# LINK PATTERNS AND ELLIPTIC HECKE ALGEBRA

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ABSTRACT. We compare the following three families of geometric objects: Schubert varieties in flag manifolds, matrix Schubert varieties, and Borel orbits of 2-nilpotent matrices. The first family is governed by permutations, the second by partial permutations, and the last one by *link patterns*. These geometric objects admit characteristic classes in equivariant elliptic cohomology obtained within the framework created by Borisov and Libgober. We construct a Hecke-type algebra for computing elliptic classes and extend its action to include partial permutations and linking patterns. A uniform point of view facilitates a better understanding of duality.

## 1. INTRODUCTION

Elliptic genus and related characteristic classes were studied since 80-ties for smooth manifolds in connection with formal group laws. It served as a tool to construct a map from the cobordism ring to the ring of modular forms. Significant extension of the theory to singular algebraic varieties started with the work of Borisov and Libgober, [BL03]. Their elliptic genus is a deformation of Hirzebruch $\chi_y$-genus. It is defined only for a class of varieties admitting mild singularities. The same applies for the underlying characteristic classes. The elliptic characteristic classes specialize to the motivic Chern classes, hence also to Chern-Schwartz-MacPherson classes. To apply Borisov-Libgober construction for Schubert varieties in a flag variety it is necessary to introduce a parameter deforming the boundary divisor of the Schubert cell, because the boundary has too bad singularities in general. The deforming parameter can be identified with a fractional line bundle, an element of the rational Picard group.

The idea of applying Hecke-type algebra to compute characteristic classes of Schubert varieties originates from [AM16, AMSS23, AMSS19], it was adapted for elliptic classes in [RiW20]. Recently further reformulation and development appeared in [LZZ23, ZZ23]. Our aim is to describe the algebra which governs computations of the elliptic characteristic classes of Schubert varieties and extend results to matrix Schubert varieties and Borel orbits of 2-nilpotent matrices. Separating purely algebraic properties from the particularities of the case of the flag variety allows to reveal a simple form of the elliptic algebra and understand the dependence of parameters.

Our construction is an extension of the methods worked out for K-theoretic classes in [RuW22] and [KW23]. We will define elements of the equivariant elliptic cohomology of $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^m)$

associated to Borel orbits of 2-nilpotent matrices. The set of 2-nilpotent matrices contains an important subset, consisting of upper-triangular block $(n \times n)$–matrices (when $m = 2n$). This allows to consider matrix Schubert varieties as a special case of Borel orbits of 2-nilpotent matrices. Our elliptic classes specialize to the K-theoretic twisted motivic Chern classes of the matrix Schubert varieties defined in [KW23]. Moreover the elliptic classes, after the normalization (essentially the same as in [RTV19] applied to weight functions) become to the elliptic classes of the Schubert varieties, which are the elliptic stable envelopes for maximal torus action, in the sense of [AO21].

In our calculus we implicitly apply localization theorem for the torus action. Thus the localized equivariant elliptic cohomology is the domain of our calculus. An element of the localized equivariant cohomology of $\mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^m)$ is a section of a line bundle over $E^{m+1}$ and depends additionally on the dynamical parameters and a free variable $h$. All together is considered as a section of a line bundle over a bigger product of elliptic curves. We might formally take a direct sum over all relevant vector bundles, as in [LZZ23], but we do not need it, since we only consider pure elements - sections of an individual line bundle.

Our main result consists of description of an algebra of Demazure type operations satisfying braid relations with dynamical parameters. In fact a version of elliptic Demazure operations was introduced in [RiW20]. Additionally we have *the flip relation* (12) proven geometrically in §16. We apply this algebra to define elliptic classes of 2-nilpotent Borel orbits. These elliptic classes unify:

- elliptic weight functions of [RTV19],
- elliptic classes of Schubert varieties, [RiW20]
- twisted motivic Chern class, [KW23],
- CSM and motivic Chern classes of upper-triangular square-zero B-orbits, [RuW22].

Embedding $\mathrm{End}(\mathbb{C}^n)$ into $\mathrm{End}(\mathbb{C}^{2n})$ as upper left corner matrices allows to consider matrix Schubert varieties as a special case of 2-nilpotent $B$-orbits. We obtain a family of elliptic functions, which satisfy two recursions related to left and right Demazure-Lusztig operations (or equivalently R-recursion and Bott Samelson recursion). This uniform point of view sheds a light on duality proven in [RiW22].

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## Contents

## 2. Background

One of the basic themes of Schubert calculus is to link homological invariants of Schubert varieties with characteristic classes of tautological bundles. The best known example of such connection is the inductive procedure of computing fundamental classes applying divided differences. The Schubert varieties in the complete flag variety $\mathcal{F}\ell_n = \mathrm{GL}_n / \mathrm{B}_n$ are indexed by permutations $w \in \mathfrak{S}_n$: the Schubert variety $X_w$ is the closure of the $\mathrm{B}_n$-orbit of the permutation matrix $M_w$. Here $\mathrm{B}_n$ denotes the Borel subgroup of $\mathrm{GL}_n$, which we choose to be the group of invertible, upper-triangular matrices. The cohomology ring of $\mathcal{F}\ell_n$ is generated by the first Chern classes of the tautological bundles. Denote by $\kappa$

$$\kappa : \mathbb{Z}[y_1, y_2, \ldots, y_n] \twoheadrightarrow H^*(\mathcal{F}\ell_n)$$

the corresponding surjection. Bernstein, Gelfand and Gelfand [BGG73] have applied the divided difference operators acting on polynomials

$$(\partial_i f)(y_1, y_2, \ldots, y_n) = \frac{f(y_1, \ldots, y_i, y_{i+1}, \ldots, y_n) - f(y_1, \ldots, y_{i+1}, y_i, \ldots, y_n)}{y_i - y_{i+1}}$$

to compute inductively the fundamental classes $[X_w]$. If a polynomial $f_w$ represents $[X_w]$, i.e.,

$$\kappa(f) = [X_w],$$

then

(1) $$\kappa(\partial_w(f)) = \begin{cases} [X_{ws_i}] & \text{if } \dim(X_{ws_i}) > \dim(X_w) \\ 0 & \text{if } \dim(X_{ws_i}) < \dim(X_w) \end{cases}.$$

Here $s_i$ is the elementary transposition $(i, i+1)$ for $0 < i < n$. The operations $\partial_i$ generate an algebra, called nil-Hecke algebra. The following relations are satisfied

$$\partial_i \partial_j = \partial_j \partial_i, \quad \text{if } |i - j| > 1,$$

$$\partial_i \partial_j \partial_i = \partial_j \partial_i \partial_j, \quad \text{if } |i - j| = 1 \qquad \text{(braid relation)},$$

$$\partial_i^2 = 0, \qquad\qquad\qquad\qquad \text{(quadratic relation)}.$$

The action of the operations $\partial_i$ can serve to compute fundamental classes in the torus–equivariant cohomology $H_{\mathbf{T}}^*(\mathcal{F}\ell_n)$. The equivariant cohomology admits the Borel presentation

(2)    $\kappa : \mathbb{Z}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n] \twoheadrightarrow H_{\mathbf{T}}^*(\mathcal{F}\ell_n) \simeq \mathbb{Z}[x_1, x_2, \ldots, x_n] \otimes_{S_n} \mathbb{Z}[y_1, y_2, \ldots y_n]$,

where $S_n$ is the ring of symmetric polynomials in $n$ variables. The $x$-variables, called the equivariant variables, are generators of $H_{\mathbf{T}}^*(pt)$. The divided differences act on both sets of variables. The operations acting on $x$-variables are denoted by $\partial_i^x$ and the operations acting on $y$-variables are denoted by $\partial_i^y$. In equivariant cohomology the formula (1) is satisfied for the operations $\partial_i^y$ while for the divided difference acting on $x$-variables we have

(3)                    $\kappa(\partial^x(f_w)) = \begin{cases} -[X_{s_i w}] & \text{if } \dim(X_{s_i w}) > \dim(X_w) \\ 0 & \text{if } \dim(X_{s_i w}) < \dim(X_w) \end{cases}$,

see [IMN11, Theorem 1.1]. The question arises: is there a geometric interpretation of the operations $\partial_i^x$ and $\partial_i^y$ acting on polynomials, not passing to the quotient algebra? It turns out that with a suitable choice of the starting point the divided difference operations compute the fundamental classes of the matrix Schubert varieties, which are closures of the orbit $\mathrm{B}_n M_w \mathrm{B}_n \subset \mathrm{End}(\mathbb{C}^n)$, see [FR03]. The equivariant cohomology

$$H_{\mathbf{T} \times \mathbf{T}}^*(\mathrm{End}(\mathbb{C}^n)) \simeq H_{\mathbf{T} \times \mathbf{T}}^*(pt) \simeq \mathbb{Z}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_n]$$

and the operations $\partial_i^y$, $\partial_i^x$ reflect certain geometric operations on matrices. Further we mix the $x$-variables with $y$-variables setting $x_{i+n} = y_i$ for $i = 1, 2, \ldots n$. We obtain a nil-Hecke algebra with generators $\partial_i^x$ for $i = 1, 2, \ldots, 2n - 1$. Its geometric meaning was described in [KZJ07, KZJ14] and [RuW22] in the way described in the next two sections.

## 3. Combinatorics governing geometry

We embed $\mathrm{End}(\mathbb{C}^n)$ into $\mathrm{End}(\mathbb{C}^{2n})$

$$\iota : A \mapsto \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}.$$

The $2n \times 2n$ matrix $\iota(A)$ is 2-nilpotent, i.e. $\iota(A)^2 = 0$. We observe that the image of the matrix Schubert cell $\mathrm{B}_n M_w \mathrm{B}_n$ is equal to the $\mathrm{B}_{2n}$ orbit of $\iota(A)$ with $\mathrm{B}_{2n}$ acting by conjugation. Therefore instead of $\mathrm{B}_n \times \mathrm{B}_n$ orbits in $\mathrm{End}(\mathbb{C}^n)$ we study $\mathrm{B}_m$ orbits in the set of 2-nilpotent matrices

$$\left\{ N \in \mathrm{End}(\mathbb{C}^m) \mid N^2 = 0 \right\}.$$

In general we do not assume that $m$ is even. By [BR12], [BP19, Th. 7.3.1] there are finitely many $B_m$ orbits. Each orbit contains exactly one matrix of a particular shape: there is at most one nonzero entry (and it is normalized to 1) at each column and row. We represent the orbits by link patterns.

**Definition 3.1.** *We fix an integers $r \geq 0$ and $m \geq 2r$. A link pattern of rank $r$ is a set of pairs $\{(a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)\}$, such that $a_i, b_i \in \{1, 2, \ldots, m\}$ are all different numbers.*

We read the link pattern $\{(7, 1), (8, 2)\}$ as $7 \mapsto 1$, $8 \mapsto 2$:

$$\mathcal{P}_{8,2}^{\min} = \; 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 \; .$$
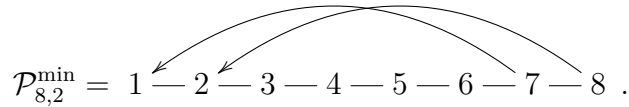
It represents the orbit of the matrix

$$N_{8,2}^{\min} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This orbit has minimal dimension among $B_8$ orbits in $\mathrm{End}(\mathbb{C}^8)$ of 2-nilpotent matrices of rank 2. The link pattern $\{(2, 5), (7, 4)\}$

$$\mathcal{P} = \; 1 - 2 - 3 - 4 - 5 - 6 - 7 - 8$$
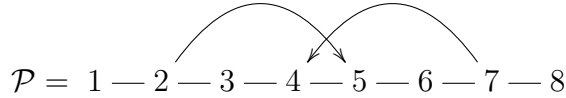
represents the orbit of the matrix

$$M_w N_{8,2}^{\min} M_w^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $w \in \mathfrak{S}_8$. In what follows we will denote the conjugation action $M_w N M_w^{-1}$ by $w \cdot N$.

