Characteristic Classes, Homogeneous Spaces in Łukęcin 2019

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Contents

Theory of characteristic classes is intimately connected with the notion of push-forward in cohomology theory. We need some basic constructions. We will start with general formalism, but primarily we are interested in the standard cohomology theory and K-theory. Many seemingly nontrivial theorems follow from abstract nonsense. The topological spaces we deal with are ,,decent", they are the complex algebraic varieties. Technically we can assume that they are of homotopy type of finite CW-complex.

1 Complex-oriented cohomology theories

A generalized cohomology theory is an assignment

$$\text{pair of topological spaces} \longrightarrow \text{graded ring}\,.$$

We can consider $\mathbb{Z}$– or $\mathbb{Z}_2$–grading. We assume that cohomology theory satisfies all of Eilenberg-Steenrod axioms (homotopy invariance, excision, long exact sequence) except the dimension axiom, i.e. we do not assume anything about $h^*(pt)$.

Let $\mathbb{P}^n = \mathbb{P}^n(\mathbb{C})$ be the complex projective space of dimension n. The complex orientation (see [Sto68, p. 61]) is a choice of elements $e_n \in h^2(\mathbb{P}^n)$ such that $1, e_n, e_n^2, \dots, e_n^n$ is a free basis of $h^*(\mathbb{P}^n)$ over the ring coefficient ring $h^*(pt)$. We assume that $e_n^{n+1} = 0$ and $e_n|\mathbb{P}^{n-1} = e_{n-1}$. Moreover e_n^n is the image of a generator of $h^*(\mathbb{P}^n, \mathbb{P}^{n-1}) \simeq h^*(S^{2n}, pt) \simeq h^{*-2n}(pt)$ with respect to the natural map $h^*(\mathbb{P}^n, \mathbb{P}^{n-1}) \to h^*(\mathbb{P}^n)$. These data allow to define

- $c_1(L)$ for a topological line bundles,
- Chern classes,
- push forward for proper maps of complex manifolds (more general: for normally nonsingular maps).

1.1 Classical cohomology. The usual choice is $e_n = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^*(\mathbb{P}^n;\mathbb{Z})$, but we might have chosen $e_n = c_1(\mathcal{O}_{\mathbb{P}^n}(-1))$.

1.2 K-theory. K-group of a compact space (CW-complex) X is defined as the

$$K_0(X) = K(X) = \text{Grothendieck group of isomorphism classes of vector bundles}\,,$$

$$[V_1] - [V_2] = [W_1] - [W_2] \quad \text{if} \quad V_1 \oplus W_2 \simeq W_1 \oplus V_2\,.$$

Of course $K(pt) \overset{\dim}{\simeq} \mathbb{Z}$.

Exercise: $K(S^1) \to K(pt)$ is an isomorphism, $K(\mathbb{P}^1) \simeq \mathbb{Z}^2$ is spanned by the trivial line bundle and the Hopf bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$ a.k.a. the tautological bundle.

The K_0-groups extends to the cohomology theory: we set

$$\widetilde{K}_0(X) = ker\big(K(X) \to K(pt)\big)\,,$$

The K-groups of a pair are defined as

$$K_0(X,A) = \widetilde{K}(X/A)\,.$$

The K theory of a pair (X,A) can be presented as a formal difference of two bundles $E - F$ with given isomorphism $E_{|A} \simeq F_{|A}$. Alternative description is via the complexes of vector bundles $0 \to E_0 \to E_1 \to \dots E_n \to 0$ which are exact on A.

The negative gradations are defined by taking the n-fold topological suspension:

$$\widetilde{K}_{-n}(X) := \widetilde{K}(\Sigma^n X)\,,$$

$(n \geq 0)$. The K theory in positive gradations is defined by the Bott Periodicity Theorem

$$\widetilde{K}(X) \simeq \widetilde{K}(\Sigma^2 X)\,.$$

Precisely, the isomorphism is given by multiplication by the Bott element

$$\beta \in \widetilde{K}(\Sigma(X \cup \{pt\}) = K(\mathbb{P}^1 \times X, \{\{\infty\} \times X\})$$

$$\beta = \mathcal{O}_{\mathbb{P}^1} - \mathcal{O}_{\mathbb{P}^1}(-1)\,,$$

which defines the isomorphism

$$[V] \mapsto \beta \times [V] = [\mathcal{O}_{\mathbb{P}^1} \boxtimes V] - [\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes V]\,.$$

Here $\mathcal{O}_{\mathbb{P}^1}$ is understood as the trivial 1-dimensional bundle and $\mathcal{O}_{\mathbb{P}^1}(-1)$. To make this definition correct we note, that $(\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes V)_{|\{\infty\}\times X}$ and $(\mathcal{O}_{\mathbb{P}^1} \boxtimes V)_{|\{\infty\}\times X}$ are canonically isomorphic.

The choice of an orientation

$$e_n = \mathcal{O}_{\mathbb{P}^n} - \mathcal{O}_{\mathbb{P}^n}(-1)$$

is motivated by the exact sequence of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0\,,$$

where H is a hyperplane in $\mathbb{P}^n$.

1.3 Complex bordism theory. We assume that X is compact a C^∞-manifold, $\Omega^k(X)$ is generated by the maps $f : M \to X$, where M is a C^∞-manifold, and the bundle $f^*(TX) - TM$ has a stable complex structure. Then $e_n = [\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n]$

1.4 Non-standard orientations in $H^*(-;\mathbb{Q})$. Instead of $e_n = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ we can take $e_n := f(c_1(\mathcal{O}_{\mathbb{P}^n}(1)))$ where $f(x) = x + a_2x^2 + a_3x^3 + \dots \in \mathbb{Q}[[x]]$ is any power series. An interesting choice is the Jacobi theta function $\theta_\tau(x)$ which depends on the parameter $\tau \in \mathbb{C}_{im(\tau)>0}$.

1.5 The first Chern class c_1. Each line bundle over a compact space is a pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ for some n. The choice of e_n allows to choose in a coherent way the first Chern class by taking the pull back from $\mathbb{P}^n$.

1.6 Construction of Chern classes. Let E be a complex vector bundle over X of dimension n. Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) \in h^2(\mathbb{P}(E))$ then $h^*(\mathbb{P}(E))$ is a free module over $h^*(X)$ (via p^*, where $p : \mathbb{P}(E) \to X$ is the projection) with the basis $1, \xi, \xi^2, \dots \xi^{n-1}$. This follows from the Leray-Hirsch theorem. The Chern classes are defined as the coefficients of the relation

$$\sum_{i=0}^{n} (-1)^i c_i(E) \xi^{n-i} = 0$$

and $c_0(E) = 1$. (Note that if $E = L$ is a line bundle, then $\mathbb{P}(E) = X$, $\xi = L$. The relation is trivial $c_1(L) - c_1(L) = 0$.)

Example: If $h^* = H^*$, then the relation $(*) = \sum (-1)^i c_i(E) \xi^{n-i} = 0$ is satisfied. It is either the definition of the Chern classes or it follows from another property:

$$(*) = c_n\left(\underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}(E)}(-1), E)\right).$$

The Hom-bundle has a section, since $\mathcal{O}_{\mathbb{P}(E)}(-1) \subset E$ is the tautological bundle. The top Chern class of any bundle with section vanishes.

Example: Let $h^* = K$ be the K-theory. Then

$$\xi = c_1(\mathcal{O}_{\mathbb{P}(E)}(-1)) = 1 - \mathcal{O}_{\mathbb{P}(E)}(1).$$

The Koszul complex

$$0 \to p^*(\Lambda^n E^*) \otimes \mathcal{O}_{\mathbb{P}(E)}(-n) \to p^*(\Lambda^{n-1} E^*) \otimes \mathcal{O}_{\mathbb{P}(E)}(1-n) \to \dots p^*(\Lambda^1 E^*) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \to \mathcal{O}_{\mathbb{P}(E)} \to 0$$

with differentials induced by the contraction

$$p^*(\Lambda^i E^*) \otimes \mathcal{O}_{\mathbb{P}(E)}(-1) \to p^*(\Lambda^{i-1} E^*)$$

is exact. Thus after tensoring with $\mathcal{O}_{\mathbb{P}(E)}(n)$ we obtain the relation

$$\sum (-1)^{n-i} [p^* \Lambda^i E^* \otimes \mathcal{O}_{\mathbb{P}(E)}(n-i)] = 0,$$

Since $(-1)^{n-i}[\mathcal{O}_{\mathbb{P}(E)}(n-i)] = (\xi - 1)^{n-i}$ we have

$$\sum (\xi - 1)^{n-i} [p^* \Lambda^i E^*] = 0.$$

Example: Let $n = 2$:

$$(\xi - 1)^2 + (\xi - 1) E^* + \Lambda^2 E^* = 0$$
$$(1 - E^* + \Lambda^2 E^*) - \xi(2 - E^*) + \xi^2 = 0,$$

Hence

$$c_1(E) = 2 - E^*, \qquad c_2(E) = 1 - E^* + \Lambda^2 E^*.$$

For arbitrary n we deduce that formally

$$c_i(E) = e_i(\delta_1, \delta_2, \dots, \delta_n),$$

provided that $E = \bigoplus_i^n L_i$, $\delta_i = 1 - L_i^*$. Here e_i is the elementary symmetric function.

Theorem: *Let $i : Y \to X$ be the inclusion map of smooth complex manifolds of codimension n, then*

$$i^* i_*(\alpha) = c_n(\nu_{Y/X}) \alpha.$$

This statement is well known in $H^*(-)$ or K-theory. In general it follows from the construction below. The top Chern class is called the Euler class and denoted by $eu(-)$, or $eu^h(-)$ to remember the cohomology theory.

1.7 Thom class and Thom isomorphism.

Lemma: *The sequence*

$$0 \to h^*(\mathbb{P}(E\oplus 1), \mathbb{P}(E)) \to h^*(\mathbb{P}(E\oplus 1)) \to h^*(\mathbb{P}(E)) \to 0$$

is exact.

The Thom class of E is the element $\tau(E) \in h^*(E, E\setminus X)) \simeq h^*(\mathbb{P}(E\oplus 1), \mathbb{P}(E))$ which maps to $\sum(-1)^{n-k}c_k(E)\xi^{n-k}$ under the natural map $h^*(\mathbb{P}(E\oplus 1), \mathbb{P}(E)) \to h^*(\mathbb{P}(E\oplus 1))$. It has the property, that restricted to the fiber is a generator $\tau(E)_{|E_x} \in h^*(E_x, E_x\setminus 0) \simeq h^*(\mathbb{C}^n, \mathbb{C}^n\setminus 0)$. The multiplication by $\tau(E)$ defines an isomorphism

$$h^*(X) \to h^*(E, E\setminus X)\,,$$

again by Leray-Hirsch theorem.

The Thom class is natural with respect to pull-back's and direct sums.

Example: if h^* is the K-theory then the Thom class of E is given by the Koszul complex $p^*(\Lambda^\bullet E^*)$ with the differential on $p^*(\Lambda^k(E^*))_v = \Lambda^k(E^*)_{p(v)}$ given by the contraction with v.

