscr{T}^{\mathrm{R}}_{i+1,s_i\lambda} \circ \mathscr{T}^{\mathrm{R}}_{i,\lambda} = \mathscr{T}^{\mathrm{R}}_{i+1,s_is_{i+1}\lambda} \circ \mathscr{T}^{\mathrm{R}}_{i,s_{i+1}\lambda} \circ \mathscr{T}^{\mathrm{R}}_{i+1,\lambda}\,.$$

We define the left operator using $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ acting on t–variables.

Definition 11.8. For $\lambda \in \mathfrak{t}_n^*$ we define operator $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ acting on $\mathbb{Z}[\underline{t}^{\pm 1}, y]$

$$\mathscr{T}^{\mathrm{L}}_{i,\lambda} = \mathfrak{T}^{\mathrm{L}}_{i,\langle \lambda, \alpha_i^\vee \rangle}\,,$$

after substitution $t_i = z_i$. We extend the action of the operator $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ to

$$\mathscr{T}^{\mathrm{L}}_{i,\lambda} \colon \mathbb{Z}[\underline{t}^{\pm 1}, \underline{x}^{\pm 1}, y] \to \mathbb{Z}[\underline{t}^{\pm 1}, \underline{x}^{\pm 1}, y]$$

linearly with respect to $\mathbb{Z}[\underline{x}^{\pm 1}]$. Explicitly for $f \in \mathbb{Z}[\underline{x}^{\pm 1}, \underline{t}^{\pm 1}, y]$ we have

$$\mathscr{T}^{\mathrm{L}}_{i,\lambda}(f) = \frac{(1+y)\left(\frac{t_i}{t_{i+1}}\right)^{1-\lceil \lambda_i-\lambda_{i+1} \rceil}}{1-\frac{t_i}{t_{i+1}}} f + \frac{\left(1+y\frac{t_{i+1}}{t_i}\right)}{1-\frac{t_{i+1}}{t_i}} s_i^t(f)\,. \tag{18}$$

where the reflection s_i^t acts on t-variables, switching t_i with t_{i+1}.

Directly from lemmas 11.1 and 11.3 we obtain braid and quadratic relations for $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ operators.

Proposition 11.9. *For a generic λ we have*

$$\mathscr{T}^{\mathrm{L}}_{i,s_i\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i,\lambda} = -y\,\mathrm{id}\,.$$

and for arbitrary λ

$$\mathscr{T}^{\mathrm{L}}_{i,s_{i+1}s_i\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i+1,s_i\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i,\lambda} = \mathscr{T}^{\mathrm{L}}_{i+1,s_is_{i+1}\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i,s_{i+1}\lambda} \circ \mathscr{T}^{\mathrm{L}}_{i+1,\lambda}\,.$$

Example 11.10. For $n = 2$ and $\lambda = (\lambda_1, \lambda_2)$ we have

$$\mathscr{T}^{\mathrm{R}}_{1,\lambda}(f)(t_1,t_2,x_1,x_2,y) =$$

$$= \frac{(1+y)\left(\frac{x_1}{x_2}\right)^{1-\lceil \boldsymbol{\lambda_2-\lambda_1} \rceil}}{1-\frac{x_1}{x_2}} f\,(t_1,t_2,x_1,x_2,y) + \frac{1+y\boldsymbol{\frac{x_1}{x_2}}}{1-\frac{x_2}{x_1}} f\,(t_1,t_2,x_2,x_1,y)\;,$$

while

$$\mathscr{T}^{\mathrm{L}}_{1,\lambda}(f)(t_1,t_2,x_1,x_2,y) = \\ = \frac{(1+y)\left(\frac{t_1}{t_2}\right)^{1-\lceil\boldsymbol{\lambda_1-\lambda_2}\rceil}}{1-\frac{t_1}{t_2}} f(t_1,t_2,x_1,x_2,y) + \frac{1+y\boldsymbol{\frac{t_2}{t_1}}}{1-\frac{t_2}{t_1}} f(t_2,t_1,x_1,x_2,y)\,.$$

11.3. **Comparison with $K_{\mathbb{T}}(\mathrm{GL}_n/B_n)$.** The main reason why we introduced the operators $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ is the following theorem:

Theorem 11.11. *The operators $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ are lifts of the operators $\mathcal{T}^{\mathrm{R}}_{s_i,\lambda}$ and the operators $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ are lifts of the operators $\mathcal{T}^{\mathrm{L}}_{s_i,\lambda}$:*

$$\begin{tikzcd} \mathbb{Z}[\underline{t}^{\pm1},\underline{x}^{\pm1},y] \arrow[r, "\mathscr{T}^{\mathrm{R}}_{i,\lambda}"] \arrow[d, "\kappa"'] & \mathbb{Z}[\underline{t}^{\pm1},\underline{x}^{\pm1},y] \arrow[d, "\kappa"] \\ K_{\mathbb{T}}(\mathrm{GL}_n/B)[y] \arrow[r, "\mathcal{T}^{\mathrm{R}}_{s_i,\lambda}"] & K_{\mathbb{T}}(\mathrm{GL}_n/B)[y], \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}[\underline{t}^{\pm1},\underline{x}^{\pm1},y] \arrow[r, "\mathscr{T}^{\mathrm{L}}_{i,\lambda}"] \arrow[d, "\kappa"'] & \mathbb{Z}[\underline{t}^{\pm1},\underline{x}^{\pm1},y] \arrow[d, "\kappa"] \\ K_{\mathbb{T}}(\mathrm{GL}_n/B)[y] \arrow[r, "\mathcal{T}^{\mathrm{L}}_{s_i,\lambda}"] & K_{\mathbb{T}}(\mathrm{GL}_n/B)[y]. \end{tikzcd}$$

Proof. We compare the equations (17) and (18) with definition 6.3. We use the fact that Kirwan map commutes with both Weyl group action i.e.

$$\kappa\circ s_i^x = s^{\mathrm{R}}_{\alpha_i}\circ\kappa\,,\qquad \kappa\circ s_i^t = s^{\mathrm{L}}_{\alpha_i}\circ\kappa$$

and formula (11), i.e.

$$\kappa(L_{i,t}) = \kappa\left(\frac{t_{i+1}}{t_i}\right) = \alpha_i^{-1}\,,\qquad \kappa(L_{i,x}) = \kappa\left(\frac{x_{i+1}}{x_i}\right) = \mathcal{L}(\alpha_i)\,.$$

□

Remark 11.12. We have four different types of operators and we denote them using different fonts:

(i) $\mathcal{T}^{\mathrm{R}}_{s,a}$ and $\mathcal{T}^{\mathrm{L}}_{s,a}$ act on the K-theory $K_{\mathbb{T}}(G/B)[y]$ (definitions 6.3.1 and 6.3.3). The parameter a is a rational number. Real parameter makes sense as well.
(ii) $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathcal{T}^{\mathrm{L}}_{s,\lambda}$ act on the K-theory $K_{\mathbb{T}}(G/B)[y]$ for a semisimple group G, (definitions 6.3.2 and 6.3.4). The group GL_n is allowed as well. The parameter λ is a weight of the fixed maximal torus $\mathbb{T}\subset B$ and s is a simple reflection.
(iii) $\mathfrak{T}^{\mathrm{R}}_{i,a}$ and $\mathfrak{T}^{\mathrm{L}}_{i,a}$ act on Laurent polynomials or rational functions, $a\in\mathbb{Q}$ is a parameter (section 11.1).
(iv) $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{i,\lambda}$ act on Laurent polynomials or rational functions, $\lambda\in\mathfrak{t}^*=\mathbb{R}^n$ is a fractional character of the standard maximal torus in GL_n (definitions 11.5 and 11.8).

Example 11.13. Let $n=2$, and consider the composition of κ with the restriction to the fixed points

$$\overline{\kappa}:\mathbb{Z}[t_1^{\pm1},t_2^{\pm1},x_1^{\pm1},x_2^{\pm1},y]\to K_{\mathbb{T}}((\mathrm{GL}_2/B)^{\mathbb{T}}[y]) = \bigoplus_{\sigma\in S_2}\mathbb{Z}[t_1^{\pm1},t_2^{\pm1},y]\,,$$

$$f(t_1,t_2,x_1,x_2,y)\mapsto \left(\,f(t_1,t_2,t_1,t_2,y)\,,\ f(t_1,t_2,t_2,t_1,y)\,\right).$$

The polynomial $f_0 = 1-\frac{x_1}{t_2}$ is sent to $(1-\frac{t_1}{t_2},0)$, which is the restriction of the class of the point X_{id}. Applying $\mathscr{T}^{\mathrm{R}}_{1,\lambda}$ we obtain

$$\frac{(1+y)\left(\frac{x_1}{x_2}\right)^{1-\lceil\lambda_2-\lambda_1\rceil}}{1-\frac{x_1}{x_2}}\Big(1-\frac{x_1}{t_2}\Big) + \frac{1+y\frac{x_1}{x_2}}{1-\frac{x_2}{x_1}}\Big(1-\frac{x_2}{t_2}\Big)\,.$$

After restriction to the fixed point set we arrive at the formula for the twisted motivic Chern class of $\mathbb{C} \subset \mathbb{P}^1$

$$\left((1+y)\left(\tfrac{t_1}{t_2}\right)^{1-\lceil \lambda_2-\lambda_1 \rceil},\, 1+y\tfrac{t_2}{t_1} \right).$$

By theorem 11.11 and the inductions given in theorem 6.9

$$\kappa(\mathscr{T}^{\mathrm{R}}_{1,\lambda}(f_0)) = \mathcal{T}^{\mathrm{R}}_{s_1,\lambda}(\kappa(f_0)) = \mathrm{mC}^{\mathbb{T}}(s_1, s_1\lambda).$$

On the other hand applying $\mathscr{T}^{\mathrm{L}}_{1,s_1\lambda}$ we obtain

$$\frac{(1+y)\left(\frac{t_1}{t_2}\right)^{1-\lceil \lambda_2-\lambda_1 \rceil}}{1-\frac{t_1}{t_2}}\left(1-\tfrac{x_1}{t_2}\right) + \frac{1+y\frac{t_2}{t_1}}{1-\frac{t_2}{t_1}}\left(1-\tfrac{x_1}{t_1}\right).$$

After restriction to the fixed point and simplifying we obtain the twisted motivic Chern class of $\mathbb{C} \subset \mathbb{P}^1$

$$\left((1+y)\left(\tfrac{t_1}{t_2}\right)^{1-\lceil \lambda_2-\lambda_1 \rceil},\, 1+y\tfrac{t_2}{t_1} \right),$$

which by theorem 8.1 is equal to

$$\kappa(\mathscr{T}^{\mathrm{L}}_{1,s_1\lambda}(f_0)) = \mathcal{T}^{\mathrm{L}}_{s_1,s_1\lambda}(\kappa(f_0)) = \mathrm{mC}^{\mathbb{T}}(s_1, \lambda).$$

Although we have equality

$$\kappa(\mathscr{T}^{\mathrm{R}}_{1,\lambda}(f_0)) = \kappa(\mathscr{T}^{\mathrm{L}}_{1,s_1\lambda}(f_0)),$$

but one can check that

$$\mathscr{T}^{\mathrm{R}}_{1,\lambda}(f_0) \neq \mathscr{T}^{\mathrm{L}}_{1,s_1\lambda}(f_0).$$

This discrepancy can be repaired by a different choice of the initial polynomial f_0. A better choice for beginning of the induction is proposed in the next section.