We summarise the connection between geometry and combinatorics with the following table

Permutations $\longleftrightarrow$ $\begin{pmatrix} \text{Schubert varieties:} \\ B\text{-orbits in } GL_n/B \text{ or} \\ (B \times B)\text{-orbits in } GL_n \end{pmatrix}$

Partial permutations $\longleftrightarrow$ $\begin{pmatrix} \text{matrix Schubert varieties:} \\ (B \times B)\text{-orbits in } \mathrm{End}(\mathbb{C}^n) \end{pmatrix}$

Directed link patterns



$\longleftrightarrow$ $\begin{pmatrix} B\text{-orbits of 2-nilpotent} \\ \text{upper-triangular matrices} \end{pmatrix}$

Link patterns



$\longleftrightarrow$ $\begin{pmatrix} B\text{-orbits of 2-nilpotent} \\ \text{matrices} \end{pmatrix}$
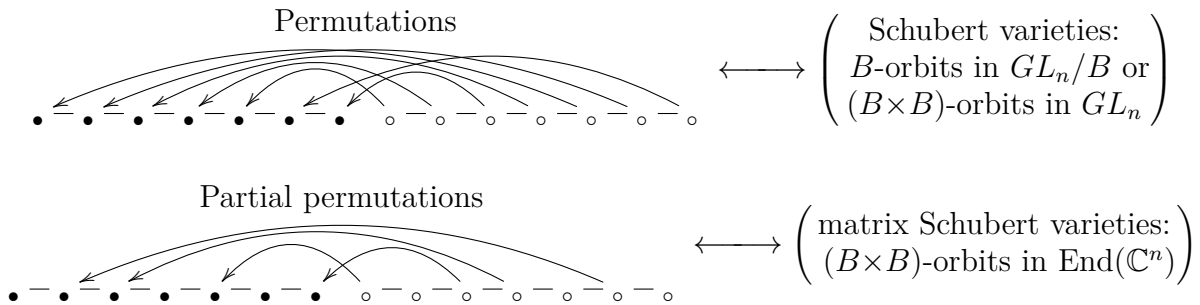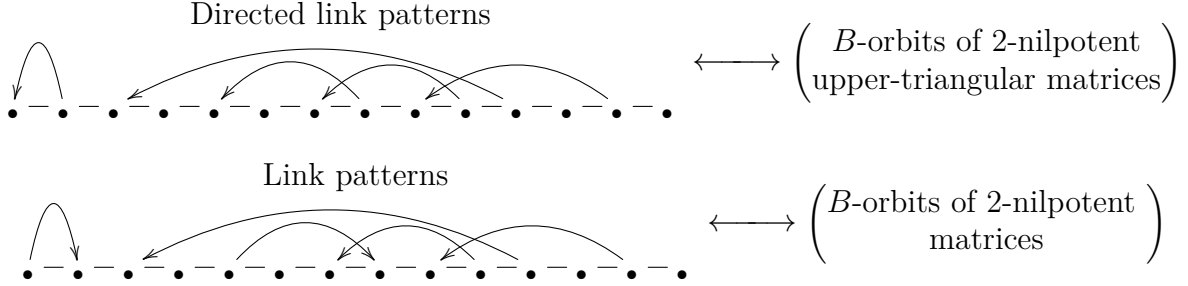
We need link patterns with labelled arrows hence we consider sequences of pairs (source, target), instead of sets of pairs. Formally we consider injective functions from $\{1, 2, \ldots, r\}$ to the set of pairs:

**Definition 3.2.** *We fix an integers $r \geq 0$ and $m \geq 2r$. A labelled link pattern of rank $r$ is a sequence of pairs $\mathcal{P} = ((a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r))$, such that $a_i, b_i \in \{1, 2, \ldots, m\}$ are all different numbers.*

Clearly, the product of permutation groups $\mathfrak{S}_r \times \mathfrak{S}_m$ acts on the set of all functions

$$\{1, 2, \ldots, r\} \longrightarrow \{1, 2, \ldots, m\}^2 \,,$$

preserving those, which represent link patterns. The permutations of the set $\{1, 2, \ldots, r\}$ will be denoted with the superscript $\mu$.

## 4. CHARACTERISTIC CLASSES OF 2-NILPOTENT ORBITS

The study of homological invariants of $B$-orbits in 2-nilpotent matrices was initiated by [KZJ07, KZJ14], where homology and K-theory classes were presented as an effect of an action of certain algebras. Further results of [RuW22] apply to more sophisticated invariants: Chern-Schwartz-MacPherson classes and motivic Chern classes of [BSY10]. The table below summarizes the Demazure-type operations in the convention used in [RuW22]:

homology fundamental class $\qquad \beta_i^{\text{coh}} f = \frac{1}{x_{i+1} - x_i} f + \frac{1}{x_i - x_{i+1}} s_i f \,, \qquad\qquad (\beta_i^{\text{coh}})^2 = 0$

K-theory fundamental class $\qquad \beta_i f = \frac{1}{1 - e^{x_{i+1} - x_i}} f + \frac{1}{1 - e^{x_i - x_{i+1}}} s_i f \,, \qquad\qquad \beta_i^2 = \beta_i$

CSM classes $\qquad \mathcal{T}_i^{\text{coh}}(f) = \frac{1}{x_{i+1} - x_i} f + \frac{1 + x_i - x_{i+1}}{x_i - x_{i+1}} s_i f \,, \qquad (\mathcal{T}^{\text{coh}})^2 = id$

motivic Chern classes $\qquad \mathcal{T}_i(f) = \frac{(1+y)e^{x_{i+1} - x_i}}{1 - e^{x_{i+1} - x_i}} f + \frac{1 + y\, e^{x_i - x_{i+1}}}{1 - e^{x_i - x_{i+1}}} s_i f \,, \quad (\mathcal{T}_i + id)(\mathcal{T}_i + y\, id) = 0.$

Note that we have corrected the sign in the formula (2)

$$\beta_i^{\text{coh}} = -\partial_i \,.$$

The above operations act on the corresponding classes of $B$-orbits of 2-nilpotent matrices belonging to $H_{\mathbf{T}}^*(End(\mathbb{C}^m))$, $K_{\mathbf{T}}(End(\mathbb{C}^m))$ or $K_{\mathbf{T}}(End(\mathbb{C}^m))[y]$. In the next section we introduce the elliptic functions on which the calculus of elliptic classes is based.

## 5. Theta, $F$ and $\delta$ functions

Let us fix notation and describe the basic properties of the elliptic functions which we will use. We start with the Jacobi theta function. For $x, \tau \in \mathbb{C}$, $\mathrm{im}(\tau) > 0$ Let $q = e^{2\pi i \tau}$.

$$\theta_\tau(x) = 2q^{\frac{1}{8}} \sin(\pi x) \prod_{n=1}^{\infty} (1 - q^n)\left(1 - q^n e^{2\pi i x}\right)\left(1 - q^n/e^{2\pi i x}\right).$$

Our theta function differs from the classical one by a constant factor, which will cancel out in further considerations. Moreover in some sources the argument $x$ is re-scaled. We have quasi-periodicity relations

$$\theta_\tau(x + 1) = -\theta_\tau(x),$$
$$\theta_\tau(x + \tau) = -q^{-1/2}\, e^{-2\pi i x}\, \theta_\tau(x).$$

The function $F_\tau(x, y)$ is defined by the formula

$$F_\tau(x, y) = \frac{\theta_\tau'(0)\, \theta_\tau(x + y)}{\theta_\tau(x)\, \theta_\tau(y)}$$

It is a meromorphic function on $\mathbb{C}^2$. The argument $\tau$ is treated as a parameter. Obviously

$$F_\tau(x, y) = F_\tau(y, x)$$

and

(4) $$F_\tau(-x, -y) = -F_\tau(x, y),$$

The quasi-periodicity relations take form

$$F_\tau(x + 1, y) = F_\tau(x, y),$$
$$F_\tau(x + \tau, y) = e^{-2\pi i y}\, F_\tau(x, y).$$

We will use the function $\delta$ satisfying

$$F_\tau(x, y) = \delta_\tau(e^{2\pi i x}, e^{2\pi i y}).$$

since $F_\tau$ descends to a function on $\mathbb{C}^* \times \mathbb{C}^*$. The function $\delta_\tau$ depends also on $q = e^{2\pi i \tau}$, but we keep it fixed and do not indicate $\tau$ in the notation. We can treat $\delta$ as an element of $\mathbb{Z}(a, b)[[q]]$. In fact except from the coefficient of $q^0$ the remaining coefficients of the $q$–expansion are Laurent polynomials in $a$ and $b$. The expansion is of the form

$$\delta(x, y) = \frac{1 - x^{-1}y^{-1}}{(1 - x^{-1})(1 - y^{-1})} + q(x^{-1}y^{-1} - xy) + q^2(x^{-2}y^{-1} + x^{-1}y^{-2} - x^2 y - xy^2) + \ldots$$

$$= \frac{1 - x^{-1}y^{-1}}{(1 - x^{-1})(1 - y^{-1})} + \sum_{n=1}^{\infty} q^n \sum_{k\ell=n} (x^{-k}y^{-\ell} - x^k y^\ell),$$

see [Zag91, §3]. The properties of the function $\delta_\tau$ are described in [MW21]. The modular properties (with respect to the transformation $\tau \mapsto -1/\tau$) will not play a significant role in the rest of the paper.

## 6. Bundle type gradation

Meromorphic functions on $\mathbb{C}^n$ can be interpreted as sections of line bundles on a product of elliptic curves, provided that they enjoy certain quasi-periodic relations. Let us recall that any $n \times n$ symmetric matrix with integer entries defines a line bundle over the $n$-fold product of a fixed elliptic curve $E = \mathbb{C}/\langle 1, \tau \rangle$. This bundle is the quotient of $\mathbb{C}^n \times \mathbb{C}$ by an action of $\mathbb{Z}^n \oplus \mathbb{Z}^n$. The action is defined by

$$(k, \ell) \cdot (z, v) = \left( z + k + \ell\tau \,,\, (-1)^{k \cdot k}(-q^{1/2})^{\ell \cdot \ell} e^{-2\pi i \ell \cdot z} v \right)$$

Here the product $\cdot$ in $\mathbb{Z}^n$ is defined by the given matrix and extended to $\mathbb{C}^n$. According to our convention $q^{1/2} = e^{\pi i \tau}$. With this interpretation the theta function is a section of the line bundle over $E$ associated to the $1 \times 1$ matrix $[\,1\,]$.

Instead of matrices it is more convenient to perform calculus of quadratic forms. The matrix $[\,1\,]$ corresponds to the quadratic form $\frac{1}{2}x^2$. We will say that $\theta(x)$ is of the type $\frac{1}{2}x^2$ and write

$$\mathrm{type}(\theta(x)) = \tfrac{1}{2}x^2 \,.$$

Products of functions result in addition of quadratic forms and quotients give differences. For example the function $F(x, y) = \frac{\theta'(0)\theta(x+y)}{\theta(x)\theta(y)}$ is a section of the bundle associated to the quadratic form

$$\tfrac{1}{2}\left( (x+y)^2 - x^2 - y^2 \right) = xy \,.$$

We write $\mathrm{type}(F(x, y)) = xy$.

We say that a combination of products of theta functions and its inverses is *pure* if all summands have the same type. A pure function defines a section of the associated line bundle.

**Important convention:** When a combination of variables appears as an argument of the $\delta$ function we use the multiplicative notation. This is to save space and to agree with the literature. In formulas involving quadratic forms we use additive notation. Multiplication of arguments of $\delta$ corresponds to addition of variables in types, for example

$$\mathrm{type}\left( \delta\Big( \frac{x_{i+1}}{x_i}, \frac{\mu_{i+1}}{\mu_i} \Big) \right) = (x_{i+1} - x_i)(\mu_{i+1} - \mu_i) \,.$$

## 7. Elliptic characteristic classes of Schubert varieties

The elliptic classes of Schubert varieties in the complete flag varieties were introduced in [RiW20]. The elliptic classes depend on an additional parameter, which, in the case of the full flag variety, is a fractional bundle, i.e. an element of

$$\mathrm{Pic}(\mathcal{F}\ell_n) \otimes \mathbb{Q} \simeq \mathfrak{t}^*_{\mathbb{Q}} \,.$$

The variables corresponding to the standard coordinates of the torus $\mathbf{T} \subset \mathrm{GL}_n$ are denoted by $\mu_i$. They are functionals on $\mathfrak{t}^*$. Our construction allows to consider combinations of the variables $\mu_i$ with complex coefficients although in the original definition of the boundary divisor had rational coefficients. There are remaining variables: the equivariant variables $x_i \in \mathfrak{t}^*$ and the topological variables $y_i \in K_{\mathbf{T}}(\mathcal{F}\ell_n)$ representing the tautological bundles. The corresponding Demazure type

operators have a parameter built in. Dual recursions are obtained by two families of operations. The localised[1] elliptic classes are subjects to the relations

(i) Bott-Samelson recursion: if $\ell(ws_i) > \ell(w)$

$$(5) \qquad \mathcal{E}\!\ell(X_{ws_i}) = \delta(\tfrac{y_{i+1}}{y_i}, \tfrac{\mu_{i+1}}{\mu_i}) \cdot s_i^\mu \mathcal{E}\!\ell(X_w) + \delta(\tfrac{y_{i+1}}{y_i}, h) \cdot s_i^y s_i^\mu \mathcal{E}\!\ell(X_w).$$

(ii) R-matrix recursion: if $\ell(s_i w) > \ell(w)$

$$(6) \qquad \mathcal{E}\!\ell(X_{s_i w}) = \delta(\tfrac{x_{i+1}}{x_i}, \tfrac{\mu_{w^{-1}(i+1)}}{\mu_{w^{-1}(i)}}) \cdot \mathcal{E}\!\ell(X_w) + \delta(\tfrac{x_i}{x_{t+1}}, h) \cdot s_i^x \mathcal{E}\!\ell(X_w) \,,$$

Here the transposition $s_i^x$ acts on $x$ variables, $s_i^y$ acts on $y$-variables, $s_i^\mu$ acts on $\mu$ variables.

There is certain asymmetry in the formulas (5–6), which is caused by different roles of variables. This asymmetry disappears when we do not divide $\mathrm{GL}_n$ by the Borel group. The full algebra of operation can be encapsulated in the construction described below in Section §9

## 8. Elliptic classes of 2-nilpotent orbits – the beginning

The elliptic characteristic classes of Borisov and Libgober are defined for certain class of singular algebraic varieties. The construction is generalized to pairs $(X, D)$, where $D$ is a divisor and $X$ is smooth away from the support of the divisor $D$. See [BL03], [RiW20, §2] for the definition. The construction involves a resolution of singularities and the pull-back of the relative canonical divisor $K_X + D$. In particular it is assumed that $K_X + D$ is a $\mathbb{Q}$-Cartier divisor. We do not have enough information about the canonical divisors of $B$-orbit closures of 2-nilpotent matrices, therefore we chose a different way.