1.8 Push forward. The case of inclusion $Y\subset X$: we assume that the normal $\nu_{X/Y}$ bundle has a structure of a complex vector bundle. We define the push-forward by the composition of the maps

$$h^{*-2\,\mathrm{codim} Y}(Y) \overset{Thom}{\simeq} h^*(\nu_{Y/X}, \nu_{Y/X}\setminus Y) \simeq h^*(Tub(Y), Tub(Y)\setminus Y) \to h^*(X, X\setminus Y) \to h^*(X)\,.$$

Every (at least in topology) proper map of smooth manifolds can be factorized as a composition $Y \hookrightarrow E \to X$ for some complex vector bundle E. The push-forward is defined when the normal bundle of X has a complex structure:

$$h^*(Y) \to h^{*+\dim E-\dim Y}(Tub(Y), Tub(Y)\setminus Y) \to h^{*+\dim E-\dim Y}(E, E\setminus X) \overset{Thom}{\simeq} h^{*+\dim X-\dim Y}(X)\,.$$

1.9 Generalized Riemann-Roch Theorem. See e.g. [FF16, §42] Suppose that we have a transformation of oriented cohomology theories $\Theta : k^* \to h^*$ in general the push forwards do not match. Since the push-forwards are defined by the Thom isomorphism, let us check what happens there. Let $\tau^k(E) \in k^*(E, E\setminus X)$ and $\tau^h(E) \in h^*(E, E\setminus X)$ be the Thom classes. The Thom isomorphism in k^* sends $\alpha \in k^*(X)$ to $\tau^k\cdot p^*(\alpha)$ where $p : E \to X$. Let $\mathcal{T}_\Theta(E) = (Thom^h)^{-1}\big(\Theta_E(\tau^k(E))\big)$. The the following diagram is commutative

$$\begin{tikzcd} k^*(X) \arrow[r, "\mathcal{T}_\Theta(E)\cdot\Theta_X"] \arrow[d, "Thom^k"'] & h^*(X) \arrow[d, "Thom^h"] \\ k^*(E, E\setminus X) \arrow[r, "\Theta_E"'] & h^*(E, E\setminus X) \end{tikzcd}\quad.$$

For $\alpha = 1 \in k^*(X)$

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "Thom^k"'] & \mathcal{T}_\Theta(E) \arrow[d, "Thom^h"] & = (Thom^h)^{-1}\big(\Theta_E(\tau^k(E))\big) \\ \tau^k(E) \arrow[r] & \Theta_E(\tau^k(E)) \end{tikzcd}$$

and in general

$$\begin{tikzcd} \alpha \arrow[r] \arrow[d, "Thom^k"'] & \mathcal{T}_\Theta(E)\cdot\Theta_X(\alpha) \arrow[d, "Thom^h"] \\ \tau^k(E)\cdot p^*\alpha \arrow[r] & \Theta_E(\tau^k(E)\cdot p^*\alpha) = \Theta_E(\tau^k(E))\cdot p^*(\Theta_X(\alpha)). \end{tikzcd}$$

1.10 Grothendieck-Riemann-Roch and Hirzebruch-Riemann-Roch. For a complex manifold X we define $Td(X) = \mathcal{T}_\Theta(\nu(X)) = \mathcal{T}_\Theta(TX)^{-1}$ (or Td_θ to underline dependence of Θ), where $\nu(X)$ is the normal bundle[1]. Let $f : Y \to X$ be a holomorphic proper map. Then

$$\begin{tikzcd} k^*(X) \arrow[r, "Td(X)\cdot\Theta_X"] \arrow[d, "f_*^k"'] & h^*(X) \arrow[d, "f_*^h"] \\ k^*(Y) \arrow[r, "Td(Y)\cdot\Theta_Y"'] & h^*(Y) \end{tikzcd} .$$

In the special case when $X = pt$

$$\Theta_{pt}(f_*^k(\alpha)) = f_*^h(Td(X) \cdot \Theta_X(\alpha)) .$$

Suppose

- k^* is the K-theory,
- $h^*(-) = H^*(-, \mathbb{Q})$,
- $\Theta = ch$ the Chern character, i.e. $ch(L) = e^{c_1(L)} = 1 + c_1(L) + \frac{1}{2}c_1(L)^2 + \cdots + \frac{1}{(\dim X)!}c_1(L)^{\dim X}$,
- $\alpha = [E]$ the class of a holomorphic vector bundle,
- $Td_K^H(X) = Td(TX)$, where $Td(-)$ is the multiplicative characteristic class associated to the power series
$$\frac{x}{1 - e^{-x}} .$$
This means that
$$Td(E \oplus F) = Td(E) \cdot Td(F) , \qquad Td(L) = \frac{c_1(L)}{1 - e^{-c_1(L)}} ,$$

then the last equality reads

$$\chi(X, E) = \int_X Td(TX) \cdot ch(E) .$$

The point is that the push forward in topological K-theory coincides with the push-forward in the algebraic K-theory

$$f_*^{alg}([E]) = \sum_{k=1}^{\dim X} (-1)^k [R^k f_*(E)] .$$

Assuming that f is a projective morphism it is enough to check the compatibility of push-forwards for inclusions of manifolds and for projections in $\mathbb{P}^n$ bundles.

2 Algebraic theories

We assume that X is a (complex) algebraic variety

2.1 Algebraic K-theories built from coherent sheaves and locally free sheaves. Consider the category of coherent sheaves Coh_X and its full subcategory $LocFree_X$ whose objects are locally free sheaves. Both are so called exact categories, i.e. there is distinguished a class of short exact sequences. In that situation we define K-theory as the free group spanned by the isomorphism classes of objects divided by the relation generated by

$$[A] + [C] = [B] \qquad \text{whenever} \qquad 0 \to A \to B \to C \to 0 \qquad \text{is a short exact sequence.}$$

Then

[1] The bundle $\nu(X)$ satisfies $\nu(X) \oplus TX$ is a trivial bundle. It always exists in topology provided that X is paracompact

$K^{alg}(X) := K(LocFree_X)\,,$ behaves well with respect to f^*,

$G(X) := K(Coh_X)\,,$ behaves well with respect to $f_*^{alg} := \sum_{k=1}^{\dim X}(-1)^k[R^k f_*(-)]$.

If X is smooth, quasiprojective, then $K^{alg}(X) = G(X)$, since every sheaf has a finite resolution consisting of locally free sheaves (Hilbert syzygy theorem).

2.2 Chow groups. This is an analogue of homology theory. The group $A_k(X)$ is generated by algebraic subvarieties of dimension k.

2.3 Comparison. Suppose X is a smooth variety of dimension n. Then there is a class map $cl : A_k(X) \to H^{2(n-k)}(X)$. If X is singular, then we have a map to Borel-Moore homology $H_{2k}^{BM}(X)$ which can be identified with $H_{2k}(\bar{X}, \partial X)$, where $\bar{X}$ is a compatification (completion) of X and $\partial X = \bar{X} \setminus X$, see §6.1.

2.4 Chern character. If X is smooth, then there is a compatibility

$$\begin{tikzcd} K^{alg}(X) \arrow[r, "\text{forgetting}"] \arrow[d, "ch^{alg}"'] & K(X) \arrow[d, "ch"] \\ A_*(X)\otimes\mathbb{Q} \arrow[r, "cl\otimes\mathbb{Q}"'] & H^{2*}(X;\mathbb{Q}) \end{tikzcd} \quad .$$

2.5 Higher Chow/K-theory. Suppose $Y \subset X$ is a smooth subvariety of a smooth variety, $U = X \setminus Y$. It is possible to extend the sequences

$$K^{alg}(Y) \xrightarrow{i_*} K^{alg}(Y) \xrightarrow{j^*} K^{alg}(U) \longrightarrow 0$$

$$A_k(Y) \xrightarrow{i_*} A_k(X) \xrightarrow{j^*} A_k(U) \longrightarrow 0.$$

The existence of these extensions is important for some proofs, but we will not discuss that.

3 Equivariant complex-oriented cohomology theories

The equivariant cohomology theory is a contravariant functor

$$(\text{compact group } G\,,\ \text{pair of } G\text{-spaces}) \mapsto \text{graded commutative ring}\,.$$

The group G and the space X are assumed to be compact. We assume usual axioms of nonequivariant theory (G-homotopy invariance, excision, long exact sequence). In addition there are natural isomorphism for a subgroup $H \subset G$ and a H-space

$$h_G^*(G \times_H X) \simeq h_H^*(X)\,.$$

In particular

$$h_G^*(G/H) \simeq h_H^*(pt)\,.$$

There is given the Thom class $\tau_G^h \in h^{2\dim E}(E, E \setminus X)$ for any complex G-bundle E. We assume that the given class is natural with respect to pull-back's and direct sums. This class allows to define a push-forward in the equivariant theory.

Remark: comparing with the nonequivariant case we impose existence of the Thom class for any equivariant vector bundle. One can view the equivariant definition, as a choice of c_1 for line bundles. This would not be enough for general G since we have to care about all equivariant bundles, which possibly do not split into line bundles.

By functoriality the ring $h_G^*(X)$ is an algebra over $h_G^*(pt)$.

3.1 Equivariant K-theory. Construction by Segal [Seg68]. This theory is more natural than equivariant cohomology, so we present it first. Let $Vect_X^G$ be the exact category of G-vector bundles on a compact topological space X. Then the equivariant K-theory of X is defined as the K-theory of that category.

$$K_G^0(X) = K_G(X) = K(Vect_X^G)\,.$$

This category is semisimple, hence the construction in §1.2 applies. The extension of K_0 to cohomology theory is possible due to Thom isomorphism (equivariant Bott periodicity)

$$\widetilde{K}_G(X) \simeq K_G(E, E \setminus X)\,,$$

given by Koszul complex (see §1.7). Straight from the definition

$$K_G^0(pt) = \mathrm{R}(G)$$

is the representation ring and

$$K_G^1(pt) = \widetilde{K}_G^0(S^1) = 0$$

If $G = \mathbb{T} = (\mathbb{C}^*)^n$ then

$$K_{\mathbb{T}}(pt) = \mathrm{R}(\mathbb{T}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]\,,$$

where t_i denotes the representation given by the projection on the i-th factor $\mathbb{T} \to S^1 \hookrightarrow \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$. Invariantly $K_{\mathbb{T}}(pt) = \mathbb{Z}[\mathbb{T}^\vee]$ where $\mathbb{T}^\vee$ denotes the group of characters of $\mathbb{T}$.

Suppose G is a (complex) algebraic group and X is a (complex) algebraic G-variety. Algebraic K-theory $K_G^{alg}(0)$ is the K-theory of the exact category of algebraic G-bundles over X.

Theorem: [FRW18] *Suppose that $G_{\mathbb{C}}$ is a linear algebraic group, $G \subset G_{\mathbb{C}}$ a maximal compact subgroup. Let X be a smooth algebraic $G_{\mathbb{C}}$-variety, consisting of a finite number of orbits. Then the natural forgetting map $\iota : K_{G_{\mathbb{C}}}^{alg}(X) \to K_G^0(X)$ is an isomorphism and $K_G^1(X) = 0$.*

3.2 Eqivariant cohomology. If G acts on X freely, then $H_G^*(X) = H^*(X/G)$. Otherwise we substitute X by a homotopy equivalent space on which G acts freely. The formal definition is the following:

$$H_G^*(X) = H^*(EG \times_G X)\,,$$

where EG is a contractible G-space with a free G-action. If $G = S^1$ then $EG = S^\infty \subset \mathbb{C}^\infty$. The classifying space BG for G is defined as EG/G. For a torus $\mathbb{T} = (S^1)^n$ we have $B\mathbb{T} = (\mathbb{P}^\infty)^n$ and $H_{\mathbb{T}}^*(pt) = H^*(B\mathbb{T}) = \mathbb{Z}[x_1, x_2, \ldots, x_n]$, where $x_i = c_1(\xi_i)$, $\xi_i = E\mathbb{T} \times_{\mathbb{T}} \mathbb{C} \to B\mathbb{T}$, with $\mathbb{T}$ acting on $\mathbb{C}$ via t_i. Invariantly: $H_{\mathbb{T}}^*(pt) = Sym(\mathbb{T}^\vee)$

If X is a manifold then it is possible to define equivariant cohomology using de Rham theory. For $G = \mathbb{T} = S^1$

$$H_{\mathbb{T}}^*(M; \mathbb{R}) \simeq H^*((\Omega_X^\bullet)^{\mathbb{T}}, d_{\mathbb{T}})\,, \quad d_{\mathbb{T}}(\omega) = d\omega + \iota_v \omega\,.$$

where v is the vector filed defined by the circle action, see [AB84].