Alternative proof of lemma 11.3. We know that the braid relations hold in $K_{\mathbb{T}}(\mathrm{GL}_n/B)$, see proposition 7.5 and example 7.6. Take $n = 3$ and apply the Kirwan map (10) composed with restriction to the fixed point corresponding to $[\mathrm{id}] \in \mathrm{GL}_3/B$

$$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}, y] \overset{z_i \mapsto x_i}{\hookrightarrow} K_{\mathbb{T}\times\mathbb{T}}(\mathrm{End}(\mathbb{C}^3))[y] \xrightarrow{\kappa}$$
$$\xrightarrow{\kappa} K_{\mathbb{T}}(\mathrm{GL}_3/B)[y] \xrightarrow{\iota^*} K_{\mathbb{T}}(pt)[y] = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, y].$$

The composition sends x_i variables to the equivariant variables t_i, therefore the map

$$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}, y] \to K_{\mathbb{T}}(\mathrm{GL}_3/B)[y]$$

is injective. The operators $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ are lifts of the twisted Demazure-Lusztig operators. Hence the braid relations hold in $\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}, y]$. □

12. Comparison with the matrix Schubert varieties

We will show that the purely algebraic definition of the operator $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ and $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ have geometric meaning. They describe how the twisted motivic Chern classes change when we pass from $\mathcal{X}_w$ to $\mathcal{X}_{ws}$ or to $\mathcal{X}_{sw}$. We start with untwisted classes and reformulate the results of [RW22b].

12.1. **Corollaries from the work on square-zero matrices.** A relation between motivic Chern classes of invariant subvarieties of a G-manifold and motivic Chern classes of quotients is studied in [FRW20, section 8]. In our situation we consider GL_n/B as a quotient $\mathrm{End}(\mathbb{C}^n)//B$ (quotient by the right action). Let

$$\mathscr{B} = \mathrm{mC}^{\mathbb{T}}(B \subset \overline{B})$$

with $\mathbb{T}$ acting on B by conjugation, expressed in x-variables. Here $\overline{B}$ is the closure of B in $\mathrm{End}(\mathbb{C}^n)$. The closure $\overline{B}$ is the vector space consisting of upper-triangular matrices. In terms of the coordinate torus characters

$$\mathscr{B} = (1+y)^n \prod_{1\le i<j\le n} (1 + y\tfrac{x_j}{x_i}) \in K_{\mathbb{T}}(\overline{B})[y] = \mathbb{Z}[\underline{x}, y]\,. \tag{19}$$

It follows that

Proposition 12.1. [FRW20, theorems 8.9 and 8.12] *For a subvariety $Y \subset \mathrm{End}(\mathbb{C}^n)$ which is invariant with respect to the right B action and left $\mathbb{T}$ action*

$$\mathrm{mC}^{\mathbb{T}}((Y \cap \mathrm{GL}_n)/B \subset \mathrm{GL}_n/B) = \kappa(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(Y \subset \mathrm{End}(\mathbb{C}^n)))\,,$$

where κ is the Kirwan map (10).

The formulation in [FRW20] is given in terms of Segre classes. That is why we divide the motivic Chern class by $\mathscr{B}$. We will give an alternative proof, which is valid for twisted motivic Chern classes as well, see proposition 12.5.

We can identify the space $\mathrm{End}(\mathbb{C}^n)$ with the subspace of square-zero upper-triangular matrices of the dimension $2n \times 2n$ of the block form $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$. The results of [RW22b] for B-orbits of square-zero matrices apply. In particular the motivic Chern classes (for the trivial slope) can be computed inductively. We translate that result to our situation. The Euler class of $\mathrm{End}(\mathbb{C}^n)$ is equal to

$$\mathscr{E} = \prod_{i=1}^{n}\prod_{j=1}^{n}(1 - \tfrac{x_j}{t_i})\,.$$

The localized motivic Chern class of a $\mathbb{T}^2$–invariant subvariety $Y \subset \mathrm{End}(\mathbb{C}^n)$ is defined as the quotient

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(Y \subset \mathrm{End}(\mathbb{C}^n)) = \mathscr{E}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(Y \subset \mathrm{End}(\mathbb{C}^n))\,.$$

By [RW22b, corollary 5.4] for $i \in \{1, 2, \dots, n-1\}$

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{O}_{s_i w} \subset \mathrm{End}(\mathbb{C}^n)) = A^t_i(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{O}_w \subset \mathrm{End}(\mathbb{C}^n)))\,, \tag{20}$$

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{O}_{w s_i} \subset \mathrm{End}(\mathbb{C}^n)) = A^x_i(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{O}_w \subset \mathrm{End}(\mathbb{C}^n)))\,. \tag{21}$$

The operations A^t_i and A^x_i are defined in a more general context of square zero matrices by a uniform formula[2]. In our situation

$$A^t_i(f) = \frac{(1+y)\frac{t_i}{t_{i+1}}}{1 - \frac{t_i}{t_{i+1}}} f + \frac{1 + y\frac{t_{i+1}}{t_i}}{1 - \frac{t_{i+1}}{t_i}} s^t_i f\,,$$

$$A^x_i(f) = \frac{(1+y)\frac{x_i}{x_{i+1}}}{1 - \frac{x_i}{x_{i+1}}} f + \frac{1 + y\frac{x_{i+1}}{x_i}}{1 - \frac{x_{i+1}}{x_i}} s^x_i f$$

[2]Setting $t_{n+i} := x_i$ we do not have to distinguish between t–variables and x–variables. Instead, in [RW22b], the superscript is used to indicate that we deal with K-theory.

for $i \in \{1, 2, \dots, n-1\}$. Above, the reflections s_i^t act on t–variables and s_i^x on x–variables.

Since $\mathscr{E}$ is symmetric with respect to both sets of variables $\{t_i\}_{i=1,2,\dots,n}$ and $\{x_i\}_{i=1,2,\dots,n}$ separately and $\mathrm{mC}^{\mathbb{T}^2}(-) = \mathscr{E} \cdot \mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(-)$, the localized classes in the formulas (20–21) can be replaced by the usual motivic Chern classes. The operation A_i^t coincides with $\mathscr{T}^{\mathrm{L}}_{i,0}$. It commutes with multiplication by $\mathscr{B}$ since $\mathscr{B}$ is expressed only by x-variables. On the other hand A_i^x differs from $\mathscr{T}^{\mathrm{R}}_{i,0}$ by the conjugation with $1 + y\frac{x_{i+1}}{x_i}$. For any f we have

$$\mathscr{T}^{\mathrm{R}}_{i,0}(\mathscr{B}^{-1} f) = \mathscr{B}^{-1} A_i^x(f) \,.$$

Therefore we obtain

Proposition 12.2. *Let $1 \le i \le n-1$. If $l(s_i w) > l(w)$, then*

$$\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{O}_{s_i w} \subset \mathrm{End}(\mathbb{C}^n)) = \mathscr{T}^{\mathrm{L}}_{i,0}\left(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{O}_w \subset \mathrm{End}(\mathbb{C}^n))\right) .$$

If $l(ws_i) > l(w)$, then

$$\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{O}_{ws_i} \subset \mathrm{End}(\mathbb{C}^n)) = \mathscr{T}^{\mathrm{R}}_{i,0}\left(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{O}_w \subset \mathrm{End}(\mathbb{C}^n))\right) .$$

Setting $f_w = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{O}_{ws_i} \subset \mathrm{End}(\mathbb{C}^n))$ we obtain the following corollary:

Corollary 12.3. *There exist rational functions f_w for $w \in S_n$, such that*

(1)
$$f_{\mathrm{id}} = \mathscr{B}^{-1}\prod_{i=1}^{n}\left((1+y)\tfrac{x_i}{t_i}\right)\prod_{1\le i<j\le n}\left(1+y\tfrac{x_j}{t_i}\right)\prod_{1\le j<i\le n}\left(1-\tfrac{x_j}{t_i}\right)$$

(2) if $l(ws_i) > l(w)$, then
$$f_{ws_i} = \mathscr{T}^{\mathrm{R}}_{i,0}(f_w)\,,$$

(3) if $l(s_i w) > l(w)$, then
$$f_{s_i w} = \mathscr{T}^{\mathrm{L}}_{i,0}(f_w)\,.$$

(4) for each permutation $\sigma \in S^n$ after the substitution $x_i = t_{\sigma(i)}$ the function f_w specializes to $\mathrm{mC}^{\mathbb{T}}(w,0)_{|\sigma}$, i.e.
$$\kappa(f_w) = \mathrm{mC}^{\mathbb{T}}(w,0)\,.$$

Remark 12.4. The function f_w differs from the weight function of [RTV19], which does not satisfy the property (2). The weight function is related to a different presentation of the flag variety: either as the quiver variety associated with A_n-quiver or as the quotient $\mathrm{Hom}(\mathbb{C}^{n-1}, \mathbb{C}^n)/B_{n-1}$. See the explanation in [RW22b, section 7].

