ON RIGIDITY OF FLAG VARIETIES

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ABSTRACT. We prove that the variety of complete flags for any semisimple algebraic group is rigid in any smooth family of Fano manifolds.

1. RESULTS

The aim of this note is to prove the following rigidity theorem.

Theorem 1.1. *Let G be a semisimple algebraic group defined over complex numbers. By $B < G$ we denote its Borel subgroup. Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of complex projective manifolds over a positive-dimensional connected base Δ. That is $\pi : \mathcal{X} \to \Delta$ is a submersion of smooth varieties and for every $t \in \Delta$ we set $X_t = \pi^{-1}(t)$. Assume that the relative anticanonical divisor $-K_{\mathcal{X}/\Delta}$ is π-ample and for $t \neq 0$ the variety $X_t = \pi^{-1}(t)$ is isomorphic to the variety of complete flags G/B. Then $X_0 \cong G/B$*

The rigidity of rational homogeneous varieties of type $G/P_{\max}$, where $P_{\max} < G$ is a maximal parabolic subgroup, was studied by Siu, Hwang and Mok, see [15], [8], [9]. Their results concern irreducible Hermitian symmetric spaces. As it was observed by Pasquier and Perrin in [14, Prop. 2.3] not all varieties of this type are rigid. Also, if a parabolic subgroup $P < G$ is neither Borel nor maximal then G/P may not be rigid. Moreover, the theorem fails to be true if we do not assume ampleness of the relative anticanonical divisor. We discuss these issues in the last section of the present paper.

The proof of the theorem is based on the characterization of varieties of complete flags by Occhetta, Sola Conde, Watanabe and the second author, from [12], and the rigidity of the nef cone for Fano manifolds from [19]. The main theorem of [12] characterizes flag manifolds of type G/B as the unique Fano manifolds whose all elementary contractions in terms of Mori theory are $\mathbb{P}^1$-bundles, see theorem 2.1. The main technical result of the paper is the following equidimensional rigidity for Fano-Mori fiber bundles.

Proposition 1.2. *Let $\pi : \mathcal{X} \to \Delta \ni 0$ be a smooth family of complex projective manifolds over a positive-dimensional connected base Δ. We set $X_t = \pi^{-1}(t)$. Let $\mathcal{Y}$ be a normal variety with a projective morphism $\mathcal{Y} \to \Delta$, we denote $Y_t = \pi^{-1}(t)$. Let $\varphi : \mathcal{X} \to \mathcal{Y}$ be a projective morphism of varieties over Δ, which commutes with the morphisms $\mathcal{X} \to \Delta$ and $\mathcal{Y} \to \Delta$. By $\varphi_t : X_t \to Y_t$ we denote the restriction of φ. We assume the following:*

(1) the relative anti-canonical divisor $-K_{\mathcal{X}/\Delta}$ is φ-ample,

(2) for every $t \neq 0$ the morphism $\varphi_t : X_t \to Y_t$ is a fiber bundle, such that the fundamental group of Y_t acts trivially on the cohomology of the fiber,

(3) *for every $t \neq 0$ the rational cohomology ring* $\mathrm{H}^*(Y_t)$ *is generated by its first and second gradation.*

Then the morphism φ is equidimensional, that is every fiber of φ is of the same dimension.

As a consequence of Fujita's theorem, [6, Thm. 2.12] we get the following.

Corollary 1.3. *In the situation of proposition 1.2 assume that for $t \neq 0$ the morphism φ_t is a Zariski projective bundle. Then φ_0 is of the same type as well.*

Here a Zariski projective bundle is understood as projectivization of a vector bundle.

We note that the condition that rational cohomology ring $\mathrm{H}^*(Y_t)$ is generated by its first and second gradation is very strong. However, it is known to be satisfied by abelian varieties, toric varieties and complete flags, among others. In order to prove the rigidity theorem for complete flags we need the following.

Theorem 1.4. *Let G be a semisimple algebraic group over complex numbers and $P_{\min} < G$ a minimal parabolic subgroup of G which contains a Borel group of G properly. Then the rational cohomology ring $\mathrm{H}^*(G/P_{\min})$ is generated as $\mathbb{Q}$-algebra by its second gradation, that is $\mathrm{H}^2(G/P_{\min})$.*

Notation. We deal with varieties defined over complex numbers. By $\cong$ we denote their isomorphism while $\simeq$ stands for homeomorphism. Unless specified otherwise, the cohomology is considered with rational coefficients; given X, a projective (compact) variety, by $\mathrm{H}^*(X)$ we denote its (graded) cohomology ring with $\smile$-product. A fiber bundle means a locally trivial bundle in the underlying classical topology.