3.3 Chern character. To define Chern character in equivariant theory we have to complete the cohomology, since in general $EG \times_G X$ is of infinite dimension:

$$\begin{array}{rcl} ch^{\mathbb{T}} : K_G(X) & \to & \hat{H}_G^*(X; \mathbb{Q}) \\ [V] & \mapsto & ch(EG \times_G V) \end{array}$$

For example

$$ch^{\mathbb{T}} : K_{\mathbb{T}}(pt) \to \hat{H}_{\mathbb{T}}^*(pt; \mathbb{Q})$$

is the map

$$\begin{array}{rcl} \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}] & \to & \mathbb{Q}[[x_1, x_2, \ldots, x_n]] \\ t_i^{\pm 1} & \mapsto & e^{\pm x_i} \end{array}$$

4 Localization

4.1 Abstract localization theorem. Let $S \subset h_G^{2*}(pt)$ be a multiplicative system. Let

$$X_S = \{x \in X \ : \ S \cap ker(h_G^*(pt) \to h_G^*(G/G_x)) = \emptyset\} .$$

Example: Suppose $G = \mathbb{T}$, $h^* = H^*(-, \mathbb{Q})$, S is generated by $h^2_{\mathbb{T}}(pt) = \mathbb{Q}[t_1, t_2, \dots t_n]_1 \setminus \{0\}$, i.e. by the nontrivial linear forms (or simply $S = H^*_{\mathbb{T}}(pt) \setminus \{0\}$). Then $X_S = X^{\mathbb{T}}$.

Proof: If $\mathbb{T}' \subsetneq \mathbb{T}$, there exists a character $0 \neq f : \mathbb{T} \to \mathbb{C}^*$ such that $\mathbb{T}' \subset ker(f)$. Then

$$f \in ker(H^2_{\mathbb{T}}(pt) \to ker(H^2_{\mathbb{T}}(\mathbb{T}/\mathbb{T}')) = ker(\mathbb{T}^\vee \to \mathbb{T}'^\vee) .$$

Example: $G = \mathbb{T}$, $h^* = K^*$, S is generated by $(1 - f)$, where f is a nontrivial character. Then $X_S = X^{\mathbb{T}}$.

Theorem: *The restriction map $S^{-1}h^*_G(X) \to S^{-1}h^*_G(X_S)$ is an isomorphism.*

Proof. If $X = G/H$, such that $s \in S \cap h^*_G(G/H)$, then $S^{-1}h^*_G(G/H) = 0$. If X is a decent topological space, (e.g. G-CW-complex), then applying an induction by equivariant cells we have

$$S^{-1}h^*_G(X, X \setminus X_S) = 0 .$$

Finally we use the the long exact sequence argument.

Example: $h^* = H^*(-, \mathbb{Q})$, $G = \mathbb{T} = (S^1)^{n+1}$, $X = \mathbb{P}^n$:

$$H^*_{\mathbb{T}}(\mathbb{P}^n) = \mathbb{Q}[t_0, t_1, \dots, t_n, h]/(\Pi(h - t_i)) , \qquad H^*_{\mathbb{T}}((\mathbb{P}^n)^{\mathbb{T}}) = \bigoplus_{i=0}^{n} \mathbb{Q}[t_0, t_1, \dots, t_n] .$$

The restriction map

$$f \ \mapsto \ (f_{|h \to -t_i})_{i=0,1,\dots n}$$

This is a monomorphism. The image consists of the sequences $(g_i)_{i=0,1,\dots n}$, such that $g_i - g_j$ is divisible by $t_i - t_j$. Hence the cokernel of the restriction map is killed by S generated by the monomials $t_i - t_j$.

For torus $\mathbb{C}^*$ action on compact algebraic manifolds the localization theorem leads to the conclusion that

$$\dim H^*(M; \mathbb{Q}) = \dim H^*(M^{\mathbb{T}}; \mathbb{Q}) .$$

For example for $M = \mathbb{P}^1$ we have $\dim H^*(\mathbb{P}^1) = \dim H^*(\{0, \infty\}) = 2$. The relation between cohomologies of M and $M^{\mathbb{T}}$ was studied by Białynicki-Birula [BB73]. The decomposition of cohomology following from the decomposition into BB-cells gives stronger stronger result

$$H^k(M; \mathbb{Q}) \simeq \bigoplus_{F \subset M^{\mathbb{T}}} H^{k-2n_F}(F; \mathbb{Q}) ,$$

where n_F is the dimension of the subbundle of $\nu(F)$ with positive weights of torus action.

The localization theorem is meaningful for $G = \mathbb{T}$. Otherwise, e.g. when G is connected, then $H^*_G(pt; \mathbb{Q}) \to H^*_G(G/H; \mathbb{Q}) = H^*_H(pt; \mathbb{Q})$ is a monomorphism, provided that H contains the maximal torus. This follows from the formula

$$H^*_G(X; \mathbb{Q}) = H^*_{\mathbb{T}}(X; \mathbb{Q})^W$$

where $W = N\mathbb{T}/\mathbb{T}$ is the Weyl group of G. For $S = H^*_G(pt) \setminus \{0\}$ and a homogeneous space $X = G/P$ we have $S^{-1}H^*_G(G/P) = 0$ which agrees with the fact that $(G/P)^G = \emptyset$.

4.2 Localization formula. Suppose X is a smooth complex $\mathbb{T}$-manifold. The composition map

$$h^*_{\mathbb{T}}(X^{\mathbb{T}}) \xrightarrow{i_*} h^*_{\mathbb{T}}(X) \xrightarrow{i^*} h^*_{\mathbb{T}}(X^{\mathbb{T}})$$

is the multiplication by the $eu(\nu_F)$ on each component of $X^{\mathbb{T}}$. If we can invert

$$eu(\nu_{X^{\mathbb{T}}}) = (eu(\nu_F))_{F\subset X^{\mathbb{T}}}\,,$$

then the composition $\frac{1}{eu(\nu_{X^{\mathbb{T}}})} i^* \circ i_*$ is an identity. Then (under some finiteness assumptions) the composition $i_* \circ \frac{1}{eu(\nu_{X^{\mathbb{T}}})} i^*$ is the identity as well.

Theorem: *If $S \subset h^*_{\mathbb{T}}(pt)$ contains $c_1(\mathbb{C}_\chi)$ for χ appearing in the normal bundles ν_F then $eu(\nu_{X^{\mathbb{T}}})$ is invertible.*

For the proof in the case of Chow groups see [EG98]. This proof generalizes easily to a wider setup. Theorem is obvious if $X^{\mathbb{T}}$ is finite. As a corollary we have:

Theorem: Localization Theorem. *For $\alpha \in h^*_{\mathbb{T}}(X)$*

$$\alpha = \sum_{F\subset X^{\mathbb{T}}} i_{F*}\left(\frac{i_F^*(\alpha)}{eu(\nu_F)}\right).$$

The equality holds in the localization of $h^*_{\mathbb{T}}(X)$.

Taking the push-forward to the point we obtain classical theorems:

Theorem: Atiyah-Bott-Berline-Vergne. *Suppose that X is a compact complex manifold, $X^{\mathbb{T}}$ is finite, $\alpha \in H^*_{\mathbb{T}}(X)$*

$$\int_X \alpha = \sum_{p\in X^{\mathbb{T}}} \frac{\alpha_{|p}}{eu(p)}$$

where $eu(p) = eu^H(p) \in H^*_{\mathbb{T}}(pt)$ is the product of weights appearing in the tangent representation at p.

Example: $X = \mathbb{P}^n$, $\alpha = c_1(\mathcal{O}(1))^k$ via residues

$$\sum_{i=0}^n \frac{(-x_i)^k}{\prod_{j\neq i}(x_j - x_i)} = (-1)^{n-k}\sum_{i=0}^n \frac{x_i^k}{\prod_{j\neq i}(x_i-x_j)} = (-1)^{n-k}\sum_{i=0}^n Res_{z=x_i}\frac{z^k}{\prod_{j=0}^n(z-x_j)} =$$

$$= -(-1)^{n-k}Res_{z=\infty}\frac{z^k}{\prod_{j=0}^n(z-x_j)} = (-1)^{n-k}\mathrm{Coeff}\left[\frac{w^{-k}}{\prod_{j=0}^n(w^{-1}-x_j)}, w, 1\right] =$$

$$= (-1)^{n-k}\mathrm{Coeff}\left[\frac{w^{n-k}}{\prod_{j=0}^n(1-wx_j)}, w, 1\right] = (-1)^{n-k}\mathrm{Coeff}\left[\frac{1}{\prod_{j=0}^n(1-wx_j)}, w, k-n\right] =$$

$$= (-1)^{k-n}\sum_{|I|=k-n+1} x^I\,,$$

i.e. the complete symmetric function of degree $k-n$.

Theorem: Atiyah-Bott, Grothendieck. *Suppose that X is a complex manifold, $X^{\mathbb{T}}$ is finite, $[E] \in K_{\mathbb{T}}(X)$*

$$\chi(X,E) = \sum_{p\in X^{\mathbb{T}}} \frac{E_p}{eu^K(p)}$$

where $eu^K(p) \in K^*_{\mathbb{T}}(pt) \simeq \mathbb{Z}[t_1^{\pm1}, t_2^{\pm1}, \dots, t_n^{\pm1}]$ is the product of factors $(1-t^{-w})$, where w is the weight appearing in the tangent representation at p.

Example: $X = \mathbb{P}(V) = \mathbb{P}^n$, $E = \mathcal{O}(k)$ for $k \geq 0$: modifying the previous calculation

$$\chi(\mathbb{P}^n;\mathcal{O}(k))_{T_i:=t_i^{-1}} = \sum_{|I|=k} T^I\,.$$

Note that we obtain the character of $Sym^k(V^*)$.

Example: $X = Gr(k,n)$, $h^* = H^*$, $\alpha = c_1(L)^{k(n-k)}$, where $L = \mathcal{O}(-1) = \Lambda^k\gamma_{k,n}$, $\gamma_{k,n}$ is the tautological bundle:

$$\int_{Gr(k,n)} c_1(L)^k = \sum_{I\subset \underline{n}\ |I|=k} \frac{\left(\sum_{i\in I} x_i\right)^{k(n-k)}}{\prod_{i\in I,\, j\in I^\vee}(x_j - x_i)}\,.$$

Example: In K-theory let us compute $\chi(Gr(k,n);L^*)$ of the previous example

$$\chi(Gr(k,n);L^*) = \sum_{I\subset \underline{n}\ |I|=k} \frac{\prod_{i\in I} t_i^{-1}}{\prod_{i\in I,\, j\in I^\vee}(1-\frac{t_i}{t_j})}\,.$$

After simplification the answer is $e_2(t_1^{-1},t_2^{-1},t_3^{-1},t_4^{-1})$. Computing further powers of L^{-1}:

$$\chi(Gr(k,n);L^{-2}) = e_2 - e_1e_3$$

$$\begin{array}{lcl}
L^{-1} & \mapsto & e_2 \\
L^{-2} & \mapsto & e_2^2 - e_1e_3 \\
L^{-3} & \mapsto & e_2^3 - 2e_1\sigma_3e_2 - e_4\sigma_2 + e_3^2 + e_1^2e_4 \\
L^{-4} & \mapsto & e_2^4 - 3e_1\sigma_3e_2^2 - 2e_4e_2^2 + 2e_3^2e_2 + 2e_1^2e_4e_2 + \sigma_1^2e_3^2 + \sigma_4^2 - 2e_1e_3e_4 \\
\vdots & &
\end{array}$$

Can you see the pattern?