12.2. **Twisted motivic Chern classes of matrix Schubert varieties.** We will describe the twisted motivic Chern classes of $\mathcal{X}_w$ with respect to a family of divisors $D_{w,\lambda}$ which we defined in section 10.3. Let us introduce notation for the images of the twisted motivic Chern classes in $K_{\mathbb{T}^2}(\mathrm{End}(\mathbb{C}^n))[y]$. Let

$$\mathrm{mC}^{\mathbb{T}^2}(w,\lambda) = \widehat{\iota}_{w*}\,\mathrm{mC}^{\mathbb{T}^2}(\mathcal{X}_w, \partial\mathcal{X}_w; D_{w,\lambda}) \in \mathbb{Z}[\underline{x}^{\pm}, \underline{t}^{\pm}, y]\,,$$

where $\widehat{\iota}_w : \mathcal{X}_w \to \mathrm{End}(\mathbb{C}^n)$ is the embedding.

First we generalize proposition 12.1 to the case of twisted classes of Schubert varieties:

Proposition 12.5. *With the notation as above and the notation of definition* (5.5)

$$\kappa\left(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)\right) = \mathrm{mC}^{\mathbb{T}}(w,\lambda)\,.$$

Proof. We compare the restrictions of both classes at the fixed points. First, we concentrate on the point [id] $\in G/B$. Let N_- be the group consisting of lower-triangular unipotent matrices. The composition of the inclusion and the quotient map

$$N_- \hookrightarrow \mathrm{GL}_n \to \mathrm{GL}_n/B$$

is an isomorphism to the image which is a neighbourhood of [id]. Over that neighbourhood the bundle $\mathrm{GL}_n \to \mathrm{GL}_n/B$ is trivial. The trivialization map

$$\begin{array}{rccc} & (n,b) & \mapsto & nb \\ \widehat{\varphi}_{\mathrm{id}}: & N_- \times B & \longrightarrow & \mathrm{GL}_n \\ & \downarrow & & \downarrow {\scriptstyle pr} \\ \varphi_{\mathrm{id}}: & N_- & \longrightarrow & \mathrm{GL}_n/B \end{array}$$

is equivariant with respect to the diagonal torus $\mathbb{T} \subset \mathbb{T}^2$ acting by conjugation on N_-, B and GL_n. Thus $N_- \hookrightarrow \mathrm{GL}_n$ is a local $\mathbb{T}$-equivariant slice of the B-bundle $pr : \mathrm{GL}_n \to \mathrm{GL}_n/B$. The inverse image $\widehat{\varphi}_{\mathrm{id}}^{-1}(\mathcal{X}_w)$ is of the product form $\varphi_{\mathrm{id}}^{-1}(X_w) \times B$. Moreover the divisor $D_{w,\lambda}$ is the preimage of $\Delta_{w,\lambda}$, see formula (16). Therefore [KW22, proposition 4.7 (ii)] implies that

$$\big(\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)_{x_i:=t_i}\big)_{|N_- \cdot B} = pr^* \,\mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|N_-} \boxtimes \mathrm{mC}^{\mathbb{T}}(id_B)\,.$$

We have $B \simeq \mathbb{T} \times N_+$ and the conjugation action on the torus is trivial. We restrict the above class to id $\in \mathrm{GL}_n$ and obtain

$$\big(\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)_{x_i:=t_i}\big)_{|\mathrm{id}} = \mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|[\mathrm{id}]} \cdot (1+y)^n \cdot \prod_{1 \le i < j \le n} (1 + \tfrac{t_j}{t_i})\,.$$

The second and the third factor give $\mathscr{B}$ with x_i substituted by t_i. Thus

$$\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)_{x_i:=t_i} = \kappa(\mathscr{B})_{|[\mathrm{id}]} \cdot \mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|[\mathrm{id}]}\,.$$

For an arbitrary fixed point $\sigma \in (G/B)^{\mathbb{T}}$ we consider the translated slice σN_-. It is invariant with respect to the subtorus $\mathbb{T}^\sigma \subset \mathbb{T}^2$ consisting of pairs $(t, \sigma^{-1} t \sigma)$. Indeed for $\sigma g \in \sigma N_-$

$$t(\sigma g)(\sigma^{-1} t \sigma)^{-1} = \sigma(\sigma^{-1} t \sigma) g (\sigma^{-1} t \sigma)^{-1} \in \sigma N_-\,.$$

In particular $g = \sigma$ is a fixed point for the action of $\mathbb{T}^\sigma$. We apply the previous computation of $\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)$ restricted to the torus $\mathbb{T}^\sigma$ at the point σ. The resulting substitution is $x_i \mapsto t_{\sigma(i)}$. Therefore we obtain

$$\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)_{x_i:=t_{\sigma(i)}} = \kappa(\mathscr{B})_{|\sigma} \cdot \mathrm{mC}^{\mathbb{T}}(w,\lambda)_{|\sigma}\,.$$

□

The main result relating the twisted motivic Chern classes with the algebra described in section 11.2 is the following:

Theorem 12.6. *The left and right recursions hold*

$$\mathscr{T}^{\mathrm{L}}_{i,w\lambda}\big(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)\big) = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(s_i w,\lambda) \quad \textit{if} \;\; l(s_i w) > l(w)\,,$$

$$\mathscr{T}^{\mathrm{R}}_{i,\lambda}\big(\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)\big) = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(w s_i, s_i\lambda) \quad \textit{if} \;\; l(w s_i) > l(w)\,,$$

where $w(\lambda_1, \lambda_2, \ldots, \lambda_n) = (\lambda_{w(1)}, \lambda_{w(2)}, \ldots, \lambda_{w(n)})$.

The theorem is proven in section 14.

Remark 12.7. A good starting point for the induction of twisted classes is $\mathcal{X}_{\mathrm{id}} = \overline{B} \subset \mathrm{End}(\mathbb{C}^n)$. The rational function

$$\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathrm{id},\lambda) = \mathscr{B}^{-1}\prod_{i=1}^{n}\left((1+y)\left(\tfrac{x_i}{t_i}\right)^{1-\lceil\lambda_i\rceil}\right)\prod_{1\le i<j\le n}\left(1+y\tfrac{x_j}{t_i}\right)\prod_{1\le j<i\le n}\left(1-\tfrac{x_j}{t_i}\right)$$

has the property that equally well one can apply left or right Demazure-Lusztig operators, obtaining distinguished representatives of higher dimensional cells. That was not so for the *easy* representative of $\mathrm{mC}^{\mathbb{T}}(\mathrm{id},\lambda)$ given in example 11.13. The price to pay is that $\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(\mathrm{id},\lambda)$ is not a Laurent polynomial.

Remark 12.8. The class $\mathrm{mC}^{\mathbb{T}^2}(\mathrm{id},\lambda)$ which is the start of induction depends on λ. On the other hand the class $\mathrm{mC}^{\mathbb{T}}(\mathrm{id},\lambda) = [X_{\mathrm{id}}] \in K^{\mathbb{T}}(G/B)[y]$ does not depend on λ.

13. Resolution of matrix Schubert varieties

Before proceeding with the proof let us describe the geometry of resolutions of $\mathcal{X}_w$. We will extend the construction of the Bott-Samelson resolution in our case, basically applying the more general method of [BP19].

13.1. **The left resolution.** For a reduced word $\underline{w} = s_{i_1}s_{i_2}\dots s_{i_l}$ let

$$\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = P_{i_1}\times_B P_{i_2}\times_B\cdots\times_B P_{i_l}\times_B \overline{B}\,,$$

where P_i is the minimal parabolic subgroup containing B and s_i. The resolution map

$$\mu_{\underline{w}} : \mathcal{Z}^{\mathrm{L}}_{\underline{w}} \longrightarrow \mathcal{X}_w\,,$$

is defined as the product of coordinates

$$\mu_{\underline{w}}([p_1,p_2,\dots,p_l,b]) = p_1\,p_2\dots p_l\,b\,,\qquad p_j\in P_{i_j}\,,\quad b\in\overline{B}\,.$$

The Borel group acts on $\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ from the left and from the right. Let

$$q_{\underline{w}} : \mathcal{Z}^{\mathrm{L}}_{\underline{w}} \longrightarrow Z_{\underline{w}} = P_{i_1}\times_B P_{i_2}\times_B\cdots\times_B P_{i_l}\times_B \{*\}$$

be the map induced by the contraction of $\overline{B}$ to a point. The map $q_{\underline{w}} : \mathcal{Z}^{\mathrm{L}}_{\underline{w}} \to Z_{\underline{w}}$ is a vector bundle over $Z_{\underline{w}}$. Denote by $\iota_{\underline{w}}$ its zero section. The $B\times B$-orbit of $[s_{i_1},s_{i_2},\dots,s_{i_l},\mathrm{id}] \in \mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ is open in $\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ and maps isomorphically to $\mathcal{O}_w$. The boundary

$$\partial\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = \mu_{\underline{w}}^{-1}(\partial X_w)$$

consists of the inverse image $q_{\underline{w}}^{-1}(\partial Z_{\underline{w}})$ and the *B-boundary*

$$\partial_B\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = P_{i_1}\times_B P_{i_2}\times_B\cdots\times_B P_{i_l}\times_B \partial B\,,\qquad \partial B = \overline{B}\setminus B\,.$$

The divisor $\partial_B\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ is the sum of n irreducible components. For $j\in\{1,2,\dots,n\}$ set

$$\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = P_{i_1}\times_B P_{i_2}\times_B\cdots\times_B P_{i_l}\times_B (\{a_{jj}=0\}\cap\overline{B})\,.$$

Moreover $q_{\underline{w}}^{-1}(\partial Z_{\underline{w}})$ decomposes into irreducible components $\partial_j\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = q_{\underline{w}}^{-1}(\partial_j Z_{\underline{w}})$ corresponding to omitting the j-th letter in the word $\underline{w}$. We obtain decomposition of the boundary

$$\partial\mathcal{Z}^{\mathrm{L}}_{\underline{w}} = \partial_B\mathcal{Z}^{\mathrm{L}}_{\underline{w}} \cup q_{\underline{w}}^{-1}(\partial Z_{\underline{w}}) = \bigcup_{j=1}^{n}\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{w}} \cup \bigcup_{j=1}^{l(w)}\partial_j\mathcal{Z}^{\mathrm{L}}_{\underline{w}}\,.$$

Note that ∂B is a SNC divisor in $\overline{B}$. It follows that $\partial\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ is a SNC divisor.