2. Proofs

In the proofs we use the notation introduced in the previous section.

2.1. **Proof of theorem 1.1.** Let us recall basic facts of Mori theory and refer the reader to more comprehensive sources, e.g. [10], for details. A contraction $\varphi : X \to Y$ is a surjective morphism of normal projective varieties with connected fibers. We will also consider a relative case when both varieties are not necessarily projective but they admit projective morphisms $X \to \Delta$, $Y \to \Delta$ and φ commutes with morphisms to Δ. The variety X will be assumed smooth (over Δ) and its anti-canonical divisor $-K_X$ or, in the relative case, $-K_{X/\Delta}$ will be assumed to be φ-ample. Then φ is called Fano-Mori contraction. We will say that φ is elementary, or extremal, if $Pic(X/Y) \cong \mathbb{Z}$. This is equivalent to say

that cohomology classes of curves contracted by φ to points span 1-dimensional subspace in $\mathrm{H}^{2\dim X-2}(X)$.

A smooth projective variety X is Fano if $-K_X$ is ample. Mori cone theorem asserts that if X is Fano then the convex cone $\mathcal{C}(X)$ spanned in $\mathrm{H}^{2n-2}(X)$ by the classes of curves is rational polyhedral. Kawamata-Shokurov contraction theorem asserts that in this case there is a bijection between faces of $\mathcal{C}(X)$ and contractions of X: given a face $\Phi \subseteq \mathcal{C}(X)$ there exists a contraction $\varphi_\Phi : X \to Y_\Phi$ which contracts to points exactly these curves whose classes are in Φ. Elementary contractions of X are associated to 1-dimensional faces of $\mathcal{C}(X)$ (extremal rays).

We note that everything which is said in the previous paragraph holds in the relative case too. In particular, if $\varphi : X \to Y$ is a contraction of a smooth variety and $-K_X$ is φ-ample, then the cone $\mathcal{C}(X/Y)$ spanned by the classes of curves contracted by φ is rational polyhedral and its faces are in bijection with contractions factoring φ.

The main ingredient of the proof of 1.1 is the following main result of [12].

Theorem 2.1. *Let X be a Fano manifold such that every elementary contraction of X is a smooth $\mathbb{P}^1$-fibration. Then X is isomorphic to a complete flag variety G/B, where G is a semisimple algebraic group and B a Borel subgroup.*

It is well known that if $X \cong G/B$ then X is Fano and every elementary contraction of X is a $\mathbb{P}^1$-bundle $G/B \to G/P_{\min}$, where $P_{\min}$ is a minimal parabolic subgroup of G which contains B properly. Given a semisimple group G, the homogeneous spaces G/P are distinguished by a set of nodes in the Dynkin diagram of G. In these terms G/B is associated to the empty set of nodes and $G/P_{\min}$ is associated to a single node. It is also known that such a presentation is unique, that is if $G_1/B_1 \cong G_2/B_2$ where G's are simply connected semisimple and B's are Borel then we can identify $G_1 = G_2$ and under this identification B's are conjugate.

Now the proof of 1.1 goes as follows.

Proof. Since $-K_{\mathcal{X}/\Delta}$ is π-ample it follows that X_0 is Fano. By the main theorem of [19] and Kawamata-Shokurov contraction theorem every elementary contraction $\varphi_0 : X_0 \to Y_0$ extends to an elementary contraction $\varphi : \mathcal{X} \to \mathcal{Y}$ relative over Δ with $\varphi_t : X_t \to Y_t$ being an elementary contraction for every t. By our assumption, for $t \neq 0$ we have $X_t \cong G/B$, and by what we have said above the resulting elementary contraction $\varphi_t : X_t \to Y_t$ is a $\mathbb{P}^1$-bundle over $Y_t \cong G/P_{\min}$. Now by theorem 1.4 for $t \neq 0$ the variety Y_t satisfies the assumptions of proposition 1.2. Therefore, by corollary 1.3, $\varphi_0 : X_0 \to Y_0$ is a $\mathbb{P}^1$-bundle and by 2.1 the theorem 1.1 follows. □

2.2. A lemma.

Lemma 2.2. *Let X_0 and X_1 be a smooth projective varieties which are homeomorphic $X_0 \simeq X_1$. Assume that:*

(1) *there exists a fiber bundle structure $\varphi_1 : X_1 \to Y_1$ to a smooth variety Y_1 such that the fundamental group of Y_1 acts trivially on the cohomology of the fiber,*