4.3 Example. Weyl character formula. Let X be the complete flag variety

$$\mathrm{GL}_n/B = \{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n \mid \dim V_i = i\}\,.$$

The maximal torus of GL_n is acting. The fixed points are indexed by permutations. The K-theoretic Euler class at σ is equal to $\prod_{i<j}(1 - t_{\sigma(i)}/t_{\sigma(j)})$. Let E be the line bundle $\mathscr{L}(\lambda) := \mathrm{GL}_n \times_B \mathbb{C}_{-\lambda}$. Suppose $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \lambda_n \geq 0)$, i.e. λ is dominant

$$\sum_{\sigma\in\Sigma_n} \frac{\prod_{i=1}^n t_{\sigma(i)}^{-\lambda_i}}{\prod_{i<j}(1-\frac{t_{\sigma(i)}}{t_{\sigma(j)}})} = \chi(V_\lambda^*)\,,$$

where V_λ is the simple representation of G with the highest weight λ. by Borel-Weyl-Bott theorem (see e.g. [FuHa91, §23.3]. For example $n=2$, $\lambda=(k,0)$:

$$\frac{t_1^{-k}}{1-\frac{t_1}{t_2}} + \frac{t_2^{-k}}{1-\frac{t_2}{t_1}} = \frac{t_1^{-k-1}}{t_1^{-1}-t_2^{-1}} + \frac{t_2^{-k-1}}{t_2^{-1}-t_1^{-1}} = \frac{t_1^{-k-1}-t_2^{-k-1}}{t_1^{-1}-t_2^{-1}} = t_1^{-k} + t_1^{-k+1}t_2^{-1} + \cdots + t_2^{-k}\,.$$

Setting $t_i := t_i^{-1}$ The formula can be transformed to

$$\sum_{\sigma\in\Sigma_n} \frac{\prod_{i=1}^n t_{\sigma(i)}^{\alpha_i}}{\prod_{i<j}(t_{\sigma(i)}-t_{\sigma(j)})} = \frac{\begin{vmatrix} t_1^{\alpha_1} & t_1^{\alpha_2} & \dots & t_1^{\alpha_n} \\ t_2^{\alpha_1} & t_2^{\alpha_2} & \dots & t_2^{\alpha_n} \\ \vdots & & & \\ t_n^{\alpha_1} & t_n^{\alpha_2} & \dots & t_n^{\alpha_n} \end{vmatrix}}{\begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \dots & 1 \\ t_2^{n-1} & t_2^{n-2} & \dots & 1 \\ \vdots & & & \\ t_n^{n-1} & t_n^{n-2} & \dots & 1 \end{vmatrix}}$$

where $\alpha_k = \lambda_k + n - k$.

4.4 Lefschetz-Riemann-Roch. Relative generalization of AB-BV localization theorem theorem is the commutativity of the following diagram:

Theorem: [Nie77],[CG97, Thm. 5.11.7] *Suppose $f : X \to Y$ is a map of complex manifolds. Then the following diagram diagram is commutative*

$$\begin{tikzcd} h^*(X) \arrow[r, "\mathrm{res}_X"] \arrow[d, "f_*"'] & S^{-1}h^*(X^{\mathbb{T}}) \arrow[d, "(f_{|X^{\mathbb{T}}})_*"] \\ h^*(Y) \arrow[r, "\mathrm{res}_Y"'] & S^{-1}h^*(Y^{\mathbb{T}}) \end{tikzcd} \quad ,$$

where

$$\mathrm{res}_X(\alpha) = \frac{i_*(\alpha)}{eu(\nu_{X^{\mathbb{T}}})} = \sum_{F \subset X^{\mathbb{T}}} \frac{i_F^*(\alpha)}{eu(\nu_F)}$$

and $S \subset h^*_{\mathbb{T}}(pt)$ is generated by $c_1(\mathbb{C}_w)$, where $w \neq 0$.

If Y is a point we obtain the localization of the previous section. One can consider this theorem as a special case of the Riemann-Roch for (equivariant) generalized cohomology theories:

$$h_1^*(-) := h^*_{\mathbb{T}}(-) \xrightarrow{i^*} h_2^*(-) := S^{-1}h^*_{\mathbb{T}}((-)^{\mathbb{T}}) .$$

5 GKM

5.1 Formality. We say that a topological G-space X is equivariantly formal (with respect to the theory h^*) if $h^*_G(X)$ is a free module over $h^*_{\mathbb{T}}(pt)$.

Theorem: *Let G be a compact group, $G_{\mathbb{C}}$ its complexification. If X is a smooth, compact algebraic variety, on which $G_{\mathbb{C}}$ acts algebraically, then X is equivariantly formal with respect to $H^*(-;\mathbb{Q})$.*

(Similarly for K-theory tensored with $\mathbb{Q}$, see [Um13] for the flag variety case.)

If X is equivariantly formal then $h^*_{\mathbb{T}}(X) \to h^*_{\mathbb{T}}(X^{\mathbb{T}})$ is a monomorphism. The following lemma is a description of the image.

5.2 Chang-Skjelbred Lemma. Suppose X is equivariantly formal, then the following sequence is exact

$$0 \to H^*_{\mathbb{T}}(X;\mathbb{Q}) \xrightarrow{i^*} H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}) \xrightarrow{\delta} H^{*+1}_{\mathbb{T}}(X_1, X^{\mathbb{T}};\mathbb{Q}) ,$$

where X_1 is the union of 0- and 1-dimensional orbits and δ is the differential in the long exact sequence of the pair $(X_1, X^{\mathbb{T}})$. The statement is equivalent to

$$im(H^*_{\mathbb{T}}(X_1;\mathbb{Q}) \to H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q})) = im(H^*_{\mathbb{T}}(X;\mathbb{Q}) \to H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}))$$

or

$$ker(H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}) \to H^{*+1}_{\mathbb{T}}(X_1, X^{\mathbb{T}};\mathbb{Q})) = ker(H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{Q}) \to H^{*+1}_{\mathbb{T}}(X, X^{\mathbb{T}};\mathbb{Q})) .$$

See the proof in [Ful, Lecture 5] for a particular case of GKM spaces. For K-theory see [RK03].

5.3 GKM spaces. Instead of compact torus let now $\mathbb{T} \simeq (\mathbb{C}^*)^n$. We assume, that there is a finite number of fixed points and 1-dimensional orbits. If X is smooth and the tangent characters at each fixed point are pairwise not proportional then the second condition holds provided that the first condition holds. The GKM graph is defined as follows

- V vertices=fixed points
- E edges=one dimensional orbits, joining fixed points: from the tail $t(e)$ to head $h(e)$
- each edge e is labeled by the weight $w(e)$ through which $\mathbb{T}$ acts on the corresponding orbits.

Having such abstract data (V, E, w) we define the GKM algebra as the kernel of the map

$$\mathscr{A}(V,E,w) = ker(\bigoplus_{v\in V}\Lambda \xrightarrow{\delta} \bigoplus_{e\in E}\Lambda/(w(e)))\,,$$

where $\Lambda = H^*_{\mathbb{T}}(pt;\mathbb{Q})$

$$\delta\big((g_v)_{v\in V}\big)_e = g_{h(e)} - g_{t(e)}\,.$$

Theorem: [GKM98] *Suppose that X is equivariantly formal, then*

$$H^*_{\mathbb{T}}(X;\mathbb{Q}) \simeq \mathscr{A}(V,E,w)\,.$$

The theorem is a reformulation of the original Chang-Skjelbred Lemma, since we have

$$H^{*+1}_{\mathbb{T}}(\mathbb{P}^1,\{0,\infty\}) = H^*_{\mathbb{T}}(\mathbb{C}^*) = \Lambda/(weight),$$

hence

$$H^{*+1}_{\mathbb{T}}(X_1, X^{\mathbb{T}}) = \bigoplus_{e\in E}\Lambda/(w(e)).$$

The K-theoretic version Λ is replaced by $K^*_{\mathbb{T}}(pt) = \mathrm{R}(\mathbb{T})$ and

$$\mathscr{K}(V,E,w) = ker(\bigoplus_{v\in V}\mathrm{R}(\mathbb{T})_{\mathbb{Q}} \xrightarrow{\delta} \bigoplus_{e\in E}\mathrm{R}(\mathbb{T})_{\mathbb{Q}}/(1-t^{-w(e)}))\,.$$

Theorem: [RK03] *Suppose that $K_{\mathbb{T}}(X)_{\mathbb{Q}}$ is a free $\mathrm{R}(\mathbb{T})_{\mathbb{Q}}$-module, then*

$$K^*_{\mathbb{T}}(X)_{\mathbb{Q}} \simeq \mathscr{K}(V,E,w)\,.$$

Geometric interpretation:

$$Spec(H^*_{\mathbb{T}}(X;\mathbb{C})) = \bigsqcup_{v\in V}\mathbb{C}^n/\text{glued along a configuration of hypersurface}\,.$$

Here $\mathbb{C}^n$ is identified with the Lie algebra $\mathfrak{t}$. The (dual of the) restriction map

$$(i^*)^* : Spec(H^*_{\mathbb{T}}(X^{\mathbb{T}};\mathbb{C}))Spec(H^*_{\mathbb{T}}(X;\mathbb{C}))$$

is the gluing map

$$\bigsqcup_{v\in V}\mathbb{C}^n \longrightarrow \bigsqcup_{v\in V}\mathbb{C}^n/\sim\;.$$

Similarly

$$Spec(K_{\mathbb{T}}(X)_{\mathbb{C}})) = \bigsqcup_{v\in V}\mathbb{T}/\text{glued along a configuration of subtori of codimension 1}\,.$$

The map $(i^*)^*$ is again is the gluing map

$$\bigsqcup_{v\in V}\mathbb{T} \longrightarrow \bigsqcup_{v\in V}\mathbb{T}^n/\sim\;.$$

One can guess, that the similar picture appears for ,,equivariant elliptic cohomology". There

$$\text{“Spec”}(E\ell\ell^*_{\mathbb{T}}(pt)) = E^n$$

for $(\mathbb{C}^*)^n$ is the product of an elliptic curve E which is an analogy to

$$Spec(H^*_{(\mathbb{C}^*)^n}(pt;\mathbb{C})) = \mathbb{C}^n\,,\qquad Spec(K_{(\mathbb{C}^*)^n}(pt)_{\mathbb{C}}) = (\mathbb{C}^*)^n\,,$$

see [Gj94]. The point is that $E\ell\ell^*_{(\mathbb{C}^*)^n}(pt)$ itself does not exist, since the expected variety is not affine. Instead we consider the algebra of sheaf sections over a space glued from the products of an elliptic curve.

5.4 Example of the Grassmannian. The associated GKM graph is the following:

- vertices are the subsets $A \subset \{1, 2, \dots, n\}$, $|A| = k$.
- the edges: if A_1 differs from A_2 by one element: $A_1 = (A_1 \cap A_2) \cup \{t_{i_1}\}$, $A_2 = (A_1 \cap A_2) \cup \{t_{i_2}\}$, then there is an edge $A_1 \to A_2$ with the label $t_{i_2} - t_{i_1}$.

For projective algebraic varieties GKM graph has a natural embedding to $\mathbb{R}^n$, where $n = \dim \mathbb{T}$, see the moment map below.

5.5 Moment map without symplectic geometry. Let $\mathfrak{t}$ be the Lie algebra of the torus $\mathbb{T}$. Suppose that L is a $\mathbb{T}$-equivariant line bundle over X. There is well defined map $\mu_0 = X^{\mathbb{T}} \to \mathfrak{t}^*$,

$$\mu_0(x) = \text{weight of the } \mathbb{T} \text{ action on } L_x\,.$$

Example: If $X = \mathbb{P}^n$, $\mu_0([\epsilon_i]) = \epsilon_i^* \in \mathfrak{t}^*_{\mathbb{R}} \simeq \mathbb{R}^{n+1}$, where ϵ_i is the standard basis vector. This map extends to

$$\mu_{\mathbb{P}^n} : \mathbb{P}^n \to \mathbb{R}^{n+1}$$

$$\mu_{\mathbb{P}^n}([z_0 : z_1 : \dots : z_n]) = \left(\frac{|z_0|^2}{|z|^2}, \frac{|z_1|^2}{|z|^2}, \dots, \frac{|z_n|^2}{|z|^2}\right)\,.$$

The image is equal to the simplex $conv(\epsilon_0, \epsilon_1, \epsilon_2, \dots, \epsilon_n)$.

Suppose that L is a $\mathbb{T}$-equivariant very ample bundle. It defines an equivariant embedding

$$X \hookrightarrow \mathbb{P} := \mathbb{P}(H^0(X; L))\,.$$

Let $\mathbb{T}_{max}$ be a maximal torus in $\mathrm{GL}(H^0(X; L))$, such that the image of $\mathbb{T}$ is contained in $\mathbb{T}_{max}$. Then we have a sequence of maps

$$X \hookrightarrow \mathbb{P} \xrightarrow{\mu_{\mathbb{P}}} \mathfrak{t}^*_{max} \longrightarrow \mathfrak{t}^*\,.$$

The composition is the moment map $\mu_X : X \to \mathfrak{t}^*$, which can be constructed straight from the symplectic form $\omega = (\omega_{Fubini-Study})_{|X}$ by the method of (real) symplectic geometry.