Each fixed point $p_\varepsilon \in (\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^{\mathbb{T}}$ belongs to the zero section $\iota_{\underline{w}}(Z_{\underline{w}})$ and it corresponds to a binary sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l)$

$$p_\varepsilon = [s_{i_1}^{\varepsilon_1}, s_{i_2}^{\varepsilon_2}, \dots, s_{i_l}^{\varepsilon_l}, 0]$$

(the last 0 denotes the zero matrix belonging to $\overline{B}$). Further we will denote the fixed points simply by ε, dropping p from the notation. We will compute the multiplicity $\mu^*_{\underline{w}}(M_{w,k})$ along the component $\partial_j \mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ (see (12) for the definition of $M_{w,k}$). Let us test the multiplicity on a curve, which is transverse at the point

$$[s_{i_1}, \dots, s_{i_{j-1}}, \mathrm{id}, s_{i_{j+1}}, \dots, s_{i_l}, \mathrm{id}]\,.$$

Let $E_i(u)$ be the elementary matrix of the form

$$\begin{array}{r} \\ \\ i \\ i+1 \\ \\ \\ \end{array}\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & 0 & & \\ & & u & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}.$$

The testing curve is defined as

$$u \mapsto [s_{i_1}, \dots, s_{i_{j-1}}, E_{i_j}(u), s_{i_{j+1}}, \dots, s_{i_l}, \mathrm{id}]\,.$$

The multiplicity is equal to the order of vanishing of the function

$$u \mapsto m_k(w^{-1} s_{i_1} \dots s_{i_{l-1}} E_{i_j}(u) s_{i_{l+1}} \dots s_{i_l}) = m_k(w_{>j}^{-1}\, s_{i_j}\, E_{i_j}(u)\, w_{>j})\,, \tag{22}$$

where $w_{>j} = s_{i_{j+1}} s_{i_{j+2}} \dots s_{i_l}$. The resulting formula is fairly explicit. The matrix $w_{>j}^{-1}\, s_{i_j}\, E_{i_j}(u)\, w_{>j}$ differs from the identity matrix only at four entries having indices in the set

$$\{w_{>j}^{-1}(i_j)\ ,\ w_{>j}^{-1}(i_j+1)\}\,.$$

The key 2×2 submatrix is equal to

$$\begin{pmatrix} u & 1 \\ 1 & 0 \end{pmatrix}.$$

It follows, that the coefficient of $\partial_j \mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ in $\mu^*_{\underline{w}}(M_{w,k})$ is 1 if

$$w_{>j}^{-1}(i_j) \le k < w_{>j}^{-1}(i_j+1)$$

and otherwise it is equal to 0. Note that since the word $\underline{w}$ is reduced,

$$w_{>j}^{-1}(i_j) < w_{>j}^{-1}(i_j+1)\,.$$

The multiplicity of $\mu^*_{\underline{w}}(M_{w,k})$ at $\partial_{B,j} \mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ is equal to 1 if $k \ge j$ and otherwise it is equal to 0. By the above discussion we find the multiplicities of the pull back of the divisor $D_{w,\lambda}$ (see the formula (15) for the definition of $D_{w,\lambda}$).

Proposition 13.1. *We have*

$$\mu^*_{\underline{w}}(D_{w,\lambda}) = \sum_{j=1}^{n} \lambda_j \cdot \partial_{B,j} \mathcal{Z}^{\mathrm{L}}_{\underline{w}} + \sum_{j=1}^{l(w)} \langle w_{>j}\lambda, \alpha^\vee_{i_j} \rangle \cdot \partial_j \mathcal{Z}^{\mathrm{L}}_{\underline{w}}\,.$$

Note that $\langle w\lambda, \alpha_i^\vee \rangle = \lambda_{w^{-1}(i)} - \lambda_{w^{-1}(i+1)}\,.$

Corollary 13.2. *Let s be a simple reflection corresponding to the root α_i. Suppose that $w \in W$ is a Weyl group element such that $l(sw) > l(w)$. Let ε be a binary sequence of length $l(w)$ and $\widehat{\varepsilon} = (\delta, \varepsilon)$ for $\delta \in \{0, 1\}$.*

(1) For $\delta = 0$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\lceil\mu^*_{\underline{sw}}(D_{sw,\lambda})\rceil)_{|\widehat{\varepsilon}} = \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\lceil\mu^*_{\underline{w}}(D_{w,\lambda})\rceil)_{|\varepsilon} \cdot \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\lceil\langle w\lambda, \alpha_i^\vee\rangle\rceil\partial_1\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}}$$

(2) For $\delta = 1$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\lceil\mu^*_{\underline{sw}}(D_{sw,\lambda})\rceil)_{|\widehat{\varepsilon}} = s^t(\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\lceil\mu^*_{\underline{w}}(D_{w,\lambda})\rceil)_{|\varepsilon})$$

Where s^t denotes the action on t–variables of $K^{\mathbb{T}^2}(pt) = \mathbb{Z}[\underline{t}^{\pm1}, \underline{x}^{\pm1}]$.

Proof. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l) \in (\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^{\mathbb{T}}$ be a fixed point. The character in the normal direction to $\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$ at the point ε, is equal to

$$t_{w^\varepsilon(j)}/x_j \,,$$

where

$$w^\varepsilon = s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} \ldots s_{i_l}^{\varepsilon_l} \,.$$

The remaining characters are the same as for the Bott-Samelson resolution Z_w. Therefore for $j \in \{1, \ldots, n\}$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} & \text{when } \delta = 0\,, \\ s^t(\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\partial_{B,j}\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon}) & \text{when } \delta = 1\,. \end{cases}$$

Moreover for $j \in \{1, \ldots, l(w)\}$, as in [RW20, section 3.2]

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\partial_{j+1}\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\partial_j\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} & \text{when } \delta = 0\,, \\ s^t(\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\partial_j\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon}) & \text{when } \delta = 1\,. \end{cases}$$

We apply proposition 13.1 to deduce the recursive formula of corollary 13.2. □

Example 13.3. Let $n = 4$, $w = s_2 s_3$

$$w(1) = 1, \quad w(2) = 3, \quad w(3) = 4, \quad w(4) = 2\,.$$

Let us analyse the divisors $M_{w,k}$

$$P_2 \times_B P_3 \times_B \overline{B} \to \mathrm{End}(\mathbb{C}^4)$$

in the neighbourhood of the empty word

$$s_3 \cdot s_2 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & v & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & ua_{22} & ua_{23} + a_{33} & ua_{24} + a_{34} \\ 0 & 0 & va_{33} & va_{34} + a_{44} \\ 0 & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

The consecutive minors are equal to

$$a_{11}, \quad u\, a_{11}a_{22}, \quad uv\, a_{11}a_{22}a_{33}, \quad a_{11}a_{22}a_{33}a_{44}$$

Therefore in the neighbourhood of the point corresponding to the empty word the divisor $D_{w,\lambda}$ is equal to

$$\mathrm{div}\left(a_{11}^{\lambda_1-\lambda_2}(u\, a_{11}a_{22})^{\lambda_2-\lambda_3}(uv\, a_{11}a_{22}a_{33})^{\lambda_3-\lambda_4}(a_{11}a_{22}a_{33}a_{44})^{\lambda_4}\right) =$$

$$= (\lambda_2 - \lambda_4)\,\mathrm{div}(u) + (\lambda_3 - \lambda_4)\,\mathrm{div}(v) + \sum_{i=1}^{4} \lambda_i\,\mathrm{div}(a_{ii})\,.$$

13.2. **The right resolution.** Multiplying from the right the upper-triangular matrices by the matrices belonging to P_i we obtain a different resolution

$$\nu_{\underline{w}} : \mathcal{Z}^{\mathrm{R}}_{\underline{w}} \longrightarrow \mathcal{X}_w \,,$$

$$\mathcal{Z}^{\mathrm{R}}_{\underline{w}} = \overline{B} \times_B P_{i_1} \times_B P_{i_1} \times_B \cdots \times_B P_{i_l} \,,$$

$$\nu_{\underline{w}}([b, p_1, p_2, \dots, p_l]) = b\, p_1\, p_2 \dots p_l \,, \qquad b \in \overline{B} \,, \quad p_j \in P_{i_j} .$$

As before the boundary $\partial \mathcal{Z}^{\mathrm{R}}_{\underline{w}} = \nu^{-1}_{\underline{w}}(\partial \mathcal{X}_w)$ is a SNC divisor. It is a sum of irreducible components

$$\partial \mathcal{Z}^{\mathrm{R}}_{\underline{w}} = \bigcup_{j=1}^{n} \partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}} \cup \bigcup_{j=1}^{l(w)} \partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{w}} \,.$$

The divisor $\partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{w}}$ is defined by the condition $p_j \in B$, the B-boundary divisor $\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}}$ is defined by $a_{jj} = 0$. The argument as before leads us to the following conclusion

Proposition 13.4. *We have*

$$\nu^*_{\underline{w}}(D_{w,\lambda}) = \sum_{j=1}^{n} (w\lambda)_j \cdot \partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}} + \sum_{j=1}^{l(w)} \langle w_{>j} \lambda, \alpha^\vee_{i_j} \rangle \cdot \partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{w}} \,.$$

Note that for $\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}}$ the indices of λ are permuted, since we compute the degree of $m_k(w^{-1} E_k(u) w)$, not $m_k(E_k(u))$ as in the previous case.