(2) *there exists a morphism $\varphi_0 : X_0 \to Y_0$ to a possibly singular variety Y_0,*

(3) *for some $k \geq 1$ the elements of degree $\leq k$, that is in $\mathrm{H}^{\leq k}(Y_1)$, generate the ring $\mathrm{H}^*(Y_1)$*

(4) *under identification given by the homeomorphism $X_0 \simeq X_1$ we have*

$$\varphi_1^*(\mathrm{H}^{\leq k}(Y_1)) \subseteq \varphi_0^*(\mathrm{H}^{\leq k}(Y_0))$$

Then the dimension of every fiber of φ_0 does not exceed $\dim X_1 - \dim Y_1$.

Remark 2.3. The assumption about the trivial action of the fundamental group of Y_1 can by replaced by any other assumption which will guarantee that $\mathrm{H}^*(X_1)$ is a free module over $\mathrm{H}^*(Y_1)$.

Proof. We let $n = \dim X_0 = \dim X_1$ and $f = \dim X_1 - \dim Y_1$. Let $Z \subset X_0$ be a fiber of φ_0 and $z = \dim Z$. By $[Z] \in \mathrm{H}^{2n-2z}(X_0)$ we denote the cohomology class of Z. To get a contradiction we assume that $z > f$.

First we claim that for every $\alpha \in \mathrm{H}^{>0}(Y_0)$ we have $[Z] \smile \varphi_0^*(\alpha) = 0$. Indeed, let us look at the diagram of morphisms

$$
\begin{array}{ccc}
 & i & \\
Z & \hookrightarrow & X_0 \\
\varphi_Z \downarrow & & \downarrow \varphi_0 \\
pt & \hookrightarrow & Y_0 \\
 & j &
\end{array}
$$

We have

$$[Z] \smile \varphi_0^*(\alpha) = i_* i^* \varphi_0^*(\alpha) = i_* \varphi_Z^* j^*(\alpha) .$$

By our assumption $\deg(\alpha) > 0$. Thus restricted to the point it vanishes, i.e. $j^*(\alpha) = 0$ and our claim follows.

Now we claim that for any class $\beta \in \mathrm{H}^{2z}(X_1)$ we have $[Z] \smile \beta = 0$. Since φ_1 is a fibration of smooth projective varieties, the Serre spectral sequence degenerates by [2, Thm. II.1.2] or [5, Prop. 2.1]. Therefore the cohomology $\mathrm{H}^*(X_1)$ is a free module over $\mathrm{H}^*(Y_1)$. Let $\{\gamma_i\}_{i \in I}$ be a homogeneous basis, with the degrees of γ_i's at most $2f$. Write

$$(2.1) \qquad\qquad \beta = \sum_I \varphi_1^*(\delta_i) \smile \gamma_i$$

with $\delta_i \in \mathrm{H}^*(Y_1)$. The elements $\varphi_1^*(\delta_i)$ belong to the subring generated by

$$\varphi_1^*(\mathrm{H}^{\leq k}(Y_1)) = \varphi_0^*(\mathrm{H}^{\leq k}(Y_0)) \subseteq \varphi_0^*(\mathrm{H}^*(Y_0))$$

Note that in the decomposition (2.1) the nonzero δ_i's are of degree at least $2(z - f) > 0$. Thus $[Z] \smile \varphi_1^*(\delta_i) = 0$ by what we proved above. Then

$$[Z] \smile \beta = \sum_I [Z] \smile \varphi_1^*(\delta_i) \smile \gamma_i = 0$$

Finally, by Poincaré duality, we conclude that $[Z] = 0$ in $\mathrm{H}^*(X_1) = \mathrm{H}^*(X_0)$. Hence, since X_0 is projective, a contradiction. $\square$

2.3. **Proof of proposition 1.2.** We will show that in the situation of 1.2 the varieties X_0 and X_t, $t \neq 0$, satisfy the assumptions of lemma 2.2, with $X_1 = X_t$ and $k = 2$.