Our geometric objects are the closures of the orbits

$$X_{m,r}^w = \overline{\mathrm{B}_m\, w \cdot N_{m,r}^{\min}} \,,$$

where $w \in \mathfrak{S}_m$ is a permutation and $N_{m,r}^{\min}$ is the minimal matrix of the size $m$ and rank $r$

$$(7) \qquad N_{m,r}^{\min} = \begin{bmatrix} 0\,0\,\ldots\,0\,1\,0\,\ldots\,0 \\ 0\,0\,\ldots\,0\,0\,1\,\ldots\,0 \\ \vdots\quad\ddots \\ 0\,0\,\ldots\,0\,0\,0\,\ldots\,1 \\ \vdots \\ 0\,0\,\ldots\,0\,0\,0\,\ldots\,0 \end{bmatrix}, \quad 2r \le m$$

(there are $r$ entries with 1 on a parallel to the diagonal). Assume that $\dim(X_{m,r}^w) = \ell(w) + \dim(X_{m,r}^{\mathrm{id}})$. According to [BP19] the variety $X_{m,r}^w$ admits a desingularization of the form

$$(8) \qquad Z_{m,r}^{\underline{w}} = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell} \times_B X_{m,r}^{\mathrm{id}} \,,$$

where $\underline{w} = s_{i_1} s_{i_2} \ldots s_{i_\ell}$ is a reduced word representing $w$, and $P_i \subset \mathrm{GL}_m$ is the minimal parabolic subgroup corresponding to simple reflection $s_i$. We will consider the push forward of the elliptic class of $(Z_{m,r}^{\underline{w}}, D)$ with $D$ defined in the (unique) way guaranteeing the push-forward is pure. We will prove that the resulting class does not depend on the reduced word. Moreover, our construction allows to consider non-reduced words and permutations, not necessarily defining a

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[1]The quotients of the elliptic class by the Euler classes of the tangent bundle.

birational map $Z_{m,r}^{\underline{w}} \to X_{m,r}^{w}$. It allows to treat elliptic classes of link patterns as effects of an action of the algebra defined in §9.

Let us take a look at beginning of the induction. The variety $X_{m,r}^{\min} = X_{m,r}^{\mathrm{id}}$ is just a vector subspace of $\mathrm{End}(\mathbb{C}^m)$ described by

$$a_{i,j} = 0 \quad \text{for } i = 1, 2, \ldots m, \ 1 \le j < m - r + i,$$

where $a_{i,j}$ are the entries of the matrix. The boundary divisor of $\mathrm{B}_m \cdot N_{m,r}$ is given by

$$\prod_{i=1}^{r} a_{i,m-r+i} = 0, \qquad \begin{bmatrix} 0\,0\ldots 0 \bullet * \ldots * \\ 0\,0\ldots 0\,0 \bullet \ldots * \\ \vdots \quad \ddots \\ 0\,0\ldots 0\,0\,0 \ldots \bullet \\ \vdots \\ 0\,0\ldots 0\,0\,0 \ldots 0 \end{bmatrix}.$$

We attach a generic multiplicity $\lambda_i$ to the component of the boundary divisor

$$D_i = \{a_{i,m-r+i} = 0\}.$$

Let $\mu_i = h^{1-\lambda_i}$. The $\mathrm{B}_m$ orbits are preserved by the torus $\mathbf{T} = (\mathbb{C}^*)^m \times \mathbb{C}^*$. The first factor is the maximal torus of $\mathrm{GL}_m$ acting by conjugation on matrices, the second factor acts by scalar multiplication. The corresponding characters are denoted by $x_i$, $i = 1, 2, \ldots, m$ and $u$. We are at the position to apply directly the definition of Borisov and Libgober, or better use the formula [RiW20, formula (5)]. No desingularization is needed here. We obtain

$$(9) \qquad \mathscr{E}\ell\ell(X_{m,r}^{\min}, \sum_{i=1}^{r} \lambda_i D_i) = \prod_{i=1}^{r} \left( \delta(u \tfrac{x_i}{x_{m-r+i}}, \mu_i) \cdot \prod_{j=m-r+i+1}^{m} \delta(u \tfrac{x_i}{x_j}, h) \right).$$

This is our starting point. The remaining elliptic classes of Borel orbits are well defined due to the property of the Hecke algebra action, in particular the braid relations (11) and the flip relation (12). This is shown in §15. We avoid any analysis of the canonical divisor.

## 9. Action of elliptic Demazure operations on functions

We will define elliptic Demazure operators in an abstract way. They act on the space of meromorphic functions in $x_1, x_2, \ldots, x_m, h \in \mathbb{C}^*$ and additional variables denoted collectively by $\mu$. The unified elliptic Demazure operations are lifts and extensions of those defined in [RiW20]. The action on a meromorphic function $f$ is defined by the formula

$$(10) \qquad \mathfrak{C}_i^{\mu}(f) = \delta\left(\tfrac{x_{i+1}}{x_i}, \mu\right) f + \delta\left(\tfrac{x_i}{x_{i+1}}, h\right) s_i f, \qquad i = 1, 2, \ldots, m-1.$$

Here $s_i$ acts only on $x$-variables. The operations $\mathfrak{C}_i^{\mu}$ satisfy braid relations with parameters

$$(11) \qquad \mathfrak{C}_i^{\mu} \, \mathfrak{C}_{i+1}^{\mu/\nu} \, \mathfrak{C}_i^{\nu} = \mathfrak{C}_{i+1}^{\nu} \, \mathfrak{C}_i^{\mu/\nu} \, \mathfrak{C}_{i+1}^{\mu}.$$

The braid relation (Yang-Baxter equation) can be rewritten as

$$\mathfrak{C}_i^{\beta/\gamma} \, \mathfrak{C}_{i+1}^{\alpha/\gamma} \, \mathfrak{C}_i^{\alpha/\beta} = \mathfrak{C}_{i+1}^{\alpha/\beta} \, \mathfrak{C}_i^{\alpha/\gamma} \, \mathfrak{C}_{i+1}^{\beta/\gamma}$$

and represented by the following picture



The operations $\mathfrak{C}_i^\mu$ satisfy the following quadratic relation

$$\mathfrak{C}_i^\mu \, \mathfrak{C}_i^{1/\mu} = \delta(h, \mu)\delta(h, 1/\mu)\mathrm{id}\,.$$

After normalization

$$\overline{\mathfrak{C}}_i^\mu = \tfrac{1}{\delta(\mu, h)}\mathfrak{C}_i^\mu$$

we obtain

$$\overline{\mathfrak{C}}_i^\mu \, \overline{\mathfrak{C}}_i^{1/\mu} = \mathrm{id}\,.$$

It should be noted that the operations $\mathcal{T}_i^{\mathrm{coh}}$ and $\mathcal{T}_i$ of the table in §4 act on the polynomial ring, which is identified with $H_{\mathbf{T}}^*(pt)$, the operations $\beta_i$ and $\mathcal{T}_i$ act on the ring of Laurent polynomials (extended by the variable $y$) – which is identified with $K_{\mathbf{T}}(pt)[y]$. The natural domain of the operations $\overline{\mathfrak{C}}_i^\mu$ is the ring of meromorphic functions in $(x, \mu, h)$ which descend to sections of line bundles over the product of elliptic curve $E$. Additionally we need a variable $u$ which is not involved in the definition of $\mathfrak{C}_i^\mu$.

The proofs are given in Section §11.

**Remark 9.1** (Weyl group representation)**.** Let us restrict our attention to the case $m = 2n$, $r = n$. We identify the variables $x_{n+i} = y_i$ for $i \leq n$ with Chern classes of the tautological bundles on the flag variety, while characters of the maximal torus in $\mathrm{GL}_n$ remain to be denoted by $x_i$ for $i \leq n$. Let $\mathbf{T}^\vee$ be the dual torus, $\mu_i\colon \mathbf{T}^\vee \to \mathbb{C}^*$ the corresponding cocharacters. In [RiW20, RiW22] we have defined the operations on $Mero(\mathbf{T}^2 \times \mathbf{T}^\vee \times \mathbb{C}^*)$

$$\mathcal{C}_i = \mathfrak{C}_i^{\mu_i^\vee} \circ s_i^\mu\,,$$

where $s_i^\mu$ is the reflection acting on $\mathbf{T}^\vee$ inducing an action on the functions on $\mathbf{T}^\vee$. Explicitly:

$$\mathcal{C}_i(f)(z, \gamma, \mu) = \delta(\tfrac{\gamma_{i+1}}{\gamma_i}, \tfrac{\mu_{i+1}}{\mu_i})\, f(z, \gamma, s_i(\mu)) + \delta(\tfrac{\gamma_i}{\gamma_{i+1}}, h)\, f(z, s_i(\gamma), s_i(\mu))$$

It is shown in [RiW20, Theorem 5.1] that

$$\mathcal{C}_i \circ \mathcal{C}_i = \delta(h, \tfrac{\mu_{i+1}}{\mu_i})\delta(h, \tfrac{\mu_i}{\mu_{i+1}})\,\mathrm{id}$$

after reduction to elliptic cohomology of the flag variety. In fact the relation holds on the level of functions in $x_i$, $y_i$ and $\mu_i$. This is a special case of Theorem 11.1. After normalization we obtain an action of the permutation group, as in [ZZ22, Proposition 4.11].

## 10. Summary of the main results

For each labelled link pattern we will construct a meromorphic function in variables $x \in (\mathbb{C}^*)^m$, $\mu \in (\mathbb{C}^*)^r$ and $u, h \in \mathbb{C}^*$

$$\text{labelled link pattern } \mathcal{P} \mapsto \mathcal{E}\ell\ell(\mathcal{P})$$

with the following properties:

(i) the function $\mathcal{E}\ell\ell(\mathcal{P})$ has a pure type, i.e. it defines a section of a line bundle, as explained in §6),

(ii) $\mathfrak{C}_i^\nu(\mathcal{E}\ell\ell(\mathcal{P})) = \mathcal{E}\ell\ell(s_i\mathcal{P})$ for $\nu \in (\mathbb{C}^*)^r$ uniquely determined by the purity condition

(iii) for the minimal orbit of rank $r$ $\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})$ is given by the formula (9)

(iv) If $m = 2n$ and the link pattern represents a permutation $\sigma \in \mathfrak{S}_n$, then (a normalization of) $\mathcal{E}\ell\ell(\mathcal{P})$ represents the equivariant elliptic class of the Schubert variety $X_\sigma$.

(v) If $m = 2n - 1$ and the link pattern represents an injective map permutation $\{1, 2, \ldots, n - 1\} \to \{1, 2, \ldots, n\}$, then the elliptic class is equal to the elliptic weight function of [RTV19] (up to a change of variables and normalization).

(vi) $\mathcal{E}\ell\ell(\mathcal{P})$ is the push forward from Bott-Samelson resolution of the elliptic class in the sense of Borisov-Libgober for a unique choice of the boundary divisor guaranteeing purity.

The conditions (1)–(3) determine the function $\mathcal{E}\ell\ell(\mathcal{P})$.

It should be noted that a change of arrow labels results in permutation of $\mu$-variables in $\mathcal{E}\ell\ell(\mathcal{P})$. The key argument of the proof of independence of the choice of a resolution is based on an application of the reduced operations $\overline{\mathfrak{C}}_i^\mu$, for the right choice of $\mu$, forced by the purity condition. We obtain a $\mathfrak{S}_m$-action, hence we do not have to focus on the reduced words representing permutations. Moreover the change of purity types is well controlled:

$$\text{type}(\mathcal{E}\ell\ell(s_i\mathcal{P})) = s_i(\text{type}(\mathcal{E}\ell\ell(\mathcal{P})) - h\rho_m) + h\rho_m \,,$$

see Theorem 12.3. Here $\rho_m$ is the half sum of positive roots. This allows to write a ready to use formula for the type of $\mathcal{E}\ell\ell(w(\mathcal{P}_{m,r}^{\min}))$ for $w \in \mathfrak{S}_m$ in combinatorial terms.

We apply Borisov and Libgober results to show that the identity

$$(12) \qquad s_k^\mu \mathfrak{C}_k^{\mu_k/\mu_{k+1}}(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})) = \mathfrak{C}_{m-r+k}^{\mu_k/\mu_{k+1}}(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})) \qquad \text{for } 1 \le k < r \,.$$

holds. We call the resulting identity of elliptic functions *the flip relation*. Although this identity can be read from the four-term relation [RTV19, eq. (2.7)] we wish to give its geometric proof. For $m = 4$, $r = 2$ the identity is illustrated by the transformation of link patterns

(13)



Here $s_1^\mu$ is the transposition of labels of the link pattern. The flip identity for $m = 4$, $r = 2$ is proven in §16. It implies identities in higher dimensions, see the proof of Theorem 15.1.

We also need the Fay trisecant relation in the form [FRV07, Thm. 5.3] or [MW21, §4.1].

## 11. Verification of the braid and quadratic identities

Let us fix $r \geq 0$ and $m \geq 2r$. The operation $\mathfrak{C}_i^\mu$ defined by (10) acts on meromorphic functions on $(\mathbb{C}^*)^{m+r+2}$ (with coordinates $x_i, u, \mu_j, h$, $i = 1, 2, \ldots m$, $j = 1, 2, \ldots, r$). The operation $\mathfrak{C}_i^\nu$ depends on a character $\mu : (\mathbb{C}^*)^r \to \mathbb{C}$ written as a combination of basic characters $\mu_1, \mu_2, \ldots, \mu_r$. We assume that $\mu$ is not tautologically equal to 1, since at $\mu = 1$ the function $\delta(x, \mu)$ has a pole. Obviously $\mathfrak{C}_i^\mu$ commutes with $\mathfrak{C}_j^\mu$ if $|i - j| > 1$. We focus on $m = 3$, the general case follows.