Corollary 13.5. *Let s be a simple reflection corresponding to the root α_i. Suppose $w \in W$ is a Weyl group element such that $l(ws) > l(w)$. Let ε be a binary sequence of length $l(w)$ and $\widehat{\varepsilon} = (\varepsilon, \delta)$ for $\delta \in \{0, 1\}$.*

(1) For $\delta = 0$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\lceil \mu^*_{\underline{ws}}(D_{ws,s\lambda}) \rceil)_{|\widehat{\varepsilon}} = \mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\lceil \mu^*_{\underline{w}}(D_{w,\lambda}) \rceil)_{|\varepsilon} \cdot \mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\lceil \langle s\lambda, \alpha^\vee_i \rangle \rceil \partial_{l(ws)} \mathcal{Z}^{\mathrm{R}}_{\underline{ws}})_{|\widehat{\varepsilon}}$$

(2) For $\delta = 1$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\lceil \nu^*_{\underline{ws}}(D_{ws,s\lambda}) \rceil)_{|\widehat{\varepsilon}} = s^x(\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\lceil \nu^*_{\underline{w}}(D_{w,\lambda}) \rceil)_{|\varepsilon}) \,,$$

where s^x denotes the action on x–variables of $K^{\mathbb{T}^2}(pt) = \mathbb{Z}[\underline{t}^{\pm 1}, \underline{x}^{\pm 1}]$.

Proof. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l) \in (\mathcal{Z}^{\mathrm{R}}_{\underline{w}})^{\mathbb{T}}$ be a fixed point. The character in the normal direction to $\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}}$ at the point ε, is equal to

$$t_j / x_{(w^\varepsilon)^{-1}(j)} \,.$$

Therefore for $j \in \{1, \dots, n\}$ we have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{ws}})_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon} & \text{when } \delta = 0 \,, \\ s^x(\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\partial_{B,j} \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon}) & \text{when } \delta = 1 \,. \end{cases}$$

Moreover for $j \in \{1, \dots, l(w)\}$ (see e.g. [RW20, section 3.2])

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{ws}})_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon} & \text{when } \delta = 0 \,, \\ s^x(\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\partial_j \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon}) & \text{when } \delta = 1 \,. \end{cases}$$

The claim follows from corollary 13.4. □

Example 13.6. For the dual resolution

$$\overline{B} \times_B P_2 \times_B P_3 \to \mathrm{End}(\mathbb{C}^4)$$

we have

$$s_3 \cdot s_2 \cdot \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & v & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11} & a_{12}+ua_{13} & a_{13}+va_{14} & a_{14} \\ 0 & ua_{33} & a_{33}+va_{34} & a_{34} \\ 0 & 0 & va_{44} & a_{44} \\ 0 & a_{22}+ua_{23} & a_{23}+va_{24} & a_{24} \end{pmatrix}$$

the determinants are equal to

$$a_{11}, \quad u\, a_{11}a_{33}, \quad uv\, a_{11}a_{33}a_{44}, \quad a_{11}a_{22}a_{33}a_{44}\,.$$

Therefore in the neighbourhood of the point corresponding to the empty word the divisor $D_{w,\lambda}$ is equal to

$$\mathrm{div}\left(a_{11}^{\lambda_1-\lambda_2}(u\, a_{11}a_{33})^{\lambda_2-\lambda_3}(uv\, a_{11}a_{33}a_{44})^{\lambda_3-\lambda_4}(a_{11}a_{22}a_{33}a_{44})^{\lambda_4}\right) =$$

$$= (\lambda_2-\lambda_4)\,\mathrm{div}(u) + (\lambda_3-\lambda_4)\,\mathrm{div}(v) + \lambda_1\,\mathrm{div}(a_1) + \lambda_4\,\mathrm{div}(a_2) + \lambda_2\,\mathrm{div}(a_3) + \lambda_3\,\mathrm{div}(a_4).$$

14. Proof of theorem 12.6

Let us recall notation

$$L_{i,x} = \frac{x_{i+1}}{x_i}\,, \qquad L_{i,t} = \frac{t_{i+1}}{t_i}\,.$$

14.1. **Left induction.** For an arbitrary element $w \in W$ with a reduced word decomposition $\underline{w}$ we define

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}}, \lambda) := \frac{\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{w}}) \cdot \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\lceil \mu^*_{\underline{w}}(D_{\sigma,\lambda})\rceil)}{eu(T\mathcal{Z}^{\mathrm{L}}_{\underline{w}})}$$

Let $s = s_i$ be a simple reflection corresponding to the root α_i. Suppose $w \in W$ is a Weyl group element such that $l(sw) > l(w)$. Fix a reduced word decomposition $\underline{w}$ of w and the resulting decomposition $\underline{sw}$ of sw.

Lemma 14.1. *Let ε be a binary sequence corresponding to a fixed point in $\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$. Let $\widehat{\varepsilon}$ be a binary sequence of the form $\widehat{\varepsilon} = (\delta, \varepsilon)$, where $\delta \in \{0,1\}$. We have following equalities in the localized K-theory of a point $S^{-1}K^{\mathbb{T}^2}(pt)[y]$.*

(1)

$$eu(T\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}} = \begin{cases} eu(T\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} \cdot (1 - L_{i,t}^{-1}) & \textit{when } \delta = 0\,, \\ s^t(eu(T\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon}) \cdot (1 - L_{i,t}) & \textit{when } \delta = 1\,. \end{cases}$$

(2)

$$O_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\lceil \mu^*_{\underline{sw}}(D_{sw,\lambda})\rceil)_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\lceil \mu^*_{\underline{sw}}(D_{w,\lambda})\rceil)_{|\varepsilon} \cdot L_{i,t}^{\lceil\langle w\lambda, \alpha_i^\vee\rangle\rceil} & \textit{when } \delta = 0 \\ s^t(\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{w}}}(\lceil \mu^*_{\underline{sw}}(D_{w,\lambda})\rceil)_{|\varepsilon}) & \textit{when } \delta = 1 \end{cases}$$

(3)

$$\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} \cdot (1+y)L_{i,t}^{-1} & \textit{when } \delta = 0 \\ s^t(\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon}) \cdot (1 + yL_{i,t}) & \textit{when } \delta = 1 \end{cases}.$$

(4)

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}, s\lambda)_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}}, \lambda)_{|\varepsilon} \cdot (1+y) \cdot \frac{L_{i,t}^{\lceil \langle w\lambda, \alpha_i^\vee\rangle \rceil - 1}}{1-L_{i,t}^{-1}} & \textit{when } \delta = 0 \\ s^t(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}}, \lambda)_{|\varepsilon}) \cdot \frac{1+yL_{i,t}}{1-L_{i,t}} & \textit{when } \delta = 1 \end{cases}$$

Proof. The following map is well defined and it is $B\times B$-equivariant:

$$\varpi : \mathcal{Z}^{\mathrm{L}}_{\underline{ws}} \longrightarrow P_i/B \simeq \mathbb{P}^1\,, \qquad \varpi([p_0, p_1, p_2, \dots, p_l, x]) = [p_0]\,,$$

Let F_{id}, F_s be the fibers of ϖ over $[\mathrm{id}], [s] \in P_i/B$ respectively. The subvariety $F_{\mathrm{id}} \subset \mathcal{Z}^{\mathrm{L}}_{sw}$ coincides with divisor $\partial_1\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}$. It is equivariantly isomorphic to $\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$. The variety F_s is also isomorphic to $\mathcal{Z}^{\mathrm{L}}_{\underline{w}}$. This isomorphism is equivariant after the twist of the $\mathbb{T}^2$-torus action by s^t. The fixed point $\widehat{\varepsilon}$ lies either in F_{id} when $\delta = 0$ or in F_s when $\delta = 1$.

1) Let T_ϖ be the relative tangent bundle of ϖ. We have

$$eu(T\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}) = eu(T_\varpi)(1 - \varpi^*[T^*P_i/B])\,.$$

We observe that

$$(T_\varpi)_{\widehat{\varepsilon}} = \begin{cases} (T\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} & \text{for } \delta = 0\,, \\ s^t(T\mathcal{Z}^{\mathrm{L}}_{\underline{w}})_{|\varepsilon} & \text{for } \delta = 1 \end{cases} \quad \text{and} \quad \varpi^*(T^*P_i/B)_{|\varpi(\widehat{\varepsilon})} = \begin{cases} L_{i,t}^{-1} & \text{for } \delta = 0\,. \\ L_{i,t} & \text{for } \delta = 1\,. \end{cases}$$

The claim follows from the above formulas.

2) We have

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\partial_1\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}) = \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(F_{\mathrm{id}}) = \varpi^*\mathcal{O}_{P_i/B}([\mathrm{id}])\,.$$

Therefore

$$\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(\partial_1\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|\widehat{\varepsilon}} = \mathcal{O}_{P_i/B}([\mathrm{id}])_{|\varpi(\widehat{\varepsilon})} = \begin{cases} L_{i,t} & \text{for } \delta = 0\,. \\ 1 & \text{for } \delta = 1\,. \end{cases}$$

The claim follows from proposition 13.2.