Firstly we note that by the theorem of Ehresmann the family $\mathcal{X}$ is topologically locally trivial. If needed we can shrink the base Δ of $\pi : \mathcal{X} \to \Delta \ni 0$ and identify topology of fibers X_t, for $t \neq 0$ with that of X_0, that is $X_0 \simeq X_t$. In fact, for the purpose of topological argument we may assume Δ to be an open (in the classical topology) neighbourhood of 0 which is contractible. Then for every $t \in \Delta$ the inclusion $\iota_{X_t} : X_t \hookrightarrow \mathcal{X}$ is a homotopy equivalence. Therefore we have isomorphisms

$$\mathrm{H}^*(X_0) \xleftarrow{\iota_{X_0}^*} \mathrm{H}^*(\mathcal{X}) \xrightarrow{\iota_{X_t}^*} \mathrm{H}^*(X_t) \,.$$

Lemma 2.4. *In the set up of proposition 1.2 let us assume that $-K_{\mathcal{X}/\Delta}$ is φ-ample (we do not assume points (2) and (3) of this proposition). Then, under the identification $\mathrm{H}^*(X_t) \simeq \mathrm{H}^*(X_0)$, for $i = 1,\ 2$ we have*

$$\varphi_t^*(\mathrm{H}^i(Y_t)) \subseteq \varphi_0^*(\mathrm{H}^i(Y_0))$$

Remark 2.5. The opposite inclusion always holds as explained in section 3.

Proof. We use results which follow from known vanishings related to Mori contractions. Since $-K_{X_t/Y_t}$ is φ_t-ample it follows that for every t we have $\varphi_t^*(\mathrm{H}^1(Y_t)) = \mathrm{H}^1(X_t)$, see e.g. [18, Prop. 2.3].

Let $\mathcal{C}(\mathcal{X}/\mathcal{Y}) \subseteq \mathcal{C}(\mathcal{X})$ be the cone of (classes of) curves contracted by φ. By e.g. [18, Thm. 2.4], if $\varphi_R : \mathcal{X} \to \mathcal{Y}_R$ is a contraction of a Mori extremal ray $R \subseteq \mathcal{C}(\mathcal{X}/\mathcal{Y})$ then $\varphi_R^*(\mathrm{H}^2(\mathcal{Y}_R)) \subseteq \mathrm{H}^2(\mathcal{X})$ is annihilated (orthogonal in terms of $\smile$-product) by the class of any curve contracted by φ_R. In fact $\varphi_R^*(\mathrm{H}^2(\mathcal{Y}_R)) = R^\perp$. Since $\mathcal{C}(\mathcal{X}/\mathcal{Y})$ is generated by Mori extremal rays whose contractions factor φ we get

$$\varphi^*(\mathrm{H}^2(\mathcal{Y})) = \bigcap_{R \subseteq \mathcal{C}(\mathcal{X}/\mathcal{Y})} R^\perp$$

Now by [17, Prop.1.3] the locus of every extremal ray contraction of $\mathcal{X}$ dominates Δ. Thus for $t \neq 0$ we have the left-hand-side inclusion

$$\varphi_t^*(\mathrm{H}^2(Y_t)) \subseteq \bigcap_{R \subseteq \mathcal{C}(\mathcal{X}/\mathcal{Y})} R^\perp \ = \ \varphi^*(\mathrm{H}^2(\mathcal{Y})) \ = \ \varphi_0^*(\mathrm{H}^2(Y_0))$$

The right-hand-side equality follows because $\iota_{Y_0} : Y_0 \hookrightarrow \mathcal{Y}$ may be assumed a homotopy equivalence (after possibly shrinking $\mathcal{Y} \to \Delta$). This concludes the proof of the lemma. $\square$

From the preceding discussion and lemma 2.4 it follows that assumptions of 1.2 imply the set up of lemma 2.2 hence proposition 1.2 follows.

To prove corollary 1.3 we note that if φ_t is a $\mathbb{P}^r$-bundle then we can choose a divisor class $H \in Pic(\mathcal{X})$ such that its restriction H_t to X_t for $t \neq 0$ is a relative $\mathcal{O}(1)$ bundle for the projective bundle $\varphi_t : X_t \to Y_t$. By the arguments which we used in the proof of lemma 2.4 we have $Pic(\mathcal{X})/\varphi^*(Pic(\mathcal{Y})) \cong \mathbb{Z} \cdot [H]$, hence $(r+1)H + K_{\mathcal{X}/\mathcal{Y}} \in \varphi^* Pic(\mathcal{Y})$. Moreover by proposition 1.2 φ has all fibers of dimension r. Thus, in view of Fujita's theorem [6, 2.12] the morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ is a $\mathbb{P}^r$-bundle and the same concerns φ_0.

2.4. **Generalities on cohomology of** G/P**.** Before giving the proof of 1.4 let us recall basic facts about cohomology ring of homogeneous spaces. Let G be a semisimple complex algebraic group. We fix a maximal torus and a Borel subgroup $\mathbb{T} < B < G$. Let $P < G$ be a parabolic subgroup containing B.