**Theorem 11.1.** *The operations $\mathfrak{C}_i^\mu$ satisfy the twisted braid relations*

(14) $$\mathfrak{C}_1^\nu \, \mathfrak{C}_2^{\mu\nu} \, \mathfrak{C}_1^\mu = \mathfrak{C}_2^\mu \, \mathfrak{C}_1^{\mu\nu} \, \mathfrak{C}_2^\nu \,.$$

*Moreover*

(15) $$\mathfrak{C}_i^\mu \, \mathfrak{C}_i^{1/\mu} = \delta(h, \mu)\delta(h, 1/\mu) \,.$$

*Proof.* We adjust the notation to the one used in [RiW20]. We set $\nu = \frac{\mu_2}{\mu_1}$ and $\mu = \frac{\mu_3}{\mu_2}$ and we rewrite the braid relation:

$$\mathfrak{C}_1^{\mu_2/\mu_1} \, \mathfrak{C}_2^{\mu_3/\mu_1} \, \mathfrak{C}_1^{\mu_3/\mu_2} = \mathfrak{C}_2^{\mu_3/\mu_2} \, \mathfrak{C}_1^{\mu_3/\mu_1} \, \mathfrak{C}_2^{\mu_2/\mu_1} \,.$$

To check that relation we look at the coefficients of $f(x_i, x_j, x_k)$. We have to prove a number of identities. Comparing the coefficients of $f(x_1, x_2, x_3)$ we have to show

(16) $$\delta\left(\frac{x_1}{x_2}, h\right) \delta\left(\frac{x_2}{x_1}, h\right) \delta\left(\frac{x_3}{x_1}, \frac{\mu_3}{\mu_1}\right) + \delta\left(\frac{x_2}{x_1}, \frac{\mu_2}{\mu_1}\right) \delta\left(\frac{x_2}{x_1}, \frac{\mu_3}{\mu_2}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_3}{\mu_1}\right) =$$
$$= \delta\left(\frac{x_2}{x_3}, h\right) \delta\left(\frac{x_3}{x_2}, h\right) \delta\left(\frac{x_3}{x_1}, \frac{\mu_3}{\mu_1}\right) + \delta\left(\frac{x_2}{x_1}, \frac{\mu_3}{\mu_1}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_2}{\mu_1}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_3}{\mu_2}\right) \,.$$

This is exactly the equality [RiW20, (32)=(33)] or [RTV19, eq. (2.20)]. The comparison of the remaining coefficients leads to the equations

$$f(x_1, x_3, x_2): \quad \delta\left(\frac{x_2}{x_3}, h\right) \left(\delta\left(\frac{x_2}{x_1}, \frac{\mu_3}{\mu_2}\right) \delta\left(\frac{x_3}{x_1}, \frac{\mu_2}{\mu_1}\right) - \delta\left(\frac{x_2}{x_3}, \frac{\mu_3}{\mu_2}\right) \delta\left(\frac{x_3}{x_1}, \frac{\mu_3}{\mu_1}\right) - \delta\left(\frac{x_2}{x_1}, \frac{\mu_3}{\mu_1}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_2}{\mu_1}\right)\right) = 0 \,,$$

$$f(x_2, x_1, x_3): \quad \delta\left(\frac{x_1}{x_2}, h\right) \left(\delta\left(\frac{x_1}{x_2}, \frac{\mu_2}{\mu_1}\right) \delta\left(\frac{x_3}{x_1}, \frac{\mu_3}{\mu_1}\right) + \delta\left(\frac{x_2}{x_1}, \frac{\mu_3}{\mu_2}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_3}{\mu_1}\right) - \delta\left(\frac{x_3}{x_1}, \frac{\mu_3}{\mu_2}\right) \delta\left(\frac{x_3}{x_2}, \frac{\mu_2}{\mu_1}\right)\right) = 0 \,,$$

which follow from the blow-up relation [RiW20, Ex. 2.10]. The reaming coefficients are equal on the nose. The quadratic relation is equivalent to:

$$f(x_1, x_2): \quad \delta\left(\frac{x_1}{x_2}, h\right) \delta\left(\frac{x_2}{x_1}, h\right) + \delta\left(\frac{x_2}{x_1}, \frac{1}{\mu}\right) \delta\left(\frac{x_2}{x_1}, \mu\right) = \delta\left(\frac{1}{\mu}, h\right) \delta(\mu, h)$$

$$f(x_2, x_1): \quad \delta\left(\frac{x_1}{x_2}, h\right) \left(\delta\left(\frac{x_1}{x_2}, \mu\right) + \delta\left(\frac{x_2}{x_1}, \frac{1}{\mu}\right)\right) = 0 \,.$$

The first one again follows from the blow-up relation (see [RiW20, Ex. 2.10], [MW21, §4.1]) and the second one is the easiest, since it follows from anti-symmetry of $\delta$, see (4). $\quad\square$

It is convenient to normalize the operations $\mathfrak{C}_i^\mu$

$$\overline{\mathfrak{C}}_i^\mu = \tfrac{1}{\delta(\mu, h)} \mathfrak{C}_i^\mu \,.$$

Then

(17) $$\overline{\mathfrak{C}}_i^\mu \, \overline{\mathfrak{C}}_i^{1/\mu} = \mathrm{id} \, .$$

The operations $\overline{\mathfrak{C}}_i^\mu$ satisfy the braid relation

(18) $$\overline{\mathfrak{C}}_1^\nu \, \overline{\mathfrak{C}}_2^{\mu\nu} \, \overline{\mathfrak{C}}_1^\mu = \overline{\mathfrak{C}}_2^\mu \, \overline{\mathfrak{C}}_1^{\mu\nu} \, \overline{\mathfrak{C}}_2^\nu \, .$$

since

$$\overline{\mathfrak{C}}_1^\nu \, \overline{\mathfrak{C}}_2^{\mu\nu} \, \overline{\mathfrak{C}}_1^\mu = \tfrac{1}{\delta(\nu,h)\delta(\mu\nu,h)\delta(\mu,h)} \mathfrak{C}_1^\nu \, \mathfrak{C}_2^{\mu\nu} \, \mathfrak{C}_1^\mu$$

and

$$\overline{\mathfrak{C}}_2^\mu \, \overline{\mathfrak{C}}_1^{\mu\nu} \, \overline{\mathfrak{C}}_2^\nu = \tfrac{1}{\delta(\mu,h)\delta(\mu\nu,h)\delta(\nu,h)} \mathfrak{C}_2^\mu \, \mathfrak{C}_1^{\mu\nu} \, \mathfrak{C}_2^\nu \, .$$

## 12. Preserving purity

We consider meromorphic functions on $(\mathbb{C}^*)^{m+r+2}$ in variables $x_i$ $(i = 1, 2, \ldots, m)$, $u$, $h$, $\mu_j$ $(j = 1, 2, \ldots, r)$.

**Lemma 12.1.** *Suppose $f \neq 0$ is a pure meromorphic function on $(\mathbb{C}^*)^{m+r+2}$ of the type $\mathrm{type}(f)$, then $\mathfrak{C}_i^\mu(f)$ is pure if and only if*

$$\mu = \partial_i(\mathrm{type}(f)) - h \, .$$

*where*

$$\partial_i(\mathrm{type}(f)) = \frac{\mathrm{type}(f) - s_i \, \mathrm{type}(f)}{x_i - x_{i+1}} \, .$$

*Proof.* The formula (10) defining $\mathfrak{C}_i^\mu(f)$ has two summands. The first one has the type

(19) $$\mathrm{type}\left(\delta\left(\frac{x_{i+1}}{x_i}, \mu\right)f\right) = \mathrm{type}(f) + (x_{i+1} - x_i)\mu$$

and the second summand

(20) $$\mathrm{type}\left(\delta\left(\frac{x_i}{x_{i+1}}, h\right)s_i f\right) = s_i \, \mathrm{type}(f) + (x_i - x_{i+1})h \, .$$

Transforming the equality of (19) and (20) we obtain

$$\mu + h = \frac{\mathrm{type}(f) - s_i \, \mathrm{type}(f)}{x_i - x_{i+1}} = \partial_i(\mathrm{type}(f)) \, .$$

$\square$

We will say that $\mu \in (\mathbb{C}^*)^m$ is admissible for the operation $\mathfrak{C}_i^\mu$ (or equivalently for $\overline{\mathfrak{C}}_i^\mu$) applied to a pure function $f$ if the result $\mathfrak{C}_i^\mu(f)$ is pure. By Lemma 12.1 $\mu$ is admissible if and only if $\mu = \partial_i(\mathrm{type}(f)) - h$.

Let $\rho_m$ be defined by the formula

$$\rho_m = -\sum_{i=1}^m i \, x_i \, .$$

It is equal to the half sum of the positive roots $x_i - x_j$, $i < j$ up to a multiple of $\sum_{i=1}^m x_i$. We have

$$\partial_i(\rho_m) = 1 \quad \text{and} \quad s_i(\rho_m) = \rho_m - (x_i - x_{i+1})$$

for $i = 1, 2, \ldots, m-1$.

**Corollary 12.2.** *Suppose $f \neq 0$ is a pure meromorphic function on $(\mathbb{C}^*)^{m+r+2}$ of the type* type$(f)$. *Let $\mu = \partial_i(\text{type}(f)) - h$, then $\mathfrak{C}_i^\mu(f)$ if pure of the type*

$$\text{type}(\mathfrak{C}_i^\mu(f)) = s_i\Big(\text{type}(f) - \rho_m h\Big) + \rho_m h \,.$$

*Proof.* We have

$$
\begin{aligned}
s_i\Big(\text{type}(f) - \rho_m h\Big) + \rho_m h &= s_i\Big(\text{type}(f)\Big) - s_i(\rho_m)h + \rho_m h \\
&= s_i\Big(\text{type}(f)\Big) - (\rho_m - (x_i - x_{i+1}))h + \rho_m h \\
&= s_i\Big(\text{type}(f)\Big) + (x_i - x_{i+1})h \,,
\end{aligned}
$$

which is equal to (20). $\square$

This means that admissible $\mathfrak{C}_i^\mu$ operations act on types as the permutation of coordinates, provided that we shift the origin to $\rho_m h$.

The operations $\overline{\mathfrak{C}}_i^\mu$ differ from $\mathfrak{C}_i^\mu$ by the factor $\frac{1}{\delta(h,\mu)}$, therefore the resulting type differs by the summand $-h\mu$. The coefficient $\mu$ is given by Lemma 12.1.

It is convenient to define the function $\phi_w : \mathbb{C}^m \to \mathbb{C}$ for each permutation $w \in \mathfrak{S}_m$. As before let $x_i$ denote the standard coordinates of $\mathbb{C}^m$ on which $\mathfrak{S}_m$ acts permuting the indices. We define

$$(21) \qquad \phi_w\left(\sum_{i=1}^m \alpha_i x_i\right) = \sum_{i < j,\, w(i) > w(j)} \alpha_i - \alpha_j \,.$$

The functions $\phi_w$ satisfy the cocycle condition

$$(22) \qquad \phi_{wv}(\alpha) = \phi_w(v(\alpha)) + \phi_v(\alpha) \,.$$

We will evaluate the function $\phi_w$ on quadratic forms, linear in $x_i$, hence formally we extend $\phi_w$ linearly, allowing $a_i$ to belong to a vector space. We summarise the consideration of types by the following theorem:

**Theorem 12.3.** *Let $w = s_{i_1} s_{i_2} \ldots s_{i_\ell}$ be a presentation of a permutation $w \in \mathfrak{S}_m$, not necessarily reduced. Suppose $f$ is a pure meromorphic function. Denote by $\overline{\mathfrak{C}}_w^\diamond(f)$ the composition*

$$\overline{\mathfrak{C}}_{i_1}^{\nu_1} \overline{\mathfrak{C}}_{i_2}^{\nu_2} \ldots \overline{\mathfrak{C}}_{i_\ell}^{\nu_\ell}(f) \,,$$

*with the coefficients $\nu_k$ chosen so that the operations preserve purity*

$$\nu_k = \partial_{i_k} \text{type}\left(\overline{\mathfrak{C}}_{i_{k+1}}^{\nu_{k+1}} \overline{\mathfrak{C}}_{i_{k+2}}^{\nu_{k+2}} \ldots \overline{\mathfrak{C}}_{i_\ell}^{\nu_\ell}(f)\right) - h \,.$$

*Assume that $\nu_k + h \neq 0$ (or with the multiplicative notation $h\nu_k \neq 1$), so that $\overline{\mathfrak{C}}_i^\mu$ is defined.*

*Then the function $\overline{\mathfrak{C}}_w^\diamond(f)$ does not depend on the presentation of $w$. Moreover suppose*

$$\mathrm{type}(f) = q_x + \rho_m h + q_\mu \,,$$

*where $q_x = \sum_{i=1}^m \alpha_i x_i$ and $q_\mu$ does not depend on the variables $x_i$. Then*

$$\mathrm{type}\left(\overline{\mathfrak{C}}_w^\diamond(f)\right) = w(q_x) + \rho_m h + q_\mu - \phi_w(q_x)h \,.$$

*Proof.* First let us compute the type of $\overline{\mathfrak{C}}_w^\diamond(f)$. For $\ell = 1$

$$\mathrm{type}\left(\overline{\mathfrak{C}}_k^{\nu_1}(f)\right) = \rho_m h + s_k(q_x) - \nu_1 h \,,$$

where

$$\nu_1 = \partial_k \left(\rho_m h + q_x + q_\mu\right) - h = \alpha_k - \alpha_{k+1} = \phi_{s_k}(q_x)$$

by Lemma 12.1, as claimed. Further we argue by induction. If $w = s_k w'$ then

$$\mathrm{type}(\overline{\mathfrak{C}}_{w'}^\diamond(f)) = \rho_m h + w'(q_x) + q_\mu - \phi_{w'}(q_x)h$$

by the inductive assumption. Then

$$\mathrm{type}(\overline{\mathfrak{C}}_w^\diamond(f)) = \rho_m h + s_k(w'(q_x)) + q_\mu - \phi_{w'}(q_x)h - \phi_{s_k}(w'(q_x))h$$
$$= \rho_m h + w(q_x) + q_\mu - \phi_w(q_x)h$$

by (22).