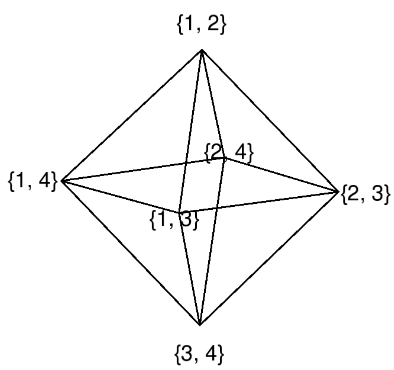
Theorem: Atiyah, Guillemin-Sternberg. *If X is a compact manifold, with $\mathbb{T}$-equivariant (very) ample bundle, then $\mu_X(X) = conv(\mu_X(X^{\mathbb{T}}))$. In particular $\mu_X(X)$ is convex.*

The original formulation of this theorem is for real symplectic manifolds with Hamiltonian action of the compact torus. See [McDSal, §5]. The moment map satisfies $\langle d\mu(x), \lambda\rangle = \iota_{v_\lambda}\omega$ for $x \in X$, $\lambda \in \mathfrak{t}$, where v_λ is a vector field on X generated by λ.

Example: Let $\mu : Gr(k, n) \to \mathbb{R}^n$, $\mu(p_I) = \sum_{i \in I} \epsilon_i$; e.g. for $G(2, 4)$ we have

$$\mu(Gr(2, 4)) = conv\{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}.$$

This is the octahedron contained in a hyperplane in $\mathbb{R}^4$.



Example: The flag variety can be embedded into the product of projective spaces

$$Fl(\mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n-1} Gr(k, n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}(\Lambda^k \mathbb{C}^n) \hookrightarrow \mathbb{P}(\mathbb{C}^N)$$

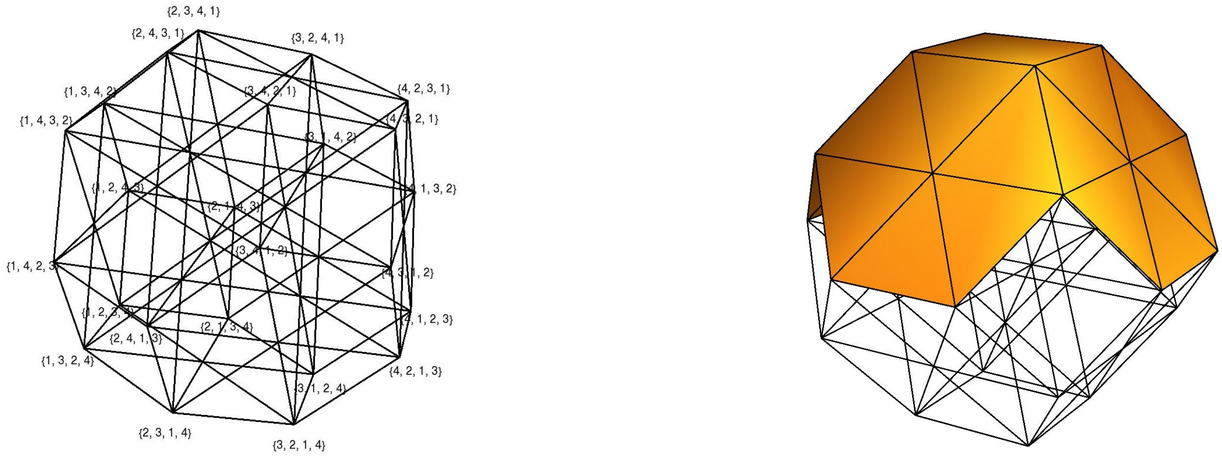
The fixed points of the homogeneous space G/B are in bijection with the Weyl group via the obvious map

$$N\mathbb{T}/\mathbb{T} \to (G/B)^{\mathbb{T}} \qquad n\mathbb{T} \mapsto nB\,.$$

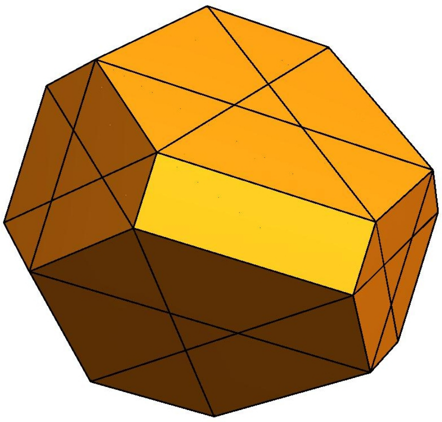
Proof: if $\mathbb{T}gB = gB$ then $g^{-1}\mathbb{T}g \subset B$. In B all tori ale conjugate, so there exists $b \in B$ such that $b^{-1}g^{-1}\mathbb{T}gb = \mathbb{T}$. Hence $gb = n \in N\mathbb{T}$ and $gB = nB$. The conclusion follows since $N\mathbb{T} \cap B = \mathbb{T}$.

Proof using compact groups: $G/B = K/T$ where $K \subset G$ is the maximal compact group, and $T \subset \mathbb{T}$ is the compact torus. Then from the general group properties: $(K/T)^T = NT/T$.

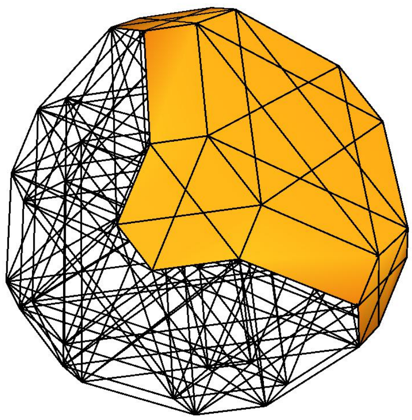
Below we give some example of moment polytopes.



The picture for $Fl(\mathbb{C}^4)$ - permutahedron.



$Fl(\mathbb{C}^4)$ with a different ample bundle.



The picture for Sp_3/B.

5.6 Kirwan surjectivity.

Theorem: *Let $G_{\mathbb{C}}$ be a reductive group. Let $M/\!\!/G_{\mathbb{C}}$ be a smooth GIT quotient of a compact manifold. Then the natural map $H^*_G(M;\mathbb{Q}) \twoheadrightarrow H^*(M/\!\!/G;\mathbb{Q})$ is surjective.*

There exists an equivariant version for a normal subgroup $G_0 \trianglelefteq G$ manifold:

$$H^*_G(M;\mathbb{Q}) \twoheadrightarrow H^*_{G/G_0}(M/\!\!/G_0;\mathbb{Q})$$

Example: : Grassmannian of k-dimensional spaces in $\mathbb{C}^n$ can be presented as the GIT-quotient.

$$Gr(k,n) = \mathrm{Mono}(\mathbb{C}^k,\mathbb{C}^n)/\mathrm{GL}_k = \mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)/\!\!/\mathrm{GL}_k(\mathbb{C}) = \mathbb{P}(\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n))/\!\!/\mathrm{PSL_k}(\mathbb{C})\,.$$

The induced homomorphism is surjective

$$H^*_{\mathbb{T}_n\times \mathrm{GL}_k}(\mathrm{Hom}(\mathbb{C}^k,\mathbb{C}^n)) \twoheadrightarrow H^*_{\mathbb{T}_n\times \mathrm{GL}_k}(\mathrm{Mono}(\mathbb{C}^k,\mathbb{C}^n)) = H^*_{\mathbb{T}}(Gr(k,n))\,,$$

$$\mathbb{Z}[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_k]^{\Sigma_k} \twoheadrightarrow H^*_{\mathbb{T}}(Gr(k,n))\,.$$

We obtain a presentation of $H^*_{\mathbb{T}}(Gr(k,n))$ as

$$\mathbb{Z}[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_k]^{\Sigma_k}/I_{k,n}\,,$$

where I is the ideal of functions such that for any $\{i_1, i_2, \dots, i_k\} \subset \{1, 2, \dots, n\}$, the substitution $f_{|\{s_j := x_{i_j}\ j=1,2,\dots,k\}} \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ vanishes.

Similarly for the flag variety

$$FL_n = \mathrm{GL}_n/B \overset{htp}{\sim} \mathrm{GL}_n/\mathbb{T} = End(\mathbb{C}^n) /\!\!/ \mathbb{T}$$

$$H^*_{\mathbb{T}\times\mathbb{T}}(End(\mathbb{C}^n)) = \mathbb{Z}[x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n] \twoheadrightarrow H^*_{\mathbb{T}}(FL_n)\,,$$

The class of the point w_0B, where w_0 is the longest permutation, is presented by the polynomial

$$\prod_{i+j\leq n} (x_i - y_j)\,.$$

For example for $n = 2$

$$x_1 - y_1 \;\mapsto\; (x_1 - x_1, x_1 - x_2) = (0, x_1 - x_2) = [w_0B]\,.$$

For $n = 3$

$$(x_1 - y_1)(x_1 - y_2)(x_2 - y_1) \;\mapsto\; (0, 0, 0, 0, 0, (x_1 - x_2)(x_1 - x_3)(x_2 - x_3))\,.$$

(To agree with the previous convention we have to switch $x_i \mapsto -x_i$)

6 Fundamental classes of singular varieties

6.1 Fundamental classes of algebraic subvarieties in $H^{BM}_*(-)$ and in K-theory. If M is a compact algebraic variety, X a closed subvariety, then X (e.g. after triangulation) can be treated as a homology cycle, and thus defines the *fundamental class* $[X] \in H_{2\dim X}(M)$. If M is not compact, then in general X is not compact, and after triangulation we obtain a cycle which is locally finite. The cohomology group built from locally finite chains is called Borel-Moore homology. If M is smooth, then canonically

$$H^{BM}_{2\dim X}(M) \simeq H^{2\,\mathrm{codim} X}(M)\,.$$

The cohomology class has the property, that

$$[X]_{|M\setminus Sing(X)} = i_*(1)\,, \qquad \text{where } i : X \setminus Sing(X) \hookrightarrow M \setminus Sing(X)\,.$$

Similarly a G-invariant subvariety defines a class $[X] \in H^{2\,\mathrm{codim} X}_G(M)$. If M is singular, then we would have to define equivariant Borel-Moore homology, whose construction is quite involving.

6.2 Fundamental class in K-theory. This class seems natural, but for bad singularities we have several choices: in algebraic K-theory of coherent sheaves take just the class of $\mathcal{O}_X$. If the ambient space M is smooth, then we can replace $\mathcal{O}_X$ by its locally free resolution, thus defining an element in $K^{alg}(M)$. In the equivariant context the locally free resolutions exist as well, hence we have a well defined element of $K^{alg}_G(M)$. But here (also in a nonequivariant case) a problem appears. Suppose $f : \widetilde{X} \to M$ is a resolution of X then $[f_*(\mathcal{O}_{\widetilde{X}})]$ in general differs from $[\mathcal{O}_X]$. The equality holds if X has rational singularities (i.e. $f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and higher images vanish). For arbitrary singularities the class $f_*\mathcal{O}_{\widetilde{X}}$ (not taking into account the higher images) does not depend on the resolution, but it is not an ultimate solution. There is another candidate „mC_0" coming from the work on motivic classes [BSY10], which will be discussed later in §11.7. It satisfies the additivity condition: whenever we have a resolution of singularities $f : \widetilde{X} \to X \subset M$ and a closed set $Y \subset X$ such that $f_{|f^{-1}(X\setminus Y}$ is an isomorphism to its image, then

$$mC_0(X) - mC_0(Y) = f_*(mC_0(\widetilde{X}) - mC_0(f^{-1}(Y))\,.$$

The above formula allows to compute $mC_0(X)$ inductively with respect to the dimension.

If X has rational singularities, then all three notions coincide. The class in the topological K-theory is defined as an image of $[\mathcal{O}_X] \in K^{alg}_G(M)$.