The roots of the Lie algebra $\mathfrak{g}$ belong to $\mathfrak{t}_{\mathbb{Z}}^* = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. Parabolic subgroups of G which contain the fixed Borel group correspond to the subsets of the set of simple roots. The empty set corresponds to the Borel group. A minimal parabolic subgroup $P = P_{\min}$ corresponds to a single root α.

Let $\mathfrak{t}_{\mathbb{Q}}^* = \mathfrak{t}_{\mathbb{Z}}^* \otimes \mathbb{Q}$. The Weyl group $W_G = W = N(\mathbb{T})/\mathbb{T}$ acts on $\mathfrak{t}_{\mathbb{Q}}^*$. We fix an invariant scalar product e.g. the negative of the Killing form. The Weyl group can be identified with the group generated by reflections in $\alpha^\perp$ where α are simple roots. The Weyl group W_P of a parabolic subgroup P is the subgroup of W generated by the simple roots defining P.

Recall that for any parabolic subgroup $P < G$ the homogeneous variety G/P admits a decomposition into Schubert cells, therefore $\mathrm{H}^*(G/P, \mathbb{Z})$ is a free abelian group generated by algebraic cycles. Further on we consider cohomology with rational coefficients. Let $S = Sym(\mathfrak{t}_{\mathbb{Q}}^*)$ be the symmetric algebra generated by $\mathfrak{t}_{\mathbb{Q}}^*$ and S^W (resp. S^{W_P}) be the subalgebra of W–invariants (resp W_P–invariants). By $S_+^W \subset S^W$ we denote the ideal of elements having positive degrees. The rational cohomology ring of G/P was computed in [3, Th. 20.6(b)] and [1, Thm. 5.5].

Theorem 2.6 ([3, 1])**.** *With the notation as above we have*

$$\text{(2.2)} \qquad\qquad \mathrm{H}^*(G/P) \simeq S^{W_P}/(S_+^W)\,,$$

where (S_+^W) *is the ideal in* S^{W_P} *generated by* S_+^W.

The isomorphism in (2.2) preserves the gradation under the convention that the linear forms $\mathfrak{t}_{\mathbb{Q}}^* \subset S$ live in the second gradation. We sketch a short proof of this formula.

Proof. Let $\mathbb{E}_G$ be a contractible space with a free G–action. We have a fibration

$$\text{(2.3)} \qquad\qquad G/P \overset{\iota}{\hookrightarrow} \mathbb{B}_P \overset{p}{\twoheadrightarrow} \mathbb{B}_G\,.$$

Here $\mathbb{B}_P = \mathbb{E}_G/P$, $\mathbb{B}_G = \mathbb{E}_G/G$ are the classifying spaces of Lie groups. The cohomology algebra of $\mathbb{B}_G$ was computed in [3, Th. 20.3]:

$$\text{(2.4)} \qquad\qquad \mathrm{H}^*(\mathbb{B}_G) \simeq S^W,$$

Similarly

$$\text{(2.5)} \qquad\qquad \mathrm{H}^*(\mathbb{B}_P) \simeq S^{W_P}.$$

The base and the fiber of (2.3) have cohomology concentrated in even degrees and therefore the Serre spectral sequence of the fibration degenerates. We obtain a surjection of algebras

$$\mathrm{H}^*(\mathbb{B}_P) \twoheadrightarrow \mathrm{H}^*(G/P)$$

induced by ι. The image of $p^*(\mathrm{H}^{>0}(\mathbb{B}_G))$ vanishes when restricted to the fiber. The resulting surjection

$$\text{(2.6)} \qquad\qquad \mathrm{H}^*(\mathbb{B}_P)/(\mathrm{H}^{>0}(\mathbb{B}_G)) \twoheadrightarrow \mathrm{H}^*(G/P)$$

of algebras is an isomorphism by counting the dimensions in each gradation. Combining (2.6) with (2.4) and (2.5) we obtain the formula (2.2). □

In particular, if $P = B$ is the Borel subgroup, then $\mathrm{H}^{2*}(G/B) \simeq S/(S_+^W)$ is generated by $\mathrm{H}^2(G/B) \simeq \mathfrak{t}_{\mathbb{Q}}^*$. An integral character $\chi \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*) = \mathfrak{t}_{\mathbb{Z}}^*$ corresponds to the first Chern class of the line bundle defined by this character, [7, Cor. 4]. The integral algebra structure $\mathrm{H}^*(G/B, \mathbb{Z})$ is much more complicated, see e.g. [16].