To show that $\overline{\mathfrak{C}}_w^\diamond(f)$ does not depend on the decomposition of $w$ let us check the braid and quadratic relation. It is enough to consider the case $n = 3$, $q_x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$, $\alpha_i \in \mathbb{C}^{r+1}$. Examining $\overline{\mathfrak{C}}_1^{\nu_1}\overline{\mathfrak{C}}_2^{\nu_2}\overline{\mathfrak{C}}_1^{\nu_3}(f)$ we find that admissible values of $\nu_i$ are the following

$$\nu_1 = \alpha_2 - \alpha_3 \,, \qquad \nu_2 = \alpha_1 - \alpha_3 \,, \qquad \nu_3 = \alpha_1 - \alpha_2 \,,$$

(written in the additive notation). For the admissible operation $\overline{\mathfrak{C}}_2^{\mu_1}\overline{\mathfrak{C}}_1^{\mu_2}\overline{\mathfrak{C}}_2^{\mu_3}(f)$ we have

$$\mu_1 = \alpha_1 - \alpha_2 \,, \qquad \mu_2 = \alpha_1 - \alpha_3 \,, \qquad \mu_3 = \alpha_2 - \alpha_3 \,.$$

The braid relation (18) applies. For the quadratic relation

$$\overline{\mathfrak{C}}_1^{\nu_1}\overline{\mathfrak{C}}_1^{\nu_2}(f) = f$$

we note that if $q_x = \alpha_1 x_1 + \alpha_2 x_2$, then $\nu_2 = \alpha_1 - \alpha_2$. Since $s_1(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_2 x_1 + \alpha_1 x_2$ then $\nu_1 = \alpha_2 - \alpha_1 = -\nu_2$. The quadratic relation (17) applies. $\qquad\square$

## 13. Elliptic classes of link patterns

We construct elliptic classes of link patterns inductively applying the action of $\mathfrak{C}_i^\diamond$. The starting point is the Borisov Libgober class

$$\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}) := \mathcal{E}\ell\ell(X_{m,r}^{\min}, \Sigma_{i=1}^r \lambda_i D_i)$$

given by (9). Suppose that a link pattern $\mathcal{P}$ is obtained by applying a permutation $w \in \mathfrak{S}_m$ to $\mathcal{P}_{m,r}^{\min}$. We assume that $w$ has a minimal length. We define $\mathcal{E}\ell\ell(\mathcal{P})$ as $\mathfrak{C}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$. To show that the definition does not depend on the choice of the permutation $w$ we analyze the elementary transformation of the elliptic class given by $\mathfrak{C}_i^\diamond$. The first step is to trace how the type of the elliptic class is affected.

**Example 13.1.** We keep the notation of §8: $\mu_i = h^{1-\lambda_i}$. The elliptic class of the link pattern $\mathcal{P}_{8,2}^{\min}$ is equal to

$$\delta\left(u\tfrac{x_1}{x_7}, \mu_1\right)\delta\left(u\tfrac{x_2}{x_8}, \mu_2\right)\delta\left(u\tfrac{x_1}{x_8}, h\right).$$

It has the type equal to

$$(u + x_1 - x_7)\mu_1 + (u + x_2 - x_8)\mu_2 + (u + x_1 - x_8)h.$$

In general

$$\text{type}(\mathcal{Ell}(\mathcal{P}_{m,r}^{\min})) = \sum_{i=1}^{r}\left((u + x_i - x_{m-r+i})\mu_i + \sum_{j=m-r+i+1}^{m}(u + x_i - x_j)h\right)$$

The coefficient of $x_i$ in

$$\text{type}(\mathcal{Ell}(\mathcal{P}_{m,r}^{\min})) - h\rho_m$$

is equal to

$$(23) \qquad \text{v}(i) = \begin{cases} rh + \mu_i & \text{for } 1 \le i \le r, \\ ih & \text{for } r + 1 \le i \le m - r, \\ (m - r + 1)h - \mu_{i-m+r} & \text{for } m - r + 1 \le i \le m. \end{cases}$$

We label the nodes of the link pattern with values $\text{v}(i)$ written multiplicatively. In the example above

$$(24) \qquad \mu_1 h^2 \cdots \mu_2 h^2 \cdots h^3 \cdots h^4 \cdots h^5 \cdots h^6 \cdots \tfrac{h^7}{\mu_1} \cdots \tfrac{h^7}{\mu_2}.$$

Action of the permutation rearranges the values $\text{v}(i)$. For example applying the permutation

$$w: \quad 1 \mapsto 3, \quad 2 \mapsto 6, \quad 3 \mapsto 1, \quad 4 \mapsto 2, \quad 5 \mapsto 5, \quad 6 \mapsto 8, \quad 7 \mapsto 4, \quad 8 \mapsto 7$$

to (24) we obtain the link pattern

$$(25) \qquad h^3 \cdots h^4 \cdots \mu_1 h^2 \cdots \tfrac{h^7}{\mu_2} \cdots h^5 \cdots \mu_2 h^2 \cdots \tfrac{h^7}{\mu_1} \cdots h^6.$$

From this presentation it is easy to read admissible $\mu$ of the operation $\mathfrak{C}_i^\mu$: if $\mathcal{P} = w \cdot \mathcal{P}_{m,r}^{\min}$, namely

$$\mu = \text{v}(w(i)) - \text{v}(w(i+1)) \qquad \text{(written additively)},$$

by Lemma 12.1. Note that if we transpose consecutive loose nodes then the operation $\overline{\mathfrak{C}}_i^\diamond$ is not defined. Indeed, $\mu = h^k/h^{k+1} = h^{-1}$ and the normalizing factor would be $\delta(\nu, h) = 0$.

**Example 13.2.** Let us apply the operation $\mathfrak{C}_3^\diamond$ to $\mathcal{Ell}(\mathcal{P}_{8,2}^{\min})$. We read weights from (24). Since $\mathcal{Ell}(\mathcal{P}_{8,2}^{\min}) = s_3(\mathcal{Ell}(\mathcal{P}_{8,2}^{\min}))$ and $\mu = h^{-1}$ we have

$$\mathfrak{C}_3^\diamond(\mathcal{Ell}(\mathcal{P}_{8,2}^{\min})) = \delta\left(\tfrac{x_4}{x_3}, h^{-1}\right)\mathcal{Ell}(\mathcal{P}_{8,2}^{\min}) + \delta\left(\tfrac{x_3}{x_4}, h\right)\mathcal{Ell}(\mathcal{P}_{8,2}^{\min}) = 0.$$

Similarly, we apply $s_1$ to the example (25). Since $s_1 w = w s_3$ we have $\mathfrak{C}_1^\diamond(\mathcal{E}\ell\ell(w\mathcal{P}_{m,r}^{\min})) = \mathfrak{C}_w^\diamond \mathfrak{C}_3^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})) = 0$.

## 14. Six moves increasing orbit dimension

We present a simple way to determine the parameter $\mu$ for which $\mathfrak{C}_i^\mu$ acting on $\mathfrak{C}_w^\diamond(\mathcal{P}_{r,m}^{\min})$ is admissible.

$$\mu = \frac{\alpha h^r}{\beta h^r} = \frac{\alpha}{\beta}$$

$$\mu = \frac{h^{m-r+1}/\alpha}{h^{m-r+1}/\beta} = \frac{\beta}{\alpha}$$

$$\mu = \frac{\alpha h^r}{h^{m-r+1}/\beta} = \frac{\alpha \beta}{h^{m-2r+1}}$$

$$\mu = \frac{\alpha h^r}{h^{m-r+1}/\alpha} = \frac{\alpha^2}{h^{m-2r+1}}$$

$$\mu = \frac{\alpha h^r}{h^k} = \frac{\alpha}{h^{k-r}}$$

$$\mu = \frac{h^k}{h^{m-r+1}/\alpha} = \frac{\alpha}{h^{m-r-k+1}}$$
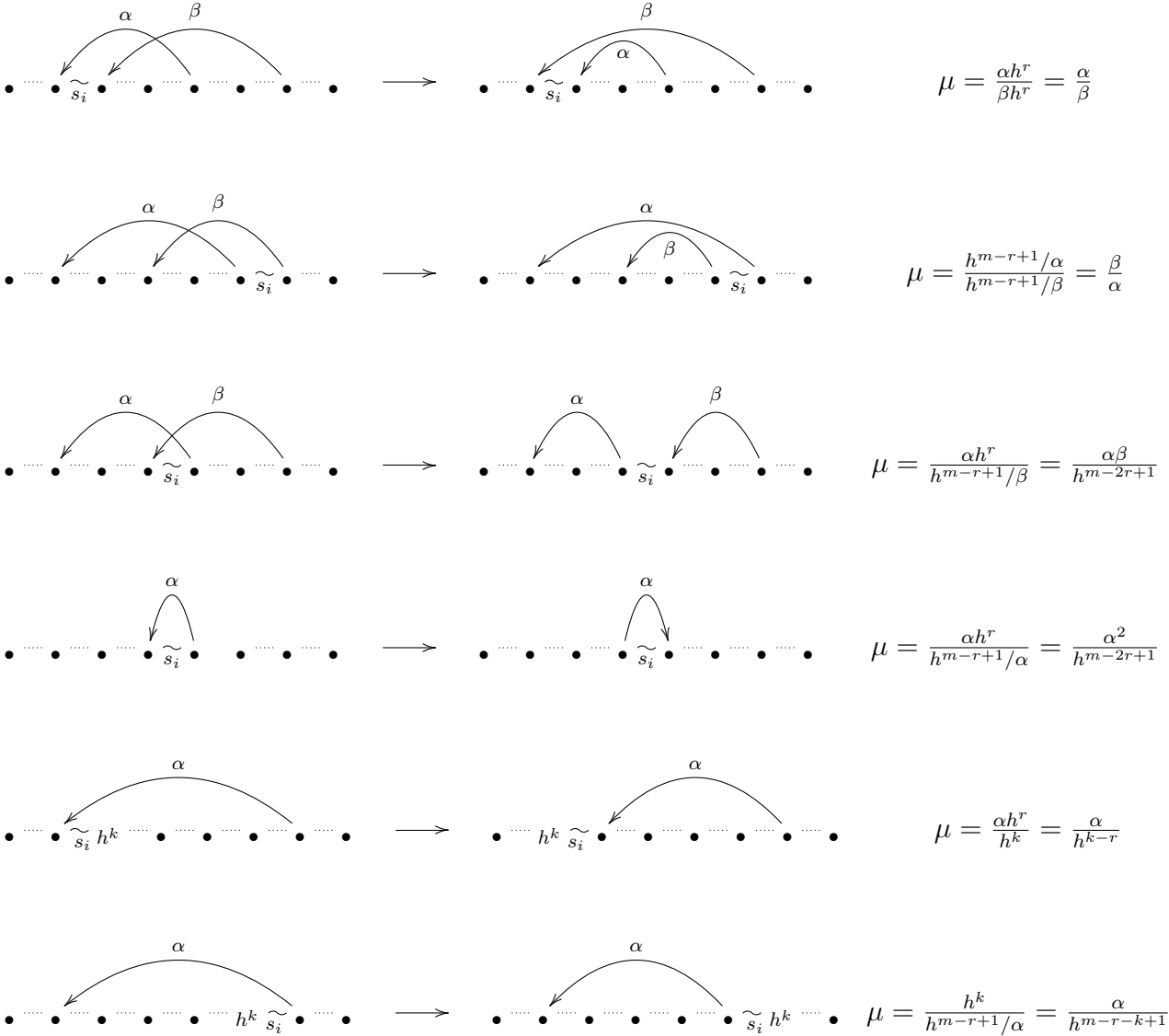
The transformations with linking patterns having reversed one or more arrows obey similar rules. Note that starting from the link pattern $\mathcal{P}_{m,r}^{\min}$ we never have $\nu = 1$, which would made the operation $\mathfrak{C}_i^\nu$ impossible.

**Corollary 14.1.** *Let $f = \mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})$. Suppose $s_{i_1} s_{i_2} \ldots s_{i_\ell}$ is a reduced decomposition of $w \in \mathfrak{S}_m$. Then the composition*

$$\mathfrak{C}_{i_1}^{\nu_1} \mathfrak{C}_{i_2}^{\nu_2} \ldots \mathfrak{C}_{i_\ell}^{\nu_\ell}(f)$$

*is well defined and does not depend on the word decomposition. Moreover if*

$$\mathrm{type}(f) = q_x + \rho_m h + q_\mu\,,$$

*then*

$$\mathrm{type}\left(\mathfrak{C}_w^\diamond(f)\right) = w(q_x) + \rho_m h + q_\mu\,.$$

*Proof.* We note that for any two reduced decompositions of $w$ one can pass from one to another applying braid relations, [BB05, Theorem 3.3.1]. □

If $w$ preserves the order of loose nodes, then transposition of strings labelled by a power of $h$ never appears. Hence the admissible parameters $\nu_i$ are never equal to $h^{-1}$. Hence the reduced operations $\overline{\mathfrak{C}}_i^{\nu_i}$ are defined. Moreover one does not have to assume that the word representing $w$ is reduced, but only that the strings coming from the loose nodes in the course of applications of $s_{i_j}$ do not cross.

**Corollary 14.2.** *Assume that $w$ preserves the order of loose nodes. Then $\overline{\mathfrak{C}}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$ is well defined and it depends only on $w$.*

## 15. Independence of the result of $\overline{\mathfrak{C}}_w^\diamond$ presentation

We prove independence of the elliptic class from the presentation of the link pattern. First we analyze the reduced operations $\overline{\mathfrak{C}}_i^\diamond$.