If $M = \mathbb{C}^n$ with linear action of the torus, and X is a hypersurface given by a homogeneous function (with respect of the torus action) of the weight w, then

$$[\mathcal{O}_X] = [\mathcal{O}_M] - [\mathcal{O}_M(-X)] = 1 - t^{-w}\,.$$

Example: Suppose $M = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2)$ with $\mathbb{T} = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^2$, the first factor acts on the source with the characters s_1, s_2 and the second factor acts on the target with the characters t_1, t_2. Let $X \subset M$ be the set of linear maps of rank ≤ 1. In local coordinates

$$X = \{x_{11}x_{22} - x_{12}x_{21} = 0\}\,.$$

The character of the equation $\frac{t_1 t_2}{s_1 s_2}$. Hence

$$[\mathcal{O}_X] = 1 - \frac{s_1 s_2}{t_1 t_2} \in K_{\mathbb{T}}(M) \simeq K_{\mathbb{T}}(pt) = \mathrm{R}(\mathbb{T}) = \mathbb{Z}[s_1^{\pm 1}, s_2^{\pm 1}, t_1^{\pm 1}, t_2^{\pm 1}]\,.$$

7 Characteristic classes of Schubert varieties in G/B

7.1 Cohomology and K-theory of G/B as representation spaces of some algebras. Let $\pi_{s_k} = \pi_k : G/B \to G/P_k$ denote the contraction associated to the simple root α_k.

Consider the following endomorphism in $h^*_{\mathbb{T}}(G/B)$

- multiplication by $a \in h^*_{\mathbb{T}}(pt)$,
- multiplication by $c_*(L_k)$ (or by $[L_k^*]$ in K-theory),
- $D_{s_k} = D_k := \pi_k^* \pi_{k*}$

These operations generate a reach structure. The resulting algebra was considered e.g. by Lusztig in [Lu85] in K-theory. Only the case of cohomology and K-theory is considered. In the remaining theories (with a small exception) the key braid relations are not satisfied. Thus the obtained formulas e.g. for a candidate for $[\Sigma_w]$ would depend on the presentation of x as a reduced word presentation of w.

7.2 Fundamental classes in $H^*(-)$: the nil-Hecke algebra.

According to [BGG73, Dem74] if $w = w's_k$, $\ell(w) = \ell(w') + 1$ then the operation $\beta_k := \pi_k^* \circ \pi_{k*}$ in cohomology satisfies

$$\beta_k([\Sigma_{w'}]) = [\Sigma_w]\,, \qquad \beta_k \circ \beta_k = 0.$$

The algebra generated by the operations β_k is called the nil-Hecke algebra.

Theorem: *The operators β_k satisfy the braid relations and $\beta_k \circ \beta_k = 0$.*

Theorem: [BGG73] *The operations β_k acting on*

$$H^*(Fl(n)) \simeq H^*_{\mathbb{T}}(End(\mathbb{C}^n))/I_n = \mathbb{Z}[x_1, x_2, \dots, x_n]/(e_1, e_2, \dots, e_n)$$

lift to the polynomial ring. The braid and square zero relations are preserved. The formulas do not depend on n.

The operations

$$\beta_i(f) = \frac{f(\dots x_i, x_{i+1}, \dots) - f(\dots x_{i+1}, x_i, \dots,)}{x_i - x_{i+1}} = \frac{f}{x_i - x_{i+1}} + \frac{s_i f}{x_{i+1} - x_i}\,.$$

are called the divided difference operators, they are known in algebraic combinatorics. The starting point is $\prod_{i=1}^n x_i^{n-i}$. For n=3 we have

$$\begin{array}{cc}
\mathrm{id} & x_1^2x_2 \\
s_1 & x_1x_2 \\
s_2 & x_1^2 \\
s_2s_1 & x_1+x_2 \\
s_1s_2 & x_1 \\
s_1s_2s_1=s_2s_1s_2 & 1
\end{array}$$

In the equivariant version we consider the presentation

$$H^*_{\mathbb{T}}(Fl(n)) \simeq H^*_{\mathbb{T}\times\mathbb{T}}(End(\mathbb{C}^n))/I_n = \mathbb{Z}[x_1,x_2,\dots,x_n,y_1,y_2,\dots,y_n]/I_n.$$

The formulas for β_i remain the same and do not involve y_i. The calculus in equivariant cohomology gives ,,double Schubert polynomials". The starting point is $\prod_{i+j\leq n}^n(x_i-y_j)$. For $n=3$ we have

$$\begin{array}{cc}
\mathrm{id} & (x_1-y_1)(x_1-y_2)(x_2-y_1) \\
s_1 & (x_1-y_1)(x_2-y_1) \\
s_2 & (x_1-y_1)(x_1-y_2) \\
s_2s_1 & x_1+x_2-y_1-y_2 \\
s_1s_2 & x_1-y_1 \\
s_1s_2s_1=s_2s_1s_2 & 1
\end{array}.$$

Let us identify elements of $H^*_{\mathbb{T}}(G/B)$ with their res-image, where

$$\mathrm{res}: H^*_{\mathbb{T}}(G/B) \to H^*_{\mathbb{T}}(G/B)\otimes\mathbb{Q}(\mathfrak{t}) = \bigoplus_{\sigma\in W}\mathbb{Q}(\mathfrak{t}), \qquad \alpha\mapsto \left\{\frac{\alpha_{|\sigma}}{eu^H(T_\sigma(G/B))}\right\}_{\sigma\in W}.$$

Here $\mathbb{Q}(\mathfrak{t})$ is the field of rational functions on $\mathfrak{t}$, and $eu^H(-)$ is the equivariant *cohomological* Euler class. The action of the Demazure operations on the right hand side is given by the formula

$$\beta_k(\{f_\bullet\})_\sigma = \tfrac{1}{c_1(L_k)_\sigma}(f_\sigma + f_{\sigma s_k}).$$

Here L_k is the line bundle, the relative tangent to the contraction $\pi_k: G/B\to G/P_k$. For $G=\mathrm{GL}_n$ we have $c_1(L_k)_\sigma = x_{\sigma(k+1)} - x_{\sigma(k)}$ (where $x_1,x_2,\dots,x_n$ are the basic weights of $\mathbb{T}\subset\mathrm{GL}_n$).

7.3 Fundamental classes in K-theory. There is an operation, which allows to compute K-theoretic fundamental class. The resulting polynomials were constructed by Lascoux and Schutzenberger, see e.g. [LaSc82, Bu02, RiSz18] and references therein. Similarly let $d_k=\pi_k^*\pi_{k*}$ in K-theory. We have: $d_k\circ d_k = d_k$.

Theorem: *The operations d_k acting on*

$$K(Fl(n))\simeq \mathbb{Z}[t_1^{\pm1},t_2^{\pm1},\dots,t_n^{\pm1}]/J_n$$

lift to the Laurent polynomial ring. The braid and idempotent relations are preserved. The formulas do not depend on n.

$$d_k(f) = \frac{z_kf(\dots,z_k,z_{k+1},\dots)-z_{k+1}f(\dots,z_{k+1},z_k,\dots)}{z_k-z_{k+1}} = \frac{f}{1-\frac{z_{k+1}}{z_k}} + \frac{s_kf}{1-\frac{z_k}{z_{k+1}}}.$$

We obtain the ,,isobaric" divided difference operators from algebraic combinatorics.

There are different distinguished elements of $K(G/B)$, namely $\xi_w=[\Sigma_w]-[\partial\Sigma_w]$. They are equal to the motivic Chern classes $mC_{y=0}(\Sigma_w^\circ\hookrightarrow G/B)$ considered in §11.7. To compute ξ_w we modify the operators d_k. Let $T_k^0 = d_k - \mathrm{id}$. We have

$$\xi_w = T_k^0(\xi_{w'}) \qquad \text{if} \qquad w=w's_k \quad \ell(w)=\ell(w')+1\,.$$

Let us note that

$$T_k^0\circ T_k^0 = -T_k^0$$

and T_k^0's satisfy the braid relation as well.

7.4 Double Grothendieck polynomials. We have a presentation

$$K^*_{\mathbb{T}}(Fl_n)=\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},\dots,z_n^{\pm1},t_1^{\pm1},t_2^{\pm1},\dots,z_n^{\pm1}]/J'_n\,.$$

Since

$$FL_n=\mathrm{GL}_n/B\overset{htp}{\sim}\mathrm{GL}_n/\mathbb{T}=End(\mathbb{C}^n)/\!\!/\mathbb{T}$$

the surjection is constructed via Kirwan map:

$$K^*_{\mathbb{T}\times\mathbb{T}}(End(\mathbb{C}^n))=\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},\dots,z_n^{\pm1},t_1^{\pm1},t_2^{\pm1},\dots,z_n^{\pm1}]\twoheadrightarrow K_{\mathbb{T}}(FL_n)\subset\bigoplus_{w\in W}K^*_{\mathbb{T}}(pt)\,,$$

$$f\quad\mapsto\quad(f_{|z_i:=t_{w(i)}})_{w\in W}\,.$$

The action of d_k lifts to the Laurent polynomial ring (given the by same formula as in nonequivariant case). The starting point is

$$f_0=\prod_{i+j\le n}(1-\tfrac{z_i}{t_j})\,.$$

Note that for $n=3$ the restrictions of f_0 are the following:

$$\begin{pmatrix}\{1,2,3\} & \{1,3,2\} & \{2,1,3\} & \{2,3,1\} & \{3,1,2\} & \{3,2,1\}\\ 0 & 0 & 0 & 0 & 0 & \left(1-\frac{t_1}{t_2}\right)\left(1-\frac{t_1}{t_3}\right)\left(1-\frac{t_2}{t_3}\right)\end{pmatrix}$$

We obtain the class of p_{321}. (To agree with our convention the variables should be inverted. Our tangent characters at p_{321} are equal $\frac{t_1}{t_2}$, $\frac{t_1}{t_3}$ and $\frac{t_2}{t_3}$, and in the formula for $[p_{321}]$ the inverses should appear.) We list the Grothendieck in that case

$$\begin{array}{cc}
\mathrm{id} & \left(1-\frac{t_1}{z_1}\right)\left(1-\frac{t_2}{z_1}\right)\left(1-\frac{t_1}{z_2}\right)\\
s_1 & \left(1-\frac{t_1}{z_1}\right)\left(1-\frac{t_1}{z_2}\right)\\
s_2 & \left(1-\frac{t_1}{z_1}\right)\left(1-\frac{t_2}{z_1}\right)\\
s_2s_1 & 1-\frac{t_1t_2}{z_1z_2}\\
s_1s_2 & 1-\frac{t_1}{z_1}\\
s_1s_2s_1=s_2s_1s_2 & 1
\end{array}$$

The restrictions to the fixed points are the following:

	id	s_1	s_2	s_2s_1	s_1s_2	$s_1s_2s_1$
id	0	0	0	0	0	$\left(1-\frac{t_1}{t_2}\right)\left(1-\frac{t_1}{t_3}\right)\left(1-\frac{t_2}{t_3}\right)$
s_1	0	0	0	$\left(1-\frac{t_1}{t_2}\right)\left(1-\frac{t_1}{t_3}\right)$	0	$\left(1-\frac{t_1}{t_2}\right)\left(1-\frac{t_1}{t_3}\right)$
s_2	0	0	0	0	$\left(1-\frac{t_1}{t_3}\right)\left(1-\frac{t_2}{t_3}\right)$	$\left(1-\frac{t_1}{t_3}\right)\left(1-\frac{t_2}{t_3}\right)$
s_2s_1	0	$1-\frac{t_2}{t_3}$	0	$1-\frac{t_1}{t_3}$	$1-\frac{t_2}{t_3}$	$1-\frac{t_1}{t_3}$
s_1s_2	0	0	$1-\frac{t_1}{t_2}$	$1-\frac{t_1}{t_2}$	$1-\frac{t_1}{t_3}$	$1-\frac{t_1}{t_3}$
w_0	1	1	1	1	1	1

In literature the polynomials are indexed by permutations in the reversed order, such that $\mathfrak{G}_{\mathrm{id}}=1$.