2.5. **Proof of theorem 1.4.** If P is a minimal parabolic subgroup corresponding to a simple root α then $W_P = \mathbb{Z}/2$ is generated by the reflection in $\alpha^\perp$. Therefore

$$S^{W_P} = Sym(\alpha^\perp) \otimes \mathbb{Q}[\alpha^2]$$

Hence S^{W_P} is generated by the linear and quadratic forms.

Our argument is based on the formula 2.2. We will show that the quotient $S^{W_P}/(S_+^W)$ is generated by the linear forms. It is enough to show that α^2 can be expressed by W_P-invariant linear forms modulo a quadratic W-invariant form. Let Q be the quadratic form corresponding to the invariant scalar product.

We choose vectors $\beta_1, \beta_2, \ldots, \beta_k$ which form an orthogonal basis of $\alpha^\perp$. Then for some nonzero numbers $a, b_1, b_2, \ldots b_k$ the quadratic form Q can be written as

$$Q = a\alpha^2 + \sum_{i=1}^{k} b_i \beta_i^2\,.$$

Thus

$$\alpha^2 \equiv -\sum_{i=1}^{k} \frac{b_i}{a} \beta_i^2 \quad \mathrm{mod} \ \ S^W\,.$$

This concludes the proof of theorem 1.4.

3. Remarks

Remark 3.1. We note that the assumption that $-K_{\mathcal{X}/\Delta}$ is π-ample in Theorem 1.1 is necessary because $X_t \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O})$ can be specialized to $X_0 \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(-a) \oplus \mathcal{O}(a))$ being an arbitrary even Hirzebruch surface. However it make sense to ask if the above rigidity theorem remains true remains true for families of projective manifolds (that is: $-K_{\mathcal{X}/\Delta}$ not necessarily assumed to be π-ample) if one assumes that G is simple (that is: with X_t irreducible complete flag variety).

Also, a naive extension of Theorem 1.1 to non-complete flags G/P, where is $P < B$ is any parabolic, does not work.

Example 3.2. The partial flag variety $X = Fl_{1,2}(V)$ of lines and planes in $V = \mathbb{C}^{2n+2}$ admits a fibration to the Grassmannian of planes $Grass_2(V)$ and to $\mathbb{P}^{2n+1}$. Let $\Omega = \Omega^1_{\mathbb{P}^{2n+1}}$ be the cotangent bundle to $\mathbb{P}^{2n+1}$. One can identify X with the Grothendieck projectivization of $\Omega(2) = \Omega \otimes \mathcal{O}(2)$. Let $\omega \in \bigwedge^2 V$

be a symplectic form. We can use the second exterior power of the dual Euler sequence

$$(3.1) \qquad\qquad 0 \longrightarrow \Omega^2(2) \longrightarrow \bigwedge^2 V \longrightarrow \Omega(2) \longrightarrow 0$$

to identify ω with a section of $\Omega(2)$ and thus obtain a surjective map $\mathcal{T}_{\mathbb{P}^{2n+1}}(-1) \to \mathcal{O}(1)$. The kernel $\mathcal{N}$ is called the null-correlation bundle; see [13, Sect. 4.2] where one can find details of this construction. In other terms $\mathcal{N}(1) \subset \mathcal{T}_{\mathbb{P}^{2n+1}}$ is the contact distribution associated with the symplectic form ω.

The (Grothendieck) projectivization $\mathbb{P}(\mathcal{N}^*(1)) \subset \mathbb{P}(\Omega(2))$ is the incidence variety of lines and ω-isotropic planes. The extension

$$(3.2) \qquad\qquad 0 \longrightarrow \mathcal{O} \longrightarrow \Omega(2) \longrightarrow \mathcal{N}^*(1) \longrightarrow 0$$

can be specialized to $\mathcal{N}^*(1) \oplus \mathcal{O}$ and the projectivization $X_t = \mathbb{P}(\Omega(2))$ specializes to $X_0 = \mathbb{P}(\mathcal{N}^*(1) \oplus \mathcal{O})$. Consequently, the $\mathbb{P}^1$-bundle

$$(3.3) \qquad\qquad Fl_{1,2}(V) = \mathbb{P}(\Omega(2)) \to Grass_2(V)$$

specializes to a map of $\mathbb{P}(\mathcal{N} \oplus \mathcal{O})$ to a cone over the isotropic Grassmannian. The latter morphism has the unique fiber Z over the vertex of the cone which is not $\mathbb{P}^1$ but $\mathbb{P}^{2n+1}$.