**Theorem 15.1.** *Let $\mathcal{P}$ be a labelled link pattern of rank $r$ and let $w \in \mathfrak{S}_m$, $\sigma \in \mathfrak{S}_r$ be permutations such that $\mathcal{P} = \sigma^\mu w \mathcal{P}_{m,r}^{\min}$. We assume that $w$ preserves the order of the loose nodes. Then $\overline{\mathfrak{C}}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$ is defined and the result*

$$\sigma^\mu \overline{\mathfrak{C}}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$$

*does not depend on the choice of $\sigma$ and $w$.*

*Proof.* The product of groups $\mathfrak{S}_m \times \mathfrak{S}_r$ acts on the set of labelled link patterns. Suppose $\sigma_1^\mu w_1(\mathcal{P}_{m,r}^{\min}) = \sigma_2^\mu w_2(\mathcal{P}_{m,r}^{\min})$ and both $w_1, w_2$ preserve the order of loose nodes. Then $w_1^{-1} w_2$ preserves the order of the loose nodes and $(\sigma_1^\mu)^{-1} \sigma_2^\mu w_1^{-1} w_2(\mathcal{P}_{m,r}^{\min}) = \mathcal{P}_{m,r}^{\min}$. Hence it is enough to show that if $(\sigma, w)$ stabilizes the labelled link pattern $\mathcal{P}_{m,r}^{\min}$ and $w$ preserves the order of the loose nodes (so it is constant on the loose nodes), then $\sigma^\mu \overline{\mathfrak{C}}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})) = \mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min})$.

The stabilizer of $\mathcal{P}_{m,r}^{\min}$ and loose nodes is generated by the simultaneous transpositions of arcs and labels:

$$\sigma = s_i\,, \quad w = s_i s_{m-r+i}\,.$$

The calculus involves only the variables $x_i, x_{i+1}, x_{m-r+i}, x_{m-r+i+1}, \mu_i, \mu_{i+1}$ it is enough to check the result of the action for $m = 4, r = 2$, $i = 1$, as in (13) since:

$$\mathcal{P}_{m,r}^{\min} = \mathcal{M} \cdot \left((\mathcal{P}_{4,2}^{\min})_{x_1 := x_i, \, x_2 := x_{i+1}, \, x_3 := x_{m-r+i}, \, x_4 := x_{m-r+i+1}, \, \mu_1 := \mu_i, \, \mu_2 := \mu_{i+1}}\right)$$

and $\mathcal{M}$ is a symmetric function with respect to $x_i \leftrightarrow x_{i+1}$, $x_{m-r+i} \leftrightarrow x_{m-r+i+1}$, does not depend on $\mu_i$, $\mu_{i+1}$. The calculation is done by an explicit check based on geometric consideration in Section 16. $\qquad\square$

**Remark 15.2.** Note that the set of permutations preserving the order of the loose nodes is not a group, hence in the argument above we had to write the composition $w_1^{-1} w_2$ to have a permutation fixing loose nodes.

## 16. The basic example – the flip relation

The purpose of this section is mainly to give a geometric proof of the four-term relation [RTV19, eq. (2.7)] which plays a role in the proof of Theorem 15.1 for the elliptic algebra.

Consider the minimal link pattern $\mathcal{P}_{4,2}^{\min}$ and the corresponding $B_4$ orbit in $\mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^4)$

$$
(X_{4,2}^{\min})^\circ = B_4 \cdot \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : a \neq 0,\ c \neq 0 \right\}.
$$

The closure $X_{4,2}^{\min}$ is isomorphic to $\mathbb{C}^3$. The effect of the actions of $s_1$ and $s_3$ are equal to the $B_4$ orbit

$$
X^\circ := B_4\, s_1 \cdot (X_{4,2}^{\min})^\circ = B_4\, s_3 \cdot (X_{4,2}^{\min})^\circ = \left\{ \begin{pmatrix} 0 & 0 & s & t \\ 0 & 0 & u & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : u \neq 0,\ sv - tu \neq 0 \right\}.
$$

The closure $X = \overline{X^\circ}$ is isomorphic to $\mathbb{C}^4$. Let $P_i \subset \mathrm{GL}_4$ be the minimal parabolic subgroup generated by $B_4$ and $s_i$. We have two Bott-Samelson type resolutions, of the pair $(X, \partial X)$:

$$
Z_1 = P_1 \times_{B_4} X \quad \text{and} \quad Z_3 = P_3 \times_{B_4} X.
$$

Let us analyze the first one. There are two charts:

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ z & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & az & bz+c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

and

$$
\begin{pmatrix} z & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & az & c+bz \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Consider the following divisor on $X$

$$
D = (1-\alpha)D_1 + (1-\beta)D_2, \quad \text{where } D_1 = \{u = 0\} \text{ and } D_2 = \{sv - tu = 0\}.
$$

The pull back of $D$ via $f : Z_1 \to X$ is a normal crossing divisor.

$$
f^*(K_X + D) = K_{Z_1} + (1-\alpha)\{az = 0\} + (1-\beta)\{ac = 0\}.
$$

Indeed, in the first chart we have

$$f^* \left( u^{\alpha-1}(sv - tu)^{\beta-1} ds \wedge dt \wedge du \wedge dv \right) = (az)^{\alpha-1}(ac)^{\beta-1} a\, da \wedge db \wedge dz \wedge dc$$

$$= z^{\alpha-1} a^{\alpha+\beta-1} c^{\beta-1} da \wedge db \wedge dz \wedge dc \,.$$

We compute the localized equivariant Borisov-Libgober class using localization formula for the torus action. We skip the computation of the torus weights and we refer to the almost identical computations for fundamental, CSM and motivic Chern classes [RuW22, Th. 4.1, Th. 5.1]. The result is

$$\mathcal{E}ll(X, D) = \delta\left(\tfrac{x_2}{x_1}, \alpha\right)\mathcal{E}ll(X, D_0) + \delta\left(\tfrac{x_1}{x_2}, h\right) s_1 \mathcal{E}ll(X, D_0) \,,$$

where

$$D_0 = (1 - \alpha - \beta)\{a = 0\} + (1 - \beta)\{c = 0\} \,.$$

Setting

(26)
$$\mu_1 = \alpha + \beta \,, \qquad \mu_2 = \beta$$

we obtain

$$\mathcal{E}ll(X, D) = \mathfrak{C}_1^{\mu_1/\mu_2}\left(\mathcal{E}ll(X, D_0)\right) \,.$$

An alternative resolution is obtained by applying the action of $s_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & z & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a - bz & b \\ 0 & 0 & -cz & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(we recall, that we act by conjugation). Then

$$f^* \left( u^{\alpha-1}(sv - tu)^{\beta-1} ds \wedge dt \wedge du \wedge dv \right) \sim (cz)^{\alpha-1}(ac)^{\beta-1} c\, da \wedge db \wedge dz \wedge dc$$

$$= z^{\alpha-1} a^{\beta-1} c^{\alpha+\beta-1} da \wedge db \wedge dz \wedge dc \,.$$

Thus

$$\mathcal{E}ll(X, D) = \mathfrak{C}_3^{\alpha}\left(\mathcal{E}ll(X, D_0')\right) \,.$$

with

$$D_0' = (1 - \alpha)\{a = 0\} + (1 - \alpha - \beta)\{c = 0\} \,.$$

Setting

(27)
$$\mu_1' = \beta \,, \qquad \mu_2' = \alpha + \beta$$

we obtain

$$\mathcal{E}ll(X, D) = \mathfrak{C}_3^{\alpha_2'/\alpha_1'}\left(\mathcal{E}ll(X, D_0')\right) \,.$$

Note that the substitutions (26) and (27) differ by the transposition $\mu_1 \leftrightarrow \mu_2'$, $\mu_2 \leftrightarrow \mu_1'$.

**Corollary 16.1.** *We have the identity*

$$\mathfrak{C}_1^{\mu_1/\mu_2}(\mathcal{A}) = s_1^\mu \mathfrak{C}_3^{\mu_2/\mu_1}(\mathcal{A}) \, ,$$

(28)

*where*

$$\mathcal{A} = \delta\left(\tfrac{x_1}{x_3}, \mu_1\right)\delta\left(\tfrac{x_2}{x_4}, \mu_2\right)\delta\left(\tfrac{x_1}{x_4}, h\right).$$

*Analogous identity holds for the reduced operations $\overline{\mathfrak{C}}_i^\nu$.*

We rewrite the identity using the definition of the operation $\mathfrak{C}_i^\mu$:

$$\delta\left(\tfrac{x_1}{x_4}, h\right)\delta\left(\tfrac{x_2}{x_1}, \tfrac{\mu_1}{\mu_2}\right)\delta\left(\tfrac{x_1}{x_3}, \mu_1\right)\delta\left(\tfrac{x_2}{x_4}, \mu_2\right) + \delta\left(\tfrac{x_1}{x_2}, h\right)\delta\left(\tfrac{x_2}{x_4}, h\right)\delta\left(\tfrac{x_2}{x_3}, \mu_1\right)\delta\left(\tfrac{x_1}{x_4}, \mu_2\right) =$$

$$= \delta\left(\tfrac{x_1}{x_4}, h\right)\delta\left(\tfrac{x_1}{x_3}, \mu_2\right)\delta\left(\tfrac{x_2}{x_4}, \mu_1\right)\delta\left(\tfrac{x_4}{x_3}, \tfrac{\mu_1}{\mu_2}\right) + \delta\left(\tfrac{x_1}{x_3}, h\right)\delta\left(\tfrac{x_3}{x_4}, h\right)\delta\left(\tfrac{x_2}{x_3}, \mu_1\right)\delta\left(\tfrac{x_1}{x_4}, \mu_2\right)$$

After applying the definition of $\delta$ and multiplying by the common denominator, simplifying by the relation $\vartheta\left(\tfrac{a}{b}\right) = -\vartheta\left(\tfrac{b}{a}\right)$, we obtain the following monstrous relation[2]

$$\vartheta\left(\tfrac{y_2}{y_1}\right)\left(\vartheta(h)\vartheta\left(\tfrac{\mu_2 x_2}{\mu_1 x_1}\right)\vartheta\left(\tfrac{x_2}{y_1}\right)\vartheta\left(\tfrac{hx_1}{y_2}\right)\vartheta\left(\tfrac{\mu_1 x_2}{y_2}\right)\vartheta\left(\tfrac{\mu_2 x_1}{y_1}\right)\right.$$

$$\left. - \vartheta\left(\tfrac{\mu_2}{\mu_1}\right)\vartheta\left(\tfrac{hx_1}{x_2}\right)\vartheta\left(\tfrac{x_1}{y_1}\right)\vartheta\left(\tfrac{hx_2}{y_2}\right)\vartheta\left(\tfrac{\mu_1 x_1}{y_2}\right)\vartheta\left(\tfrac{\mu_2 x_2}{y_1}\right)\right) =$$

$$= \vartheta\left(\tfrac{x_2}{x_1}\right)\left(\vartheta(h)\vartheta\left(\tfrac{x_2}{y_1}\right)\vartheta\left(\tfrac{\mu_2 y_2}{\mu_1 y_1}\right)\vartheta\left(\tfrac{hx_1}{y_2}\right)\vartheta\left(\tfrac{\mu_1 x_1}{y_1}\right)\vartheta\left(\tfrac{\mu_2 x_2}{y_2}\right)\right.$$

$$\left. - \vartheta\left(\tfrac{\mu_2}{\mu_1}\right)\vartheta\left(\tfrac{hy_1}{y_2}\right)\vartheta\left(\tfrac{x_2}{y_2}\right)\vartheta\left(\tfrac{hx_1}{y_1}\right)\vartheta\left(\tfrac{\mu_1 x_1}{y_2}\right)\vartheta\left(\tfrac{\mu_2 x_2}{y_1}\right)\right).$$

## 17. Unnormalized elliptic classes

Before showing that the unnormalized elliptic classes of link patterns are well defined let us analyse the first nontrivial example. We have to check that the normalizing factor $\prod \delta(\nu_i, h)$ does not change when we change the arrow labels, and accordingly the permutation of nodes.

**Example 17.1.** Let $m = 4$, $r = 2$. We present the lattice of orbits. We mark the nodes with the coefficients of type $\left(\mathcal{E}\!\ell(\mathcal{P})\right) - \rho h$. The arrows are labelled by elementary transpositions $s_i$, $i = 1, 2, 3$ or $\hat{s}_1 = s_1^\mu s_1$. The arrow with a label indicates that one link pattern is obtained from the previous one by applying the given operation. It happens twice that two operations lead to the same link pattern. The link patterns in boxes admit two presentations as $w\mathcal{P}_{4,2}^{\min} = s_1^\mu w'\mathcal{P}_{4,2}^{\min}$ with $w$ and $w'$ of the shortest possible length $\ell(w) = \ell(w')$.