7.5 Affine Hecke algebra. The folowing algebra was considered by Lusztig in [Lu85]. He considers the equivariant K-theory of G/B with respect to the group $G\times\mathbb{C}^*$, where the factor $\mathbb{C}^*$ acts trivially on G/B. Then

$$K_{G\times\mathbb{C}^*}(G/B)\simeq K_G(G/B)\otimes\mathbb{Z}[q,q^{-1}]\,,$$

where q is the fundamental representation of $\mathbb{C}^*$. Lusztig defines the operations

$$T_s:K_{G\times\mathbb{C}^*}(G/B)\to K_{G\times\mathbb{C}^*}(G/B)$$

$$T_s(x)=D_s\big((1-q\Omega^1_s)x\big)-x,\quad x\in K_{G\times\mathbb{C}^*}(G/B)$$

for each simple reflections $s = s_k \in W$. Here $\Omega^1_s = \mathscr{L}(-\alpha_s)$ is the line bundle of the differentials along the fiber of $\pi_s : G/B \to G/P_s$. For $\lambda \in \mathfrak{t}^*$ the multiplication by $\mathscr{L}(\lambda)$ is denoted by θ_λ.

Theorem: [Lu85] *Let $\widetilde{W} = \mathfrak{t}_\mathbb{Z} \rtimes W$ be the affine Weyl group. (We use multiplicative notation in $\mathfrak{t}_\mathbb{Z} = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$.) The operations T_s and θ_λ satisfy:*

$$(T_s + 1)(T_s - q) = 0 \tag{1}$$

$$T_s T_t T_s \dots = T_t T_s T_t \dots \tag{2}$$

(number of factors on the both sides is equal to the order of $st \in W$),

$$\theta_\lambda \theta_{\lambda'} = \theta_{\lambda'} \theta_\lambda \tag{3}$$

$$T_s \theta_\lambda = \theta_\lambda T_s = 0 \qquad \text{if} \quad s(\lambda) = \lambda \in \mathfrak{t}^*\,, \quad i.e. \quad s\lambda = \lambda s \in \widetilde{W} \tag{4}$$

$$T_s \theta_{s(\lambda)} T_s = q\theta_\lambda \qquad \text{if} \quad s(\lambda) = \lambda - \alpha_s \in \mathfrak{t}^*\,, \quad i.e. \quad s\lambda s^{-1}\lambda^{-1} = \alpha_s^{-1} \in \widetilde{W}\,, \tag{5}$$

where α_s is the simple (positive) root corresponding to the reflection s.

The resulting algebra is called the affine Hecke algebra. We can check the above formulas using localization formula for $\mathbb{T} \subset G$, since $K_G(G/B) = K_\mathbb{T}(G/B)^W$.

Example: Let $G = SL_2$. Lets look how the operation T_1 acts on images of the restrictions $K_\mathbb{T}(\mathbb{P}^1)[q] \to K_\mathbb{T}((\mathbb{P}^1)^\mathbb{T}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, q] \oplus \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, q]$:

$$T_1(a_1, a_2) = \left(\frac{a_1 - \frac{a_1 q t_1}{t_2}}{1 - \frac{t_1}{t_2}} + \frac{a_2 - \frac{a_2 q t_2}{t_1}}{1 - \frac{t_2}{t_1}} - a_1, \frac{a_1 - \frac{a_1 q t_1}{t_2}}{1 - \frac{t_1}{t_2}} + \frac{a_2 - \frac{a_2 q t_2}{t_1}}{1 - \frac{t_2}{t_1}} - a_2 \right) =$$

$$\left(\frac{a_1 q t_1 - a_1 t_1 - a_2 q t_2 + a_2 t_1}{t_1 - t_2}, \frac{-a_1 q t_1 + a_1 t_2 + a_2 q t_2 - a_2 t_2}{t_2 - t_1} \right)$$

In [AMSS17] it is shown that the motivic Chern classes of the Schubert cells can be computed using the dual operations

$$T_s^\vee(x) = (1 - q\Omega^1_s) D_s(x) - x, \quad x \in K_{G \times \mathbb{C}^*}(G/B)$$

The operations $T_s^\vee$ satisfy the same relations as the original Lusztig operations T_s.

7.6 Motivic Chern classes.

(The basics about motivic Chern classes are given in the appendix.)

To agree with the standard notation set $q = -y$. Then

$$T_k^\vee = (1 + yL_k^{-1})\pi_k^* \pi_{k*} - \mathrm{id}\,,$$

where $L_k = \mathscr{L}(\alpha_k)$ is the line bundle, the relative tangent to the contraction $\pi_k : G/B \to G/P_k$. Suppose $w = w' s_k$, $\ell(w) = \ell(w') + 1$ then

$$T_k^\vee(mC_y(\Sigma^\circ_{w'})) = mC_y(\Sigma^\circ_w)\,, \tag{6}$$

see [AMSS19]. In the local presentation, i.e. after restriction to the fixed points and division by the K-theoretic Euler class, the operator $T_k^\vee$ takes the form

$$T_k^\vee(\{f_\bullet\})_\sigma = \frac{(1+y)(L_k^{-1})_\sigma}{1 - (L_k^{-1})_\sigma} f_\sigma + \frac{1 + y(L_k^{-1})_\sigma}{1 - (L_k^{-1})_\sigma} f_{\sigma s_k}.$$

Here for example, for $G = \mathrm{GL}_n$ we have $(L_k)_\sigma^{-1} = t_{\sigma(k)}/t_{\sigma(k+1)}$.

Example: Let $G = SL_2$. Lets look how the operation T_1 acts on images of the restrictions $K_\mathbb{T}(\mathbb{P}^1)[y] \to K_\mathbb{T}((\mathbb{P}^1)^\mathbb{T}) = (\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, y])^2$ (without division by the Euler class). We have

$$T_1^\vee \left(1 - \frac{t_1}{t_2}, 0\right) = \left((1-q)\frac{t_1}{t_2}, 1 - \frac{qt_2}{t_1}\right)\,.$$

Since the motivic Chern classes were defined via resolution of singularities, we to show the formula (6) analyse the Bott-Samelson resolution.

7.7 CSM-classes: the group ring $\mathbb{Z}[W]$. A similar result holds for CSM-classes. It is shown in [AlMi16, AMSS17] that the CSM classes of Schubert cells satisfy the recursion: if $w = w's_k$, $\ell(w) = \ell(w')+1$ then

$$A_k(c^{sm}(\Sigma^\circ_{w'})) = c^{sm}(\Sigma^\circ_w)$$

where

$$A_k = (1 + c_1(L_k))D_k - \mathrm{id}\,.$$

In terms of res-images

$$A_k(\{f_\bullet\})_\sigma = \tfrac{1}{c_1(L_k)_\sigma} f_\sigma + \tfrac{1+c_1(L_k)_\sigma}{c_1(L_k)_\sigma} f_{\sigma s_k}.$$

Theorem: [AlMi16, AMSS17] *or by straightforward calculation we find that* $A_k \circ A_k = \mathrm{id}$ *and the operators* A_k *satisfy the braid relations.*

We obtain a representation of the group ring $\mathbb{Z}[W]$.

8 The end

8.1 Continuation. There are several extensions possible. One choice is to force the Schubert calculus to other cohomology theories. The elliptic cohomology is of special interest. Other possibility is to develop theory for the Kac-Moody groups. A lot of work was done for loop groups/affine Grassmanians by Thomas Lam and his coauthors. See for example [LaSh13].

9 Incomplete references

For sure lots of important papers are missing.

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10 Some exercises

10.1 Compute $K_G^{alg}(pt)$ for $G = B$ being the group of invertible upper-triangular matrices 2×2.

10.2 Let S^1 act on $X = S^1$ by $(\xi, z) \mapsto \xi^n z$. Compute $K_{S^1}(\mathbb{C}^*)$.

10.3 Compute $K^*_{S^1}(\mathbb{P}^1)$ for all possible linear actions.

10.4 Let X be a $\mathbb{T}$-manifold, $F \subset X^{\mathbb{T}}$ a component of the fixed poit set. Show that $eu(\nu_F)$ is invertible in $S^{-1}K_{\mathbb{T}}(F)$, wher S contains the elements $(1 - t^{-w})$ for the weghts appearing in ν_F.

10.5 Let $Gr(k, n) = A_{n-1}(k)$ be the Grassmanian of k-dimensional spaces in $V = \mathbb{C}^n$. Prove that the tangent vector bundle $T_{Gr(k,n)} \simeq \underline{\mathrm{Hom}}(\gamma_{k,n}, V/\gamma_{k,n}) = Q \otimes S$.

10.6 Compute $\chi(\mathbb{P}^n; \mathcal{O}(k))$ by localization formula.

10.7 Compute $\chi(Gr(2, 4); \mathcal{O}(1))$ by localization formula.

10.8 Prove $\chi(Gr(k, n); \gamma^*_{k,n}) = \sum_{i=1}^n t_i^{-1}$ by localization formula.

10.9 Compute $\chi(Gr(k, n); T_{Gr(k,n)})$.

10.10 Show that there are finitely many orbits of the torus action for Grassmannians, complete flag varieties, and in general for homogeneous spaces G/P.

10.11 Draw GKM graphs and their maps corresponding to the contarctions for $B_2 = C_2$ and G_2.

10.12 Check the affine Hecke relations for $G = SL_2$.

11 Appendix 1: Characteristic classes of singular varieties

An important one-parameter deformation of the notion of the fundamental class is the equivariant Chern-Schwartz-MacPherson (CSM, in notation $c^{sm}(-)$) class.

11.1 Constructible functions. The topological Euler characteristic of a complex algebraic variety satisfies

$$\chi_{top}(X) = \chi_{top}(Y) + \chi_{top}(X \setminus Y)$$

whenever Y is a closed subvariety of X. Moreover if X is smooth and compact, then

$$\chi_{top}(X) = \int_X c_{\dim X}(TX)\,.$$

The question arises: how to define in a natural way a (co)homology class $c(X)$ which after pushing to the point would give $\chi_{top}(X)$. This question seems to be ambiguous.

11.2 Question of Grothendieck and Verdier about transformation of functions. Instead we will be looking for a homology class

$$c(X) = [X] + \textit{lower terms} \in H_*^{BM}(-)\,.$$

which behaves well with respect to proper push forwards. To say what it mans, we consider another object, the constructible functions

$$F(X) = \{\phi : X \to \mathbb{Z} \mid \forall n \in \mathbb{Z}\ \ \phi^{-1}(n) \text{ is a constructible set}\}\,.$$

The group $F(X)$ is generated by the characteristic functions 1_Y of subvarieties $Y \subset X$. Such functions behave covariantly. For a map $f : X_1 \to X_2$ we define

$$f_* 1_Y(x) = \chi(f^{-1}(x) \cap Y)\,.$$

Clearly for $f : X \to pt$ we have $f_*(1_X) = \chi(X)$. Grothendieck and Verdier asked if there exist a natural transformation of functors

$$c : F(-) \to H_*^{BM}(-)$$

Such that for a smooth variety X

$$c(1_X) = c_*(TX) \cap [X]\,. \tag{7}$$

Here $c_*(TX) = \sum_{k=0}^{\dim X} c_k(X)$ denotes the total Chern class of the tangent bundle. This formula makes sense only when X is nonsingular. Otherwise there is no good candidate for the tangent bundle[2]. The functorial class c (if exists) is unique: Let $f : \widetilde{X} \to X$ be a resolution. Then

$$f_*(1_{\widetilde{X}}) = 1_X + \text{something supported by the set of dimension} < \dim X\,.$$

Proceeding inductively we find that if c exists, then it is unique. It exists by the work of Schwartz and MacPherson and it will be denoted c^{sm}.

The question has been positively answered by MacPherson. Later it came out that the classes constructed by MacPherson coincide with earlier constructed classes of M. H. Schwartz. Now this class, which can be constructed in several ways, is called Chern-Schwartz-MacPherson classes (CSM) and denoted by c^{sm}.