One easily checks that here the image of $H^*(Y_0)$ in $H^*(X_0)$ is smaller than the image of $H^*(Y_t)$ e.g. computing the ranks of these groups. At this point the proof of proposition 1.2 breaks in our example.

Let us see what happens with the class $[Z]$. The $\mathbb{P}^1$-bundle (3.3) is defined by global sections of $\Omega(2)$. Thus the pull-back of the ample generator of Picard group of $Grass_2(V)$ is the relative $\mathcal{O}_{\mathbb{P}(\Omega(2)/\mathbb{P}^{2n+1})}(1)$ on $\mathbb{P}(\Omega(2))$. We denote it by D. On the other hand by H we denote the hyperplane class from $\mathbb{P}^{2n+1}$; it is the relative $\mathcal{O}(1)$ on the projectivization of the universal bundle over $Grass_2(V)$ which makes the incidence variety $Fl_{1,2}(V)$. Let $\sum_{i=0}^{2n+1} c_i H^i$, with $c_i \in \mathbb{Z}$ be the Chern class of $\Omega(2)$. The numbers c_i can be calculated from the Euler sequence. By Leray-Hirsch formula we have

$$(3.4) \qquad\qquad H^{2*}(\mathbb{P}(\Omega(2)) \simeq \mathbb{Q}[H, D]/(H^{2n+2}, f(H, D))$$

where the polynomial f is given by the formula:

$$(3.5) \qquad\qquad f(H, D) = \sum_{i=0}^{2n+1} (-1)^i c_i H^i D^{2n+1-i} \,.$$

Note that $c_{2n+1} = 0$ since the bundle $\Omega(2)$ has a nowhere vanishing section. We can write

$$(3.6) \qquad\qquad f(H, D) = f_0(H, D) \cdot D$$

for some homogeneous polynomial $f_0 \in \mathbb{Z}[H, D]$ of degree $2n$. Comparing the formula (3.6) with the splitting sequence (3.2) we see that class of the fiber $Z \cong \mathbb{P}^{2n+1}$ in $H^{4n}(\mathbb{P}(\mathcal{N} \oplus \mathcal{O}))$ via the isomorphism (3.4) it can be identified with $f_0(H, D)$. In particular it is annihilated by the class D. Since H is the class of the relative $\mathcal{O}(1)$ on $Fl_{1,2}(V)$ treated as $\mathbb{P}^1$ over $Grass_2(V)$ we can write

$$[Z] = \varphi_t^*(\alpha) + \varphi_t^*(\beta) \smile H \,,$$

for some classes $\alpha \in \mathrm{H}^{4n}(Grass_2(\mathbb{C}^{2n+2}))$ and $\beta \in \mathrm{H}^{4n-2}(Grass_2(\mathbb{C}^{2n+2}))$. Since $[Z] \smile D = 0$, the classes α and β are annihilated by D. Applying Hard Lefschetz theorem we conclude that α belongs to the primitive cohomology and $\beta = 0$. In terms of Schubert classes α is proportional to $\sum_{i=0}^{n}(-1)^i S_{[n+i,n-i]}$. Here $S_{[n+i,n-i]}$ is the cohomology class of the Schubert variety defined by the partition $[n+i, n-i]$. This shows that $[Z] = \varphi_t^*(\alpha)$ and α is not effective.

In the example above we observe that the cohomology of Y_0 can be smaller than that of a general fiber. In fact

$$\varphi_0^*(\mathrm{H}^*(Y_0)) \subseteq \varphi_t^*(\mathrm{H}^*(Y_t)).$$

This is a general rule, but it rather concerns the image of $\mathrm{H}^*(Y_0)$ in $\mathrm{H}^*(X_0)$. Suppose Δ is a disc in $\mathbb{C}$. We have a commutative diagram:

(3.7)
$$
\begin{array}{ccccc}
\mathrm{H}^*(X_0) & \xleftarrow{\simeq} & H^*(\mathcal{X}) & \xrightarrow{\simeq} & H^*(X_t) \\
\varphi_0^* \big\uparrow & & \varphi^* \big\uparrow & & \varphi_t^* \big\uparrow \\
\mathrm{H}^*(Y_0) & \xleftarrow{\simeq} & \mathrm{H}^*(\mathcal{Y}) & \longrightarrow & \mathrm{H}^*(Y_t)
\end{array}
$$