---

[2]We apply the multiplicative notation for the arguments of the theta function.

$$\alpha^{\mathrm{s}} \cdots \beta^{\mathrm{s}} \cdots \beta^{\mathrm{t}} \cdots \alpha^{\mathrm{t}}$$
$$\left\{\frac{\alpha}{\beta}, \frac{\alpha^2}{h}, \frac{\alpha\beta}{h}, \frac{\alpha\beta}{h}, \frac{\beta^2}{h}\right\}$$

$s_2 \nearrow$ $\hat{s}_1, s_3 \nwarrow\!\!\!\nwarrow$

$$\alpha^{\mathrm{s}} \cdots \beta^{\mathrm{t}} \cdots \beta^{\mathrm{s}} \cdots \alpha^{\mathrm{t}}$$
$$\left\{\frac{\alpha}{\beta}, \frac{\alpha^2}{h}, \frac{\alpha\beta}{h}, \frac{\alpha\beta}{h}\right\}$$

$$\alpha^{\mathrm{s}} \cdots \beta^{\mathrm{s}} \cdots \alpha^{\mathrm{t}} \cdots \beta^{\mathrm{t}}$$
$$\left\{\frac{\alpha^2}{h}, \frac{\alpha\beta}{h}, \frac{\alpha\beta}{h}, \frac{\beta^2}{h}\right\}$$

$\hat{s}_1 \nearrow$ $s_3 \nwarrow$ $s_2 \nwarrow$

$$\alpha^{\mathrm{t}} \cdots \beta^{\mathrm{s}} \cdots \alpha^{\mathrm{s}} \cdots \beta^{\mathrm{t}}$$
$$\left\{\frac{\beta}{\alpha}, \frac{\alpha\beta}{h}, \frac{\beta^2}{h}\right\}$$

$$\alpha^{\mathrm{s}} \cdots \beta^{\mathrm{t}} \cdots \alpha^{\mathrm{t}} \cdots \beta^{\mathrm{s}}$$
$$\left\{\frac{\alpha}{\beta}, \frac{\alpha^2}{h}, \frac{\alpha\beta}{h}\right\}$$

$$\alpha^{\mathrm{s}} \cdots \alpha^{\mathrm{t}} \cdots \beta^{\mathrm{s}} \cdots \beta^{\mathrm{t}}$$
$$\left\{\frac{\alpha^2}{h}, \frac{\alpha\beta}{h}, \frac{\beta^2}{h}\right\}$$

$\uparrow s_3$ $\hat{s}_1 \nearrow$ $s_2 \nwarrow$ $s_1 \nearrow$ $s_2 \nwarrow$ $\uparrow s_3$

$$\alpha^{\mathrm{t}} \cdots \beta^{\mathrm{s}} \cdots \beta^{\mathrm{t}} \cdots \alpha^{\mathrm{s}}$$
$$\left\{\frac{\beta}{\alpha}, \frac{\beta^2}{h}\right\}$$

$$\alpha^{\mathrm{t}} \cdots \alpha^{\mathrm{s}} \cdots \beta^{\mathrm{s}} \cdots \beta^{\mathrm{t}}$$
$$\left\{\frac{\alpha\beta}{h}, \frac{\beta^2}{h}\right\}$$

$$\alpha^{\mathrm{s}} \cdots \alpha^{\mathrm{t}} \cdots \beta^{\mathrm{t}} \cdots \beta^{\mathrm{s}}$$
$$\left\{\frac{\alpha^2}{h}, \frac{\alpha\beta}{h}\right\}$$

$s_2 \nwarrow$ $s_3 \nwarrow$ $s_1 \nearrow$

$$\alpha^{\mathrm{t}} \cdots \beta^{\mathrm{t}} \cdots \beta^{\mathrm{s}} \cdots \alpha^{\mathrm{s}}$$
$$\left\{\frac{\beta}{\alpha}\right\}$$

$$\alpha^{\mathrm{t}} \cdots \alpha^{\mathrm{s}} \cdots \beta^{\mathrm{t}} \cdots \beta^{\mathrm{s}}$$
$$\left\{\frac{\alpha\beta}{h}\right\}$$

$\hat{s}_1, s_3 \nwarrow\!\!\!\nwarrow$ $\nearrow s_2$

$$\alpha^{\mathrm{t}} \cdots \beta^{\mathrm{t}} \cdots \alpha^{\mathrm{s}} \cdots \beta^{\mathrm{s}}$$
$$\{\}$$
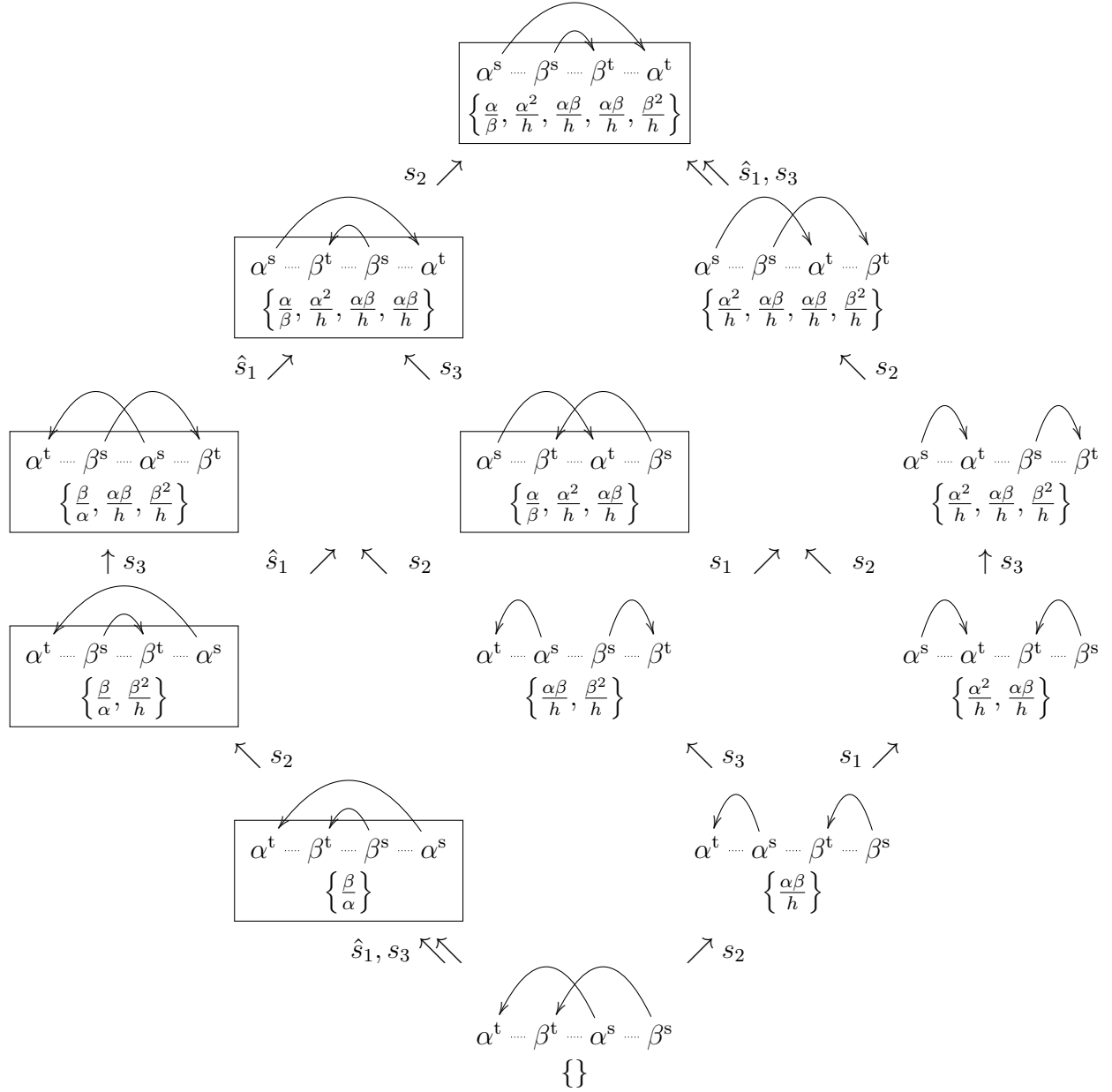
The variables $\alpha$ and $\beta$ are associated to arrows. Instead of indicating arc labels we list the coefficients at the nodes: $\alpha^{\mathrm{t}}$ and $\beta^{\mathrm{t}}$ at the targets of the arrows and $\alpha^{\mathrm{s}}$ and $\beta^{\mathrm{s}}$ at the sources

$$\alpha^{\mathrm{t}} = \alpha h^2, \quad \beta^{\mathrm{t}} = \beta h^2, \quad \alpha^{\mathrm{s}} = \frac{h^3}{\alpha}, \quad \beta^{\mathrm{s}} = \frac{h^3}{\beta}.$$

Below the patterns in braces we give the $\mu$-coefficients lists for all link patterns. A discrepancy between the lists can originate from two places, where the images $\hat{s}_1$ and $s_3$ are the same. We check directly that the $\mu$-coefficient lists agree. In fact it is enough to check it for $s_3 \mathcal{P}_{4,2}^{\min} = \hat{s}_1 \mathcal{P}_{4,2}^{\min}$.

Before giving a proof of independence of the normalizing factor from the link presentation we argue that it is enough to consider link patterns of the rank $r$ and the size $m = 2r$. If $m > 2r$ then we extend the link pattern adding $m - 2r$ nodes and arrows from the new nodes to loose nodes, preserving the order. We set $\mu_i' = \frac{\mu_i}{h^{m/2-r}}$. Then
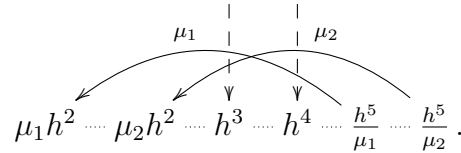
$$\mu_i h^r = \mu_i' h^{m/2}, \qquad \frac{h^{m-r+1}}{\mu_i} = \frac{h^{m/2+1}}{\mu_i'} \qquad \text{for } 1 \le i \le r.$$

We choose the remaining variables $\mu_j'$ for $j = r+1, \ldots, m-r$ so that
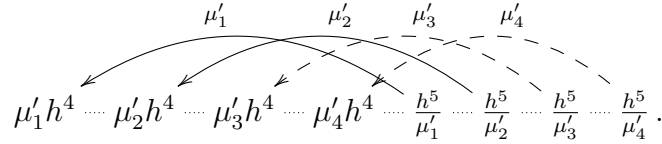
$$\mu_j' h^{m/2} = h^j.$$

Multiplying all the coefficients by $h^{m/2-r}$ we obtain the distribution of coefficients as for the link pattern $\mathcal{P}_{2(m-r),m-r}^{\min}$. Such operation does not change the quotients of coefficients we have to determine.
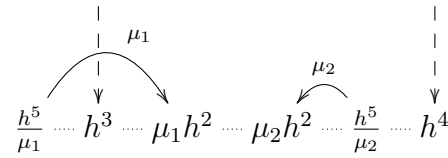
**Example 17.2.** Extending the link pattern $\mathcal{P}_{m,r}^{\min}$ to $\mathcal{P}_{2(m-r),m-r}^{\min}$:



Our procedure leads to



Similarly we extend an arbitrary link pattern. For example



is extended to
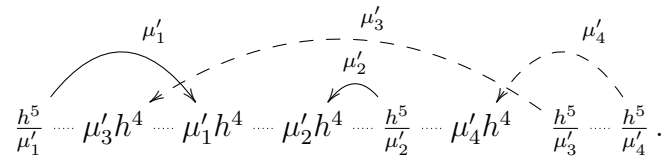


Now we are at the position to prove:

**Theorem 17.3.** *Let $\mathcal{P}$ be a labelled link pattern of the size $m$ and rank $r$. The function $\sigma^\mu \mathfrak{C}_w^\diamond(\mathcal{Ell}(\mathcal{P}_{m,r}^{\min}))$ does not depend on $(\sigma, w) \in \mathfrak{S}_r \times \mathfrak{S}_m$, provided that $w$ has minimal length among all the possible pairs $(\sigma, w)$ satisfying*

$$(29) \qquad \mathcal{P} = \sigma^\mu w \mathcal{P}_{m,r}^{\min}.$$

*Proof.* We can assume that $m$ is even and the rank $r = m/2$. Suppose the length of $w$ is equal to $\ell$ and let $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ be a reduced decomposition. The functions $\mathfrak{C}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$ and $\overline{\mathfrak{C}}_w^\diamond(\mathcal{E}\ell\ell(\mathcal{P}_{m,r}^{\min}))$ differ by the product $\prod_{i=k}^\ell \delta(\nu_k, h)$ where $\nu_k$ is the parameter of $k$-th operation $\mathfrak{C}_{i_k}^{\nu_k}$. The list of $\nu_k$ parameters can be read from the picture representing the link pattern. If the arrow with the label $\alpha$ is reversed then $\frac{\alpha^2}{h}$ appears. To determine which of coefficients $\frac{\alpha}{\beta}$ or $\frac{\alpha\beta}{h}$ appears, it is enough to analyze the relative position of the corresponding pair of arrows. The situation is reduced to the case $m = 4$, $r = 2$, where independence of the presentation (29) is verified directly. The list of parameters is given in the table of Example 17.1. $\qquad\square$

## 18. Relation with the elliptic classes of Schubert varieties

Our initial aim was to associate elliptic classes to link patterns so that in the case $m = 2n$, $r = n$ and when a link pattern represents a permutation of $n$ elements, after normalization and restriction as in [RTV19, eq (2.34)] (see [RiW20, §6.1]) we would recover elliptic classes of Schubert varieties. The corresponding procedure division and restriction is simply related to division by the Borel group $B_n$, analogously as in the case of the twisted motivic Chern classes, [KW23] .

Let us assume that all the arrows of the link patterns have targets at nodes with positions $i \le n$. Such link patterns $\mathcal{P}$ define a permutation $w_\mathcal{P}$. We rename the equivariant variables: we do not change the equivariant variables $x_i$ for $i \le n$ and let

$$x_{j+n} = y_j \qquad \text{for } j = 1, 2, \dots, n.$$

The additional variable $u$ is specialized to 1. We introduce the normalization

$$eu_M^{ell} = \prod_{i=1}^n \prod_{j=1}^n \vartheta\left(\tfrac{x_i}{y_j}\right).$$

This is the equivariant elliptic Euler class of the space of $n \times n$-matrices with $(\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ action by left and right multiplication. Let

$$eu_{\mathcal{F}\ell}^{ell} = \prod_{n \ge i > j \ge 1} \vartheta\left(\tfrac{y_i}{y_j}\right)$$

be another normalizing factor, which after the application of the Kirwan map (2) becomes the equivariant elliptic Euler class of the tangent bundle to the flag variety. Furthermore we need

$$\mathcal{B} = \prod_{1 \le i < j \le n} \frac{\vartheta\left(\tfrac{y_i}{y_j} h\right)}{\vartheta(h)},$$

which stands for the elliptic Chern class of the tangent bundle to the unipotent part of the Borel group.