3) Suppose that $\delta = 0$. Then $\widehat{\varepsilon} \in F_{\mathrm{id}}$. The divisor $\partial Z_{\underline{sw}}$ is SNC. Therefore (cf. [KW22, lemma 9.7])

$$\begin{aligned} \mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|F_{\mathrm{id}}} &= (1+y)\cdot \mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(-F_{\mathrm{id}})_{|F_{\mathrm{id}}} \cdot \mathrm{mC}^{\mathbb{T}}(F^\circ_{\mathrm{id}} \subset F_{\mathrm{id}}) \\ &= (1+y)L_{t,i}^{-1} \cdot \mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{w}})\,. \end{aligned}$$

Suppose that $x = 1$. The divisor $\partial Z_{\underline{sw}} + F_s$ is SNC, therefore

$$\begin{aligned} \mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{sw}})_{|F_s} &= (1 + y\mathcal{O}_{\mathcal{Z}^{\mathrm{L}}_{\underline{sw}}}(-F_s)_{|F_s}) \cdot \mathrm{mC}^{\mathbb{T}}(F^\circ_s \subset F_s) \\ &= (1+yL_{t,i}) \cdot s^t(\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{L}}_{\underline{w}}))\,. \end{aligned}$$

4) The last point follows from (2), (3) and definition 5.2. □

Lemma 14.2. *We have*

$$\mathscr{T}^{\mathrm{L}}_{i,w\lambda}\big(\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)\big) = \mathrm{mC}^{\mathbb{T}^2}(sw,\lambda)\,.$$

Proof. Since the Euler $\mathscr{E}$ class is symmetric, hence the multiplication by $\mathscr{E}$ comutes with $\mathscr{T}^{\mathrm{L}}_{i,w\lambda}$, it is enough to prove the equality

$$\mathscr{T}^{\mathrm{L}}_{i,w\lambda}\big(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda)\big) = \mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(sw,\lambda)\,.$$

By definition 6.3

$$\mathrm{mC}^{\mathbb{T}^2}(sw,\lambda)=\mu_{\underline{sw}*}\left(\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})^\circ\subset \mathcal{Z}^{\mathrm{L}}_{\underline{sw}})\cdot\mathcal{O}_{Z_{\underline{sw}}}(\lceil\mu^*_{\underline{ws}}D_{sw,\lambda}\rceil)\right)\,.$$

The LRR formula (theorem 2.3) implies that the local twisted motivic Chern class $\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(sw,\lambda)$ is equal to the sum

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(sw,\lambda)=\sum_{\widehat{\varepsilon}\in(\mathcal{Z}^{\mathrm{L}}_{\underline{sw}})^{\mathbb{T}}}\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{sw}},\lambda)_{|\widehat{\varepsilon}}.$$

The above sum splits as the sum of two subsums: those with $\widehat{\varepsilon}=(0,\varepsilon)$ and with $\widehat{\varepsilon}=(1,\varepsilon)$, cf. proof of lemma 6.17. The first summand is equal to

$$\sum_{\varepsilon\in(\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^{\mathbb{T}}}\frac{(1+y)L_{i,t}^{\lceil\langle w\lambda,\alpha_i^\vee\rangle\rceil-1}}{1-L_{i,t}^{-1}}\cdot\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}},\lambda)_{|\varepsilon}=\frac{(1+y)\cdot L_{i,t}^{\lceil\langle w\lambda,\alpha_i^\vee\rangle\rceil-1}}{1-L_{i,t}^{-1}}\cdot\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda)\,. \tag{23}$$

The second summand is equal to

$$\sum_{\varepsilon\in(\mathcal{Z}^{\mathrm{L}}_{\underline{w}})^{\mathbb{T}}}\frac{1+yL_{i,t}}{1-L_{i,t}}\cdot s^t(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}},\lambda))_{|\varepsilon}=\frac{1+yL_{i,t}}{1-L_{i,t}}\cdot s^t\,\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda) \tag{24}$$

The sum (23)+(24) is exactly the definition of $\mathscr{T}^{\mathrm{L}}_{i,\lambda}\big(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda)\big)$. □

Proof of 12.6 (1). The operator $\mathscr{T}^{\mathrm{L}}_{i,w\lambda}$ commutes with multiplication by the classes $\mathscr{E}$ and $\mathscr{B}$. Therefore the result follows from lemma 14.2. □

Remark 14.3. It is possible to write a closed formula for $\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{L}}_{\underline{w}},\lambda)$. The restriction at the point $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_l)\in\{0,1\}^l$ is equal to

$$(1+y)^n\cdot w^\varepsilon\left(\prod_{j=1}^n\left(\tfrac{x_j}{t_j}\right)^{1-\lceil\lambda_i\rceil}\cdot\prod_{1\le i<j\le n}\frac{1+y\,x_j/t_i}{1-x_j/t_i}\right)\cdot\prod_{j=1}^l w^\varepsilon_{<j}(\psi(\underline{w},\varepsilon,j))\,,$$

$$\psi(\underline{w},\varepsilon,j)=\begin{cases}\frac{(1+y)(t_{i_j}/t_{i_j+1})^{1-\lceil a\rceil}}{1-t_{i_j}/t_{i_j+1}} & \text{if } \varepsilon_j=0\,,\ a=\lambda_{w^{-1}_{>j}(i_j)}-\lambda_{w^{-1}_{>j}(i_j+1)},\\ \frac{1+y\,t_{i_j+1}/t_{i_j}}{1-t_{i_j+1}/t_{i_j}} & \text{if } \varepsilon_j=1,\end{cases}$$

w^ε and $w^\varepsilon_{<j}=s_{i_1}^{\varepsilon_1}s_{i_2}^{\varepsilon_2}\dots s_{i_{j-1}}^{\varepsilon_{j-1}}$ are acting on the indices of t–variables.

The proof of this formula is a direct application of lemma 14.1.

14.2. **Right induction.** For an arbitrary element $w\in W$ with a reduced word decomposition $\underline{w}$ we define

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{R}}_{\underline{w}},\lambda):=\frac{\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{R}}_{\underline{w}})^\circ\subset\mathcal{Z}^{\mathrm{R}}_{\underline{w}})\cdot\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\lceil\nu^*_{\underline{w}}(D_{w,\lambda})\rceil)}{eu(T\mathcal{Z}^{\mathrm{R}}_{\underline{w}})}\,.$$

Let $s=s_i$ be a simple reflection corresponding to the root α_i. Suppose $w\in W$ is a Weyl group element such that $l(ws)>l(w)$. Fix a reduced word decomposition $\underline{w}$ of w. It induces a reduced word decomposition $\underline{ws}$ of ws.

Lemma 14.4. *Let ε be a binary sequence corresponding to a fixed point in $\mathcal{Z}^{\mathrm{R}}_{\underline{w}}$. Let $\widehat{\varepsilon}$ be a binary sequence of the form $\widehat{\varepsilon}=(\varepsilon,\delta)$, where $\delta\in\{0,1\}$. We have following equalities in the localized K-theory of a point $S^{-1}K^{\mathbb{T}^2}(pt)[y]$.*

(1)

$$eu(T\mathcal{Z}^{\mathrm{R}}_{\underline{ws}})_{|\widehat{\varepsilon}}=\begin{cases}eu(T\mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon}\cdot(1-L_{i,x}^{-1}) & \textit{when } \delta=0\,,\\ s^t(eu(T\mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon})\cdot(1-L_{i,x}) & \textit{when } \delta=1\,.\end{cases}$$

(2)

$$O_{\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}}(\lceil \nu^*_{\underline{ws}}(D_{ws,s\lambda})\rceil)_{|\widehat{\varepsilon}} = \begin{cases} \mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\lceil \nu^*_{\underline{ws}}(D_{w,\lambda})\rceil)_{|\varepsilon} \cdot L_{i,x}^{\lceil -\langle \lambda, \alpha_i^\vee\rangle\rceil} & \text{when } \delta = 0 \\ s^x(\mathcal{O}_{\mathcal{Z}^{\mathrm{R}}_{\underline{w}}}(\lceil \nu^*_{\underline{ws}}(D_{w,\lambda})\rceil)_{|\varepsilon}) & \text{when } \delta = 1 \end{cases}$$

(3)

$$\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{R}}_{\underline{ws}})^\circ \subset \mathcal{Z}^{\mathrm{R}}_{\underline{ws}})_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{R}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon} \cdot (1+y)L_{i,x}^{-1} & \text{when } \delta = 0 \\ s^x(\mathrm{mC}^{\mathbb{T}}((\mathcal{Z}^{\mathrm{R}}_{\underline{w}})^\circ \subset \mathcal{Z}^{\mathrm{R}}_{\underline{w}})_{|\varepsilon}) \cdot (1+yL_{i,x}) & \text{when } \delta = 1 \end{cases}.$$

(4)

$$\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}, s\lambda)_{|\widehat{\varepsilon}} = \begin{cases} \mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{R}}_{\underline{w}}, \lambda)_{|\varepsilon} \cdot (1+y) \cdot \frac{L_{i,x}^{\lceil -\langle \lambda, \alpha_i^\vee\rangle\rceil - 1}}{1-L_{i,x}^{-1}} & \text{when } \delta = 0 \\ s^x(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(\mathcal{Z}^{\mathrm{R}}_{\underline{w}}, \lambda)_{|\varepsilon}) \cdot \frac{1+yL_{i,x}}{1-L_{i,x}} & \text{when } \delta = 1 \end{cases}$$

Proof. The proof differs from the proof of 14.1 by switching right with left actions. We employ the natural map to the left coset $B\backslash P_i \simeq \mathbb{P}^1$:

$$\varpi : \mathcal{Z}^{\mathrm{R}}_{\underline{ws}} \longrightarrow B\backslash P_i\,, \qquad \varpi([x, p_1, p_2, \dots, p_{l(w)+1}]) = [p_{l(w)+1}]\,.$$

The map ϖ is $B \times B$-equivariant. Let F_{id}, F_s be the fibers of ϖ over $[\mathrm{id}], [s] \in P_i/B$ respectively. The subvariety $F_{\mathrm{id}} \subset \mathcal{Z}^{\mathrm{R}}_{ws}$ coincides with the divisor $\partial_{l(ws)}\mathcal{Z}^{\mathrm{R}}_{\underline{ws}}$. It is equivariantly isomorphic to $\mathcal{Z}^{\mathrm{R}}_{\underline{w}}$. The variety F_s is also isomorphic to $\mathcal{Z}^{\mathrm{R}}_{\underline{w}}$. This isomorphism is equivariant after the twist of the $\mathbb{T}^2$-torus action by s^x. The fixed point $\widehat{\varepsilon}$ lies either in F_{id} when $\delta = 0$ or in F_s when $\delta = 1$.