In general, for fibrations over $\Delta \setminus \{0\}$ there is a monodromy operator (usually denoted by T) given by the action of the generator of $\pi_1(\Delta \setminus \{0\}, t) \simeq \mathbb{Z}$ on $\mathrm{H}^*(Y_t)$. The image of

$$\mathrm{H}^*(Y_0) \to \mathrm{H}^*(Y_t)$$

is contained in the invariant subspace of monodromy. By the "invariant cycle theorem" [4], [11] if $\mathcal{Y}$ is a Kähler manifold then we have an equality

$$\mathrm{im}\,(\mathrm{H}^*(Y_0) \to \mathrm{H}^*(Y_t)) = \mathrm{H}^*(Y_t)^T.$$

For the maps $\mathcal{Y} \to \Delta$ obtained by contraction of a smooth map $\varphi : \mathcal{X} \to \Delta$ the monodromy is trivial, since the cohomology of Y_t embeds into the cohomology of X_t where clearly the monodromy is trivial. Hence $\varphi_0(\mathrm{H}^*(Y_0)) = \varphi_t(\mathrm{H}^*(Y_t))$. Therefore by lemma 2.2 we get the following.

Corollary 3.3. *With the notation of proposition 1.2 assume (2) and that $\mathcal{Y}$ is smooth. Then the dimension of every fiber of φ_0 does not exceed $\dim X_0 - \dim Y_0$.*

We conclude with the remark that further understanding deformation in the context of Mori contraction undoubtedly leads to study the vanishing cycle sheaf on the contracted space $\mathcal{Y}$.

References

[1] I. N. Bernšteĭn, I. M. Gel′fand, and S. I. Gel′fand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973.

[2] André Blanchard. Sur les variétés analytiques complexes. *Ann. Sci. Ecole Norm. Sup. (3)*, 73:157–202, 1956.

[3] Armand Borel. *Topics in the homology theory of fibre bundles*, volume 1954 of *Lectures given at the University of Chicago*. Springer-Verlag, Berlin-New York, 1967.

[4] C. H. Clemens. Degeneration of Kähler manifolds. *Duke Math. J.*, 44(2):215–290, 1977.

[5] P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.*, (35):259–278, 1968.

[6] Takao Fujita. On polarized manifolds whose adjoint bundles are not semipositive. In *Algebraic geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 167–178. North-Holland, Amsterdam, 1987.

[7] Alexander Grothendieck. Torsion homologique et sections rationelles. In *Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications*, volume 3, pages exp no 5, 1–29. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958.

[8] Jun-Muk Hwang and Ngaiming Mok. Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. *Invent. Math.*, 131(2):393–418, 1998.

[9] Jun-Muk Hwang and Ngaiming Mok. Deformation rigidity of the rational homogeneous space associated to a long simple root. *Ann. Sci. École Norm. Sup. (4)*, 35(2):173–184, 2002.

[10] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[11] David R. Morrison. The Clemens-Schmid exact sequence and applications. In *Topics in transcendental algebraic geometry (Princeton, N.J., 1981/1982)*, volume 106 of *Ann. of Math. Stud.*, pages 101–119. Princeton Univ. Press, Princeton, NJ, 1984.

[12] Gianluca Occhetta, Luis E. Solá Conde, Kiwamu Watanabe, and Jarosław A. Wiśniewski. Fano manifolds whose elementary contractions are smooth $\mathbb{P}^1$-fibrations. to appear in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Preprint arXiv:1407.3658, DOI: 10.2422/2036-2145.201508_007.

[13] Christian Okonek, Michael Schneider, and Heinz Spindler. *Vector bundles on complex projective spaces*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 2011. Corrected reprint of the 1988 edition, With an appendix by S. I. Gelfand.

[14] Boris Pasquier and Nicolas Perrin. Local rigidity of quasi-regular varieties. *Math. Z.*, 265(3):589–600, 2010.

[15] Yum Tong Siu. Errata: "Nondeformability of the complex projective space" [J. Reine Angew. Math. **399** (1989), 208–219; MR1004139 (90h:32048)]. *J. Reine Angew. Math.*, 431:65–74, 1992.

[16] Hirosi Toda. On the cohomology ring of some homogeneous spaces. *J. Math. Kyoto Univ.*, 15:185–199, 1975.

[17] Jarosław A. Wiśniewski. On deformation of nef values. *Duke Math. J.*, 64(2):325–332, 1991.

[18] Jarosław A. Wiśniewski. Cohomological invariants of complex manifolds coming from extremal rays. *Asian J. Math.*, 2(2):289–301, 1998.

[19] Jarosław A. Wiśniewski. Rigidity of the Mori cone for Fano manifolds. *Bull. Lond. Math. Soc.*, 41(5):779–781, 2009.

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