**Theorem 18.1.** *The reduction of the class $\mathcal{E}\ell\ell^{\mathrm{red}}(\mathcal{P})$ to the flag varieties is equal to elliptic characteristic classes of the Schubert variety $X_{w_\mathcal{P}}$ defined in* [RiW20] *after substitution $\mu_i := \mu_i^{-1}$ and $x_{i+n} := y_i$ for $1 \le i \le n$.*

*Proof.* It is more convenient to work with localized classes

$$\mathcal{E}\ell\ell^{\mathrm{loc.red}}(\mathcal{P}) = \frac{\mathcal{E}\ell\ell^{\mathrm{red}}(\mathcal{P})}{eu^{ell}_{\mathcal{F}\ell}} = \frac{eu^{ell}_M}{eu^{ell}_{\mathcal{F}\ell}\mathcal{B}}\mathcal{E}\ell\ell(pa)$$

and check the R-matrix recursion (6). Since the function $eu^{ell}_M$ is symmetric with respect to permutation of $x_i$ variables and $eu^{ell}_{\mathcal{F}\ell}$, $\mathcal{B}$ do not depend on $x_i$ variables at all, the functions $\mathcal{E}\ell\ell^{\mathrm{loc.red}}(\mathcal{P})$ satisfy the recursion

$$\mathcal{E}\ell\ell^{\mathrm{loc.red}}(s_i\mathcal{P}) = \mathfrak{C}^\diamond_i(\mathcal{E}\ell\ell^{\mathrm{loc.red}}(\mathcal{P}))$$

whenever $i < n$ and $\ell(s_i w_{\mathcal{P}}) = \ell(w_{\mathcal{P}}) + 1$. We find the value of the admissible coefficient $\nu$ for the operation $\mathfrak{C}^\diamond_i = \mathfrak{C}^\nu_i$. It is equal to the quotient of the labels of arrows pointing to $i$-th and $(i+1)$-th node, that is

$$\frac{w^{-1}_{\mathcal{P}}(\mu_i)}{w^{-1}_{\mathcal{P}}(\mu_{i+1})}\,.$$

We note that here to agree with the formula (6) we have to change variables $\mu_i$ to $\mu_i^{-1}$. It remains to examine the restriction of $\mathcal{E}\ell\ell^{\mathrm{loc.red}}(\mathcal{P}^{\mathrm{min}}_{2n,n})$ to the fixed points of the flag variety. The torus fixed points in the flag variety are identified with the permutations $\sigma \in \mathfrak{S}_n$. The restriction to the fixed point $\sigma$ is a function depending only on $x_i$ and $\mu_i$. It is obtained by the substitution $y_i := x_{\sigma(i)}$ for $1 \le i \le n$. We have to show that

$$\mathcal{E}\ell\ell^{\mathrm{1red}}(\mathcal{P}^{\mathrm{min}}_{2n,n})_{|\{y_i:=x_{\sigma(i)}\}} = \begin{cases} 1 & \text{if } \sigma = \mathrm{id} \\ 0 & \text{if } \sigma \ne \mathrm{id}\,. \end{cases}$$

We expand

$$\mathcal{E}\ell\ell^{\mathrm{loc.red}}(\mathcal{P}^{\mathrm{min}}_{2n,n}) = \frac{\prod^n_{i=1}\prod^n_{j=1}\vartheta\left(\frac{x_i}{y_j}\right) \cdot \prod^n_{i=1}\delta\left(\frac{x_i}{y_i}, \mu_i\right) \cdot \prod_{i<j}\delta\left(\frac{x_i}{y_j}, h\right)}{\prod_{i<j}\frac{\vartheta\left(\frac{y_i}{y_j}h\right)}{\vartheta(h)} \cdot \prod_{i>j}\vartheta\left(\frac{y_i}{y_j}\right)}$$

$$= \frac{\prod^n_{i=1}\prod^n_{j=1}\vartheta\left(\frac{x_i}{y_j}\right) \cdot \prod^n_{i=1}\frac{\vartheta\left(\frac{x_i}{y_i}\mu_i\right)}{\vartheta\left(\frac{x_i}{y_i}\right)\vartheta(\mu_i)} \cdot \prod_{i<j}\frac{\vartheta\left(\frac{x_i}{y_j}h\right)}{\vartheta\left(\frac{x_i}{y_j}\right)\vartheta(h)}}{\prod_{i<j}\frac{\vartheta\left(\frac{y_i}{y_j}h\right)}{\vartheta(h)} \cdot \prod_{i>j}\vartheta\left(\frac{y_i}{y_j}\right)}$$

$$= \prod_{i>j}\frac{\vartheta\left(\frac{x_i}{y_j}\right)}{\vartheta\left(\frac{y_i}{y_j}\right)} \cdot \prod^n_{i=1}\frac{\vartheta\left(\frac{x_i}{y_i}\mu_i\right)}{\vartheta\left(\mu_i\right)} \cdot \prod_{i<j}\frac{\vartheta\left(\frac{x_i}{y_j}h\right)}{\vartheta\left(\frac{y_i}{y_j}h\right)}\,.$$

The expression above is equal to 1 after substitution $y_i := x_i$. For other substitutions the first factor specializes to zero. $\qquad\square$

## 19. Elliptic weight function

The elliptic weight function of [RTV19] in the form given in [RiW20, §6] can be identified with the elliptic class of link patterns. It is a function in $n$ parameters $z_i$ and $n-1$ parameters $\gamma_i$. Set

$$x_i = z_i \,, \text{ for } 1 \leq i \leq n \text{ and } x_{j+n} = \gamma_j \,, \text{ for } 1 \leq j \leq n-1 \,.$$

Consider link patterns with $m = 2n-1$ nodes and the rank $r = n-1$, such that the arrows have sources in the last $n-1$ nodes. Such link patterns parameterize the orbits which are contained in the upper-right $n \times (n-1)$-rectangle. Each orbit coincides with an orbit with respect to $B_n \times B_{n-1}$ acting on $\mathrm{Hom}(\mathbb{C}^{n-1}, \mathbb{C}^n)$. Since the considered link patterns have $n-1$ arrows, the orbits are of maximal rank, i.e. are contained in the set of injective maps $\mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^{n-1}, \mathbb{C}^n)$. The quotient $\mathrm{Hom}^{\mathrm{inj}}(\mathbb{C}^{n-1}, \mathbb{C}^n)/B_{n-1}$ is just the flag variety $\mathcal{F}\ell_n$ and the $B_n \times B_{n-1}$-orbits are mapped by the quotient map to the Schubert cells. As in the previous section we multiply the elliptic classes of link patterns by the elliptic class of the matrix block

$$eu_{M'}^{ell} = \prod_{i=1}^{n} \prod_{j=1}^{n-1} \vartheta\left(\tfrac{z_i}{\gamma_j}\right)$$

and divide by the Chern class of the unipotent part of $B_{n-1}$

$$\mathcal{B}' = \prod_{n > i > j \geq 1} \frac{\vartheta\left(\tfrac{\gamma_i}{\gamma_j} h\right)}{\vartheta(h)} \,.$$

The resulting quotients

$$(30) \qquad \frac{\mathcal{E}\ell\ell(\mathcal{P}) \cdot eu_{M'}^{ell}}{\mathcal{B}'}$$

satisfies R-matrix recursion (6) and for $\mathcal{P}_{2n-1,n-1}^{\min}$ the restrictions to the fixed points of $\mathcal{F}\ell_n$ are equal to 0 for $\sigma \neq \mathrm{id}$ and the elliptic Euler class for $\sigma = \mathrm{id}$. It is an exercise to check that the functions agree before restricting, provided that we introduce a substitution as below:

**Corollary 19.1.** *The considered quotient* (30) *is equal to the elliptic weight function of* [RiW20, §6] *provided that we substitute*

$$(31) \qquad \mu_i := \frac{h\mu_n}{\mu_i} \,, \text{ for } 1 \leq i < n \,.$$

**Example 19.2.** (Compare [RiW20, Example 6.2].) Let $n = 3$, $m = 5$, $r = 2$. Then

$$\mathcal{E}\ell\ell(\mathcal{P}_{5,2}^{\min}) = \delta\left(\tfrac{z_1}{\gamma_1}, \mu_1\right) \delta\left(\tfrac{z_2}{\gamma_2}, \mu_2\right) \delta\left(\tfrac{z_1}{\gamma_2}, h\right),$$

$$\frac{\mathcal{E}\ell\ell(\mathcal{P}_{5,2}^{\min}) \cdot eu_{M'}^{ell}}{\mathcal{B}'} = \vartheta\left(\tfrac{z_2}{\gamma_1}\right) \vartheta\left(\tfrac{z_3}{\gamma_1}\right) \vartheta\left(\tfrac{z_3}{\gamma_2}\right) \frac{\vartheta\left(\tfrac{z_1}{\gamma_2} h\right)}{\vartheta\left(\tfrac{\gamma_1}{\gamma_2} h\right)} \frac{\vartheta\left(\tfrac{z_1}{\gamma_1} \mu_1\right)}{\vartheta\left(\mu_1\right)} \frac{\vartheta\left(\tfrac{z_2}{\gamma_2} \mu_2\right)}{\vartheta\left(\mu_2\right)} \,.$$

After the substitution (31) we obtain $\hat{\mathbf{w}}_{123}$ of *loc.cit.*

## 20. Geometric meaning of $\mathcal{E}\ell\ell(\mathcal{P})$

In the whole paper we have avoided to use directly the construction of elliptic characteristic classes as defined by Borisov and Libgober [BL03], except for the computation for $\mathcal{E}\ell\ell(X_{m,r}^{\min})$. The starting case was trivial from the geometric point of view, and further we did not have to compute the boundary divisor discrepancies of the resolution maps (8). We applied the principle, that the elliptic class is pure. Therefore we had to keep track of the coefficients $\nu_i$ and prove that the resulting class does not depend on the choice of a reduced word representing permutation. Now we can go back and recover the multiplicities of the divisors. Suppose $\mathcal{P} = s_{i_1} s_{i_2} \ldots s_{i_\ell} \mathcal{P}_{m,r}^{\min}$ is a reduced presentation of a link pattern. Let

$$\pi : Z = Z_{m,r}^{w} = P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_\ell} \times_B X_{m,r}^{\mathrm{id}} \to X_{m,r}^{w}$$

be the associated resolution and let

$$Z' = P_{i_2} \times_B P_{i_3} \times_B \cdots \times_B P_{i_\ell} \times_B X_{m,r}^{\mathrm{id}} \,.$$

We have a fibration $Z = P_{i_1} \times_B Z' \to \mathbb{P}^1 = P_{i_1}/B$ with the fiber $Z'$. The fiber over $\mathrm{id}B$ is a component of the boundary divisor $\partial Z$, other components are obtained from the components of $\partial Z'$ by application of the associated bundle construction $P_{i_1} \times_B -$. We define the coefficients of the boundary of the resolution inductively. We start with the multiplicities $\lambda_i$ attached to the components of $\bigcup_{i=1}^m D_i = \partial X_{m,r}^{\min}$. Suppose the multiplicities $\alpha_i'$ of $\partial_i Z'$ are defined and we do not change them applying the associated bundle construction, only shifting the indices by 1. We define the coefficient $\alpha_1$ of the component $B \times_B Z' \subset \partial Z$. If the admissible coefficient of the operation $\mathfrak{C}_{i_1}^{\diamond}$ is equal to $\frac{\mu_a^s \mu_b^t}{h^k}$ we set

$$\alpha_1 = 1 + k + s\lambda_a + t\lambda_b - s - t \,.$$

We recall that in §8 we have fixed the notation $\mu_i = h^{1-\lambda_i}$. Hence

$$\frac{\mu_a^s \mu_b^t}{h^k} = h^{s(1-\lambda_a)+t(1-\lambda_b)-k} = h^{1-(1+k+s\lambda_a+t\lambda_b-s-t)} \,.$$

It follows that

$$\mathcal{E}\ell\ell(\mathcal{P}) = \pi_*(\mathcal{E}\ell\ell(Z, \Sigma_{i=1}^{\ell} \alpha_i \partial Z_i + \Sigma_{j=1}^m \lambda_j \widetilde{D}_j)) \,.$$

Here by $\widetilde{D}_j$ we mean the result of the associated bundle construction applied $\ell$ times to the $j$-th component of $D_j \subset \partial X_{m,r}^{\min}$.

**Example 20.1.** Let

$$\mathcal{P} = s_1 s_2 \mathcal{P}_{3,1}^{\min} = \begin{array}{c} \overset{\frown}{\phantom{x}} \searrow \\ \frac{h^3}{\alpha} \dashind \alpha h \dashind h^2 \end{array} \,.$$

The link pattern $\mathcal{P}$ represents the orbit of the matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The resolution (in one of the maps) has the form

$$(x,y,z) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ z & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} xyz & -xy & x \\ xyz^2 & -xyz & xz \\ 0 & 0 & 0 \end{pmatrix} \,.$$

Here the boundary divisors are the following:

$$\widetilde{D}_1 = \{x = 0\},$$

$$\partial_2 Z = \{y = 0\},$$

$$\partial_1 Z = \{z = 0\}.$$

We set the multiplicity of $\widetilde{D}_1$ to be $\lambda_1 = \lambda$. The multiplicities of $\partial_i Z$ are dictated by the diagram



$$1 - \alpha_1 = 2(1 - \lambda) - 2, \quad 1 - \alpha_2 = (1 - \lambda) - 1,$$

i.e.

$$\alpha_1 = 2\lambda + 1, \qquad \alpha_2 = \lambda + 1.$$

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