The rest of the proof is essentially a repetition of the proof for the left induction. The small differences are caused by the fact that the divisors $D_{w,\lambda}$ are not symmetric with respect to switch from left to right actions. We leave details to the reader. $\square$

Definition 14.5. Let

$$\widetilde{\mathscr{T}}^{\mathrm{R}}_{i,\lambda}(f) = \frac{1 + y\, L_{i,x}}{1 - L_{i,x}} \cdot s^x + \frac{(1+y) \cdot L_{i,x}^{\lceil -\langle \lambda, \alpha_i^\vee\rangle\rceil - 1}}{1 - L_{i,x}^{-1}} \cdot \mathrm{id}\,.$$

Note that $\widetilde{\mathscr{T}}^{\mathrm{R}}_{i,\lambda}$ is equal to $\mathfrak{T}^{\mathrm{L}}_{i,-\langle\lambda,\alpha_i^\vee\rangle}$ acting on x–variables.

Remark 14.6. The operator $\widetilde{\mathscr{T}}^{\mathrm{R}}_{i,\lambda}$ differs from $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$ by conjugation by $1 + y\, L_{i,x}$, thus it satisfies

$$\mathscr{B}^{-1}\widetilde{\mathscr{T}}^{\mathrm{R}}_{r,\lambda}(f) = \mathscr{T}^{\mathrm{R}}_{r,\lambda}(\mathscr{B}^{-1}f)\,. \tag{25}$$

Repeating the proof of lemma 14.2 we obtain

Lemma 14.7. *We have*

$$\widetilde{\mathscr{T}}^{\mathrm{R}}_{i,\lambda}\big(\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda)\big) = \mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(ws, s\lambda)\,.$$

Proof of 12.6 (2). We apply proposition 14.7 and the formula (25). The Euler class $\mathscr{E}$ commutes with the operator $\mathscr{T}^{\mathrm{R}}_{i,\lambda}$, hence we obtain the recursion for $\mathscr{E}\mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}_{\mathrm{loc}}(w,\lambda) = \mathscr{B}^{-1}\,\mathrm{mC}^{\mathbb{T}^2}(w,\lambda)$. $\square$

15. Twisted double Hecke algebra of the general type

Let G be a semisimple, simply-connected Lie group with a maximal torus $\mathbb{T}$. Consider the representation ring of $\mathbb{T}^2$. The characters of the first copy are decorated with the superscript t, the characters of the second copy with x. For a simple reflection s let

$$L_{s,t} = \alpha_s^{-1} \otimes 1 \in R(\mathbb{T}) \otimes R(\mathbb{T}) \simeq R(\mathbb{T}^2)\,,$$

$$L_{s,x} = 1 \otimes \alpha_s^{-1} \in R(\mathbb{T}) \otimes R(\mathbb{T}) \simeq R(\mathbb{T}^2)\,,$$

$$\mathfrak{T}^{\mathrm{L}}_{s,a}(f) = \frac{1 + y\, L_{s,t}}{1 - L_{s,t}} s^t f + \frac{(1+y)L_{s,t}^{\lceil a\rceil - 1}}{1 - L_{s,t}^{-1}} f\,,$$

$$\mathfrak{T}^{\mathrm{R}}_{s,a}(f) = \frac{1 + y\, L_{s,x}^{-1}}{1 - L_{s,x}} s^x f + \frac{(1+y)L_{s,x}^{\lceil a\rceil - 1}}{1 - L_{s,x}^{-1}} f\,,$$

$$\mathscr{T}^{\mathrm{L}}_{s,\lambda} = \mathfrak{T}^{\mathrm{L}}_{s,\langle\lambda,\alpha_s^\vee\rangle}\,, \qquad \mathscr{T}^{\mathrm{R}}_{s,\lambda} = \mathfrak{T}^{\mathrm{R}}_{s,-\langle\lambda,\alpha_s^\vee\rangle}\,.$$

As in the A_n case the right operators commute with left operators. By construction $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ are lifts of the operators $\mathcal{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathcal{T}^{\mathrm{L}}_{s,\lambda}$ given in definition 6.3. Precisely, let

$$\kappa : R(\mathbb{T}^2) \to K_{\mathbb{T}}(G/B)$$

be the surjection sending $L_{s,x}$ to $\mathcal{L}(\alpha_s)$ and $L_{s,t}$ to $\alpha_s^{-1} = \mathbb{C}_{-\alpha_s}$ viewed as an element of the coefficient ring $K_{\mathbb{T}}(pt) \simeq R(\mathbb{T})$. Then

$$\mathcal{T}^{\mathrm{R}}_{s,\lambda} \circ \kappa = \kappa \circ \mathscr{T}^{\mathrm{R}}_{s,\lambda}\,, \qquad \mathcal{T}^{\mathrm{L}}_{s,\lambda} \circ \kappa = \kappa \circ \mathscr{T}^{\mathrm{L}}_{s,\lambda}\,.$$

The operators $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ satisfy quadratic relations, because when only one reflection is involved, one can assume that $G = \mathrm{SL}_2$.

Theorem 15.1. *Suppose $a, b \in \mathbb{Q}$. If $a \notin \mathbb{Z}$, then*

$$\mathfrak{T}_{s,b} \circ \mathfrak{T}_{s,a} = (y+1)\frac{L^{\lceil -a\rceil} - L^{\lceil b\rceil}}{1-L}\mathfrak{T}_{s,a} - y\,\mathrm{id}\,.$$

If $a \in \mathbb{Z}$

$$\mathfrak{T}_{s,b} \circ \mathfrak{T}_{s,a} = (y+1)\frac{L^{\lceil 1-a\rceil} - L^{\lceil b\rceil}}{1-L}\mathfrak{T}_{s,a} - y\,\mathrm{id}\,,$$

where $L = L_{s,x}$ or $L_{s,t}$ and $\mathfrak{T}_{s,c} = \mathfrak{T}^{\mathrm{R}}_{s,c}$ or $\mathfrak{T}^{\mathrm{L}}_{s,c}$ correspondingly, $c \in \{a, b\}$,

Proof. Directly by the definition above

$$\mathfrak{T}_{s,b} = \mathfrak{T}_{s,-a} + (y+1)\frac{L^{\lceil -a\rceil} - L^{\lceil b\rceil}}{1-L}\mathrm{id}\,.$$

Hence, if $a \notin \mathbb{Z}$, by lemma 11.1

$$\mathfrak{T}_{s,b} \circ \mathfrak{T}_{s,a} = \mathfrak{T}_{s,-a} \circ \mathfrak{T}_{s,a} + (y+1)\frac{L^{\lceil -a\rceil} - L^{\lceil b\rceil}}{1-L}\mathfrak{T}_{s,a} = -y\,\mathrm{id} + (y+1)\frac{L^{\lceil -a\rceil} - L^{\lceil b\rceil}}{1-L}\mathfrak{T}_{s,a}\,.$$

For $a \in \mathbb{Z}$ we replace $-a$ by $1-a$ in the above formula. □

The braid relations for simply-laced groups already follow from the SL_3 case. It remains to check the braid relations for the C_2 and G_2 to obtain the general type. The proofs are straightforward generalizations of the proof of proposition 11.3. We state the braid relation without the superscripts L, R:

15.1. **Type C_2.**

$$\mathscr{T}_{1,s_2s_1s_2\lambda}\,\mathscr{T}_{2,s_1s_2\lambda}\,\mathscr{T}_{1,s_2\lambda}\,\mathscr{T}_{2,\lambda} = \mathscr{T}_{2,s_1s_2s_1\lambda}\,\mathscr{T}_{1,s_2s_1\lambda}\,\mathscr{T}_{2,s_1\lambda}\,\mathscr{T}_{1,\lambda}\,.$$

In the standard coordinates of $\mathfrak{t} \subset \mathfrak{sp}_2$ the simple roots are the following

$$\alpha_1 = (1,-1)\,, \qquad \alpha_2 = (0,2)\,.$$

Writing the weight $\lambda = (a,b)$ in coordinates we have the identity

$$\mathfrak{T}_{1,a-b}\,\mathfrak{T}_{2,a}\,\mathfrak{T}_{1,a+b}\,\mathfrak{T}_{2,b} \;=\; \mathfrak{T}_{2,b}\,\mathfrak{T}_{1,a+b}\,\mathfrak{T}_{2,a}\,\mathfrak{T}_{1,a-b}\,.$$

15.2. **Type G_2.**

$$\mathscr{T}_{1,s_2s_1s_2s_1s_2\lambda}\mathscr{T}_{2,s_1s_2s_1s_2\lambda}\mathscr{T}_{1,s_2s_1s_2\lambda}\mathscr{T}_{2,s_1s_2\lambda}\mathscr{T}_{1,s_2\lambda}\mathscr{T}_{2,\lambda} =$$
$$= \mathscr{T}_{2,s_1s_2s_1s_2s_1\lambda}\mathscr{T}_{1,s_2s_1s_2s_1\lambda}\mathscr{T}_{2,s_1s_2s_1\lambda}\mathscr{T}_{1,s_2s_1\lambda}\mathscr{T}_{2,s_1\lambda}\mathscr{T}_{1,\lambda}\,.$$

For the weight $\lambda = (a,b)$ written in the basis of simple roots α_1, α_2 with $|\alpha_1| > |\alpha_2|$ we obtain an equivalent identity

$$\mathfrak{T}_{1,a}\,\mathfrak{T}_{2,3a+3b}\,\mathfrak{T}_{1,2a+3b}\,\mathfrak{T}_{2,3a+6b}\,\mathfrak{T}_{1,a+3b}\,\mathfrak{T}_{2,3b} = \mathfrak{T}_{2,3b}\,\mathfrak{T}_{1,a+3b}\,\mathfrak{T}_{2,3a+6b}\,\mathfrak{T}_{1,2a+3b}\,\mathfrak{T}_{2,3a+3b}\,\mathfrak{T}_{1,a}\,.$$

A geometric meaning $\mathscr{T}^{\mathrm{R}}_{s,\lambda}$ and $\mathscr{T}^{\mathrm{L}}_{s,\lambda}$ will be explained in a forthcoming paper. This issue goes far beyond the scope of the current paper.

References

[AM16] P. Aluffi and L. C. Mihalcea. Chern-Schwartz-MacPherson classes for Schubert cells in flag manifolds. *Compos. Math.*, 152(12):2603–2625, 2016.

[AMSS17] P. Aluffi, L. C. Mihalcea, J. Schuermann, and Ch. Su. Shadows of characteristic cycles, Verma modules, and positivity of Chern-Schwartz-MacPherson classes of Schubert cells, 2017. arXiv:1709.08697, to appear in Duke Math. J.

[AMSS19] P. Aluffi, L. C. Mihalcea, J. Schürmann, and Ch. Su. Motivic Chern classes of Schubert cells, Hecke algebras, and applications to Casselman's problem, 2019. arXiv:1902.10101, to appear in *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série.*

[AMSS22] Paolo Aluffi, Leonardo C. Mihalcea, Jörg Schürmann, and Changjian Su. From motivic Chern classes of Schubert cells to their Hirzebruch and CSM classes, 2022. arXiv:2212.12509.

[AO21] M. Aganagic and A. Okounkov. Elliptic stable envelopes. *J. Amer. Math. Soc.*, 34(1):79–133, 2021.

[BFM75] P. Baum, W. Fulton, and R. MacPherson. Riemann-Roch for singular varieties. *Inst. Hautes Études Sci. Publ. Math.*, (45):101–145, 1975.

[BGG73] I. N. Bernšteĭn, I. M. Gelfand, and S. I. Gelfand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973.

[BK05] M. Brion and S. Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 2005.

[BL03] L. Borisov and A. Libgober. Elliptic genera of singular varieties. *Duke Math. J.*, 116(2):319–351, 2003.

[BP19] M. Bender and N. Perrin. Singularities of closures of B-conjugacy classes of nilpotent elements of height 2. *Transform. Groups*, 24(3):741–768, 2019.

[Bri97] M. Brion. Equivariant Chow groups for torus actions. *Transform. Groups*, 2(3):225–267, 1997.

[Bri05] Michel Brion. Lectures on the geometry of flag varieties. In *Topics in cohomological studies of algebraic varieties*, Trends Math., pages 33–85. Birkhäuser, Basel, 2005.

[Bri15] M. Brion. On linearization of line bundles. *J. Math. Sci. Univ. Tokyo*, 22(1):113–147, 2015.

[BSY10] J.-P. Brasselet, J. Schürmann, and S. Yokura. Hirzebruch classes and motivic Chern classes for singular spaces. *J. Topol. Anal.*, 2(1):1–55, 2010.

[Car05] R. W. Carter. *Lie algebras of finite and affine type*, volume 96 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2005.

[CG10] N. Chriss and V. Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2010. Reprint of the 1997 edition.

[Dem74] M. Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974.

[Deo87] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan-Lusztig polynomials. *J. Algebra*, 111(2):483–506, 1987.

[EV98] Pavel Etingof and Alexander Varchenko. Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups. *Comm. Math. Phys.*, 196(3):591–640, 1998.

[Fel95] Giovanni Felder. Conformal field theory and integrable systems associated to elliptic curves. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 1247–1255. Birkhäuser, Basel, 1995.

[FR18] L. Fehér and R. Rimányi. Chern-Schwartz-MacPherson classes of degeneracy loci. *Geom. Topol.*, 22(6):3575–3622, 2018.

[FRW20] László M. Fehér, Richárd Rimányi, and Andrzej Weber. Characteristic classes of orbit stratifications, the axiomatic approach. In *Schubert calculus and its applications in combinatorics and representation theory*, volume 332 of *Springer Proc. Math. Stat.*, pages 223–249. Springer, Singapore, [2020] ©2020.

[FRW21] L. Fehér, R. Rimányi, and A. Weber. Motivic Chern classes and K-theoretic stable envelopes. *Proc. Lond. Math. Soc. (3)*, 122(1):153–189, 2021.

[GKV97] Victor Ginzburg, Mikhail Kapranov, and Eric Vasserot. Residue construction of Hecke algebras. *Adv. Math.*, 128(1):1–19, 1997.

[GN17] E. Gorsky and A. Neguţ. Infinitesimal change of stable basis. *Selecta Math. (N.S.)*, 23(3):1909–1930, 2017.

[Hir56] F. Hirzebruch. *Neue topologische Methoden in der algebraischen Geometrie*. Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

[Hum90] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[Kem76] G. R. Kempf. Linear systems on homogeneous spaces. *Ann. of Math. (2)*, 103(3):557–591, 1976.

[KK90] Bertram Kostant and Shrawan Kumar. T-equivariant K-theory of generalized flag varieties. *J. Differential Geom.*, 32(2):549–603, 1990.

[Knu03] A. Knutson. A Schubert calculus recurrence from the noncomplex W-action on G/B, 2003. arXiv:math/0306304.

[Kon22] J. Koncki. Comparison of motivic Chern classes and stable envelopes for cotangent bundles. *J. Topol.*, 15(1):168–203, 2022.

[KRW20] S. Kumar, R. Rimányi, and A. Weber. Elliptic classes of Schubert varieties. *Math. Ann.*, 378(1-2):703–728, 2020.

[KW22] J. Koncki and A. Weber. Twisted motivic Chern class and stable envelopes. *Adv. Math.*, 404:Paper No. 108374, 2022.

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(11):629–633, 1982.

[Lus85] George Lusztig. Equivariant K-theory and representations of Hecke algebras. *Proc. Amer. Math. Soc.*, 94(2):337–342, 1985.

[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. *Annals of Mathematics*, 100(2):423–432, 1974.

[MNS22] L. C. Mihalcea, H. Naruse, and Ch. Su. Left Demazure–Lusztig Operators on Equivariant (Quantum) Cohomology and K-Theory. *Int. Math. Res. Not. IMRN*, (16):12096–12147, 2022.

[MO19] D. Maulik and A. Okounkov. Quantum groups and quantum cohomology. *Astérisque*, (408):ix+209, 2019.

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[MS22] Leonardo C. Mihalcea and Changjian Su. Whittaker functions from motivic Chern classes. *Transform. Groups*, 27(3):1045–1067, 2022. With an appendix by Mihalcea, Su and Dave Anderson.

[Ohm06] T. Ohmoto. Equivariant Chern classes of singular algebraic varieties with group actions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 140(1):115–134, 2006.

[Oko17] A. Okounkov. Lectures on K-theoretic computations in enumerative geometry. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 251–380. Amer. Math. Soc., Providence, RI, 2017.

[Oko21] A. Okounkov. Inductive construction of stable envelopes. *Lett. Math. Phys.*, 111(6):Paper No. 141, 56, 2021.

[OS22] A. Okounkov and A. Smirnov. Quantum difference equation for Nakajima varieties. *Invent. Math.*, 229(3):1203–1299, 2022.

[RS18] R. Rimányi and A. Szenes. Residues, Grothendieck polynomials and K-theoretic Thom polynomials, 2018. arXiv:1811.02055, Int. Math. Res. Not. IMRN, rnac345, https://doi.org/10.1093/imrn/rnac345.

[RSVZ19] R. Rimányi, A. Smirnov, A. Varchenko, and Z. Zhou. Three-dimensional mirror self-symmetry of the cotangent bundle of the full flag variety. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 15:Paper No. 093, 22, 2019.

[RTV15] R. Rimányi, V. Tarasov, and A. Varchenko. Trigonometric weight functions as K-theoretic stable envelope maps for the cotangent bundle of a flag variety. *Journal of Geometry and Physics*, 94:81 – 119, 2015.

[RTV19] R. Rimányi, V. Tarasov, and A. Varchenko. Elliptic and K-theoretic stable envelopes and Newton polytopes. *Selecta Math. (N.S.)*, 25(1):Art. 16, 43, 2019.

[RV18] R. Rimányi and A. Varchenko. Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae. In *Schubert varieties, equivariant cohomology and characteristic classes—IMPANGA 15*, EMS Ser. Congr. Rep., pages 225–235. Eur. Math. Soc., Zürich, 2018.

[RW20] R. Rimányi and A. Weber. Elliptic classes of Schubert varieties via Bott–Samelson resolution. *Journal of Topology*, 13(3):1139–1182, 2020.

[RW22a] R. Rimányi and A. Weber. Elliptic classes on Langlands dual flag varieties. *Commun. Contemp. Math.*, 24(1):Paper No. 2150014, 15, 2022.

[RW22b] Piotr Rudnicki and Andrzej Weber. Characteristic classes of Borel orbits of square-zero upper-triangular matrices. *J. Algebra*, 598:351–384, 2022.

[Smi21] A. Smirnov. Quantum differential and difference equations for $\mathrm{Hilb}^n(\mathbb{C}^2)$, 2021. arXiv:2102.10726.

[Su17] Changjian Su. Restriction formula for stable basis of the Springer resolution. *Selecta Math. (N.S.)*, 23(1):497–518, 2017.

[SY07] J. Schürmann and S. Yokura. A survey of characteristic classes of singular spaces. In *Singularity theory*, pages 865–952. World Sci. Publ., Hackensack, NJ, 2007.

[SZ20] Ch. Su and Ch. Zhong. Stable bases of the springer resolution and representation theory. In *Schubert Calculus and Its Applications in Combinatorics and Representation Theory*, pages 195–221, Singapore, 2020. Springer Singapore.

[SZZ20] Ch. Su, G. Zhao, and Ch. Zhong. On the K-theory stable bases of the Springer resolution. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(3):663–711, 2020.

[SZZ21] Ch. Su, G. Zhao, and Ch. Zhong. Wall-crossings and a categorification of K-theory stable bases of the Springer resolution. *Compos. Math.*, 157(11):2341–2376, 2021.

[Tho92] R. W. Thomason. Une formule de lefschetz en K-théorie équivariante algébrique. *Duke Math. Journal*, 68:447–462, 1992.

[Uma13] V. Uma. Equivariant K-theory of flag varieties revisited and related results. *Colloq. Math.*, 132(2):151–175, 2013.

[Web16] A. Weber. Equivariant Hirzebruch class for singular varieties. *Selecta Mathematica*, 22(3):1413–1454, 2016.

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