11.3 Chern-Schwartz-MacPherson: Aluffi approach via logarithmic tangent bundle. Aluffi have shown that if $V \subset M$ is a smooth subvariety (not necessarily closed) then c^{sm} can be computed in the following way: Let $f : Y \to M$ be a resolution of $\bar{U}$, such that $f_{|f^{-1}(U)}$ is an isomorphism on its image. Let $U' = f^{-1}(U)$, $D = Y \setminus U'$. We assume that D is a simple divisor with normal crossings, $D = \bigcup D_i$. Then

$$c^{sm} 1_U = f_*(c^{sm}(U')) \tag{8}$$

$$c^{sm}(U') = c^{sm}(1_Y) - \sum_i c^{sm}(1_{D_i}) + \sum_{i,j} c^{sm}(1_{D_i \cap D_j}) - \dots\,. \tag{9}$$

The later class is the Poincaré dual of the Chern class of the vector bundle $\Omega^1_X(\log D)^*$, whose associated sheaf is the sheaf of the vector fields which are tangent to the stratification given by D.

11.4 Generalization: $K(Var/M)$ instead of constructible functions. Brasselet, Shürmann and Yokura were searching for the characteristic class, which admit a functorial generalization for singular varieties. That is, one would like to assume a normalization condition different than (7). It turns out, that except Chern-Schwartz-MacPherson classes there does not exists anything essentially different. Instead, it is interesting to consider a very formal object: the Grothendieck group of varieties over M denoted by $K(Var/M)$. It is generated by maps $Y \to X$ (not just inclusions) and the additive relations hold

$$[f : X \to M] = [f_{|Y} : Y \to M] + [f_{|X \setminus Y} : X \setminus Y \to M]$$

[2] In fact there are too many choices and none of them has all expected properties.

for a closed subvariety $Y \subset X$. It is a highly nontrivial result of Bittner (following from the weak factorization theorem) that $K(Var/M)$ is generated by proper maps with smooth domain and the relations are generated by the blow-up relation

$$[f : X \to M] - [f_{|Y} : Y \to M] = [\widetilde{f} : Bl_Y X \to M] - [\widetilde{f}_{|E} : E \to M] \tag{10}$$

for a closed submanifold $Y \subset X$. The corresponding results hold in the equivariant category as well.

11.5 Hirzebruch class constructed by Brasselet-Schurman-Yokura in $H_*(-)[y]$. Brasselet, Shürmann and Yokura have constructed a transformation of functors

$$td_y : K(Var/M) \to H_*^{BM}(M; \mathbb{Q})[y]\,,$$

where y is indeterminate. The normalization condition is given by

$$td_y(f : X \to M) = f_*(Td(TM) \cdot ch(\lambda_y T^*X) \cap [X])\,,$$

where

$$\lambda_y(E) = 1 + yE + y^2\Lambda_2 E + \dots\,.$$

This class is called the Hirzebruch class. For smooth varieties it appears in the Hirzebruch work from 50-ties on topological methods in algebraic geometry, [Hir56]. If M is smooth, then

$$td_y(id_M) = Td_y(TM)\,,$$

where $Td_y(-)$ is a multiplicative characteristic class such, that for line bundles

$$Td_y(L) = c_1(L)\frac{1 + y\, e^{-c_1(L)}}{1 - e^{-c_1(L)}}\,.$$

By Hirzebruch-Riemann-Roch theorem

$$\int_M Td_y(TM) = \sum_{k=0}^{\dim M} \chi(M; \Omega^k_M) y^k = \sum_{p,q} (-1)^q h^{p,q} y^p\,.$$

This is the Hirzebruch χ_y-genus. For example

$$\chi_y(Gr(k,n)) = \left(\frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})}\right)_{q:=-y}\,.$$

11.6 Special values: $y = -1,\ 0,\ 1$. For the special value $y = 0$ we have the Todd class. If $y = 1$ then we obtain the L-class

$$\int_M Td_{y=1}(TM) = \mathrm{signature}(M)\,.$$

After the normalization

$$\widetilde{Td}_y(L) = c_1(L)\frac{1 + e^{-(y+1)c_1(L)}}{1 - e^{-(y+1)c_1(L)}}$$

in the limit with $y \to -1$ we obtain the Chern-Schwartz-MacPherson class. That is because

$$\lim_{y \to -1} x\frac{1 + e^{-(y+1)x}}{1 - e^{-(y+1)x}} = 1 + x\,.$$

11.7 Motivic Chern classes of BSY. Instead of considering the characteristic classes with values in $H_*(X)[y]$ it is natural to define the classes in K-theory of coherent sheaves, called ,,motivic Chern class" with the normalization condition

$$mC(f : X \to M) = f_*(\lambda_y(\Omega^1_X)) \in K(M)[y]$$

for proper morphism from a smooth domain. The existence of such class can be shown by checking the blow-up relation (10).

For $y=0$ we obtain a candidate for the fundamental class. If X has rational singularities, then

$$mC_0(X \hookrightarrow M) = [\mathcal{O}_X]\,.$$

There can be given a meaning to the variable y. Let $y=-h^{-1}$. Consider the equivariant K theory for $\mathbb{C}^*$ action. Let q be the character of the natural representation. Any vector bundle is automatically $\mathbb{C}^*$-bundle with respect to the scalar multiplication. In that theory consider a characteristic class with the the normalization condition

$$mc(id_M) = \text{equivariant K-theoretic Euler class of } TM\,.$$

Then

$$mc(id_M) \in K_{\mathbb{C}^*}(M) = K(M)[h,h^{-1}]$$

if $\mathbb{C}^*$ acts trivially on M. That class is equal to

$$\begin{aligned} mc(id_M) &= 1 - h^{-1}T^*M + h^{-2}\Lambda^2 T^*M - \dots \pm h^{-n}\Lambda^n T^*M \\ &= 1 - h^{-1}\Omega^1_M + h^{-2}\Omega^2_M - \dots \pm h^{-n}\Omega^n_M \end{aligned}$$

12 Appendix 2: Equivariant classes of singular varieties

12.1 Simplification. For convenience assume that X is contained in (or mapped to) a smooth ambient space, otherwise we would have to deal with equivariant homology or equivariant K-theory built from coherent sheaves. For smooth variety M (not necessarily complete) we obtain the transformations:

$$c^{sm}_G : F_G(M) \longrightarrow H^*_G(M,\mathbb{Z})\,,$$

$$(td_y)_G : K_G(Var/M) \longrightarrow H^*_G(M,\mathbb{Q})[y]\,,$$

$$mC_G : K_G(Var/M) \longrightarrow K^*_G(M)[y]\,,$$

where $F_G(M)$ and $K_G(Var/M)$ denote the group of equivariant constructible functions and the Grothendieck group of equivariant maps. For introduction to equivariant characteristic class see, [Ohm06, Web12, AMSS17] for CSM-classes, [Web16] for td_y, [FRW18, AMSS19] for mC.

12.2 Fundamental computation. Let $M=\mathbb{C}$ with the standard $\mathbb{C}^*$-action. Then

	mC	td_y	c^{sm}
$\mathbb{C}\to\mathbb{C}$	$1+y\,t^{-1}$	$x\frac{1+y\,e^{-x}}{1-e^{-x}}$	$1+x$
$\{0\}\to\mathbb{C}$	$1-t^{-1}$	x	x
$\mathbb{C}^*\to\mathbb{C}$	$(1+y)t^{-1}$	$x\frac{(1+y)e^{-x}}{1-e^{-x}}$	1

This formula allows to compute locally characteristic classes of a resolution.

12.3 Rigidity. The class mC (and hence the remaining ones) has the following property

Theorem: *Let $G=\mathbb{T}$, $M=pt$. Then*

$$mC_{\mathbb{T}}(X\to pt) = mC(X\to pt)\,.$$

In other words the equivariant global invariants do not see the group action.

12.4 Example: Schubert variety $\Sigma_1 \subset Gr(2,4)$. The canonical neighborhood of the point $p_{1,2}$ in $Gr_2(\mathbb{C}^4)$ is identified with

$$\mathrm{Hom}(span(\varepsilon_1,\varepsilon_2), span(\varepsilon_3,\varepsilon_4))$$

and the variety Σ_1 intersected with this neighbourhood is identified with

$$\{\phi \in \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2) \,:\, \det(\phi)=0\}\,.$$

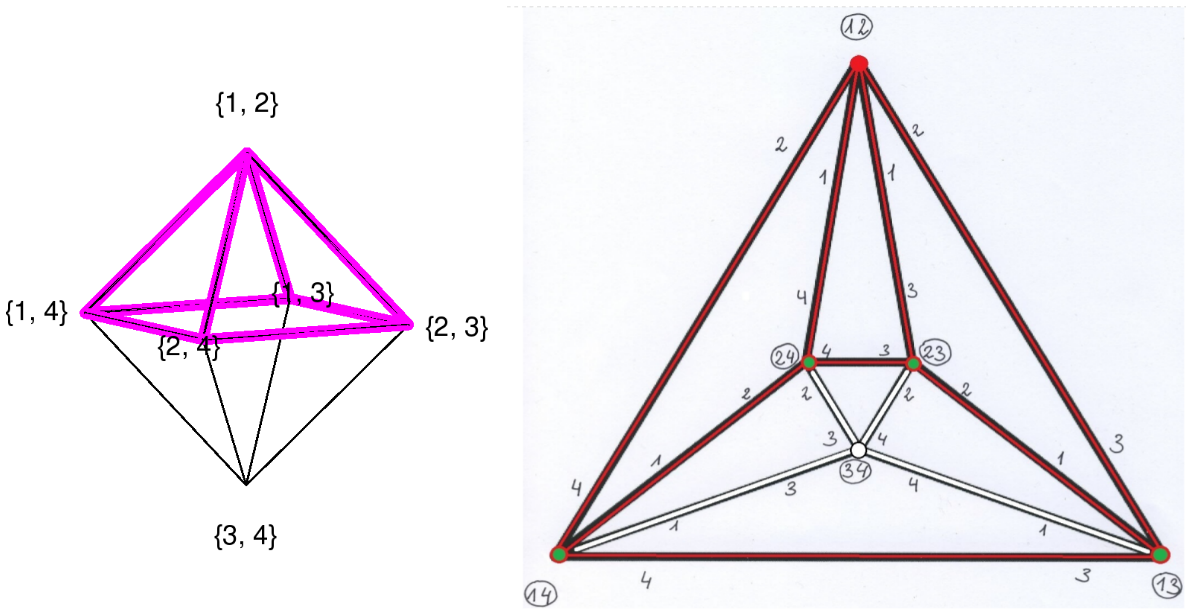
The corresponding elements of Σ_1 are the planes spanned by the row-vectors of the matrix

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

The variety Σ_1 is defined by vanishing of the determinant of the 2×2 matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\,.$$

Before performing computations let us draw the Goresky-Kottwitz-MacPherson graph[3] ([GKM98, Th. 7.2]) with the variety Σ_1 displayed.



Schubert variety Σ_1 in $Gr_2(\mathbb{C}^4)$.

The numbers attached to the edges indicate the weights of the $\mathbb{T}$ actions along the one dimensional orbits. For example at the point $p_{1,3}$ in the direction towards $p_{1,2}$ the action is by the character $x_2 - x_3$. The variety Σ_1 is singular at the point $p_{1,2}$ and it is smooth at the remaining points. For example at the point $p_{1,3}$ the coordinates are

$$\begin{pmatrix} 1 & a & 0 & b \\ 0 & c & 1 & d \end{pmatrix}$$

and the equation of Σ_1 is $b=0$. For that point the local equivariant Chern class is equal to

$$(x_4-x_1)(1+x_2-x_1)(1+x_2-x_3)(1+x_4-x_3)\,.$$

By AB-BV localization theorem we have

$$\begin{aligned} 5 = \chi_{top}(\Sigma_1) &= \\ &= \boxed{\mathrm{res}(c^{sm}(\Sigma_1))_{p_{12}}} + \mathrm{res}(c^{sm}(\Sigma_1))_{p_{13}} + \mathrm{res}(c^{sm}(\Sigma_1))_{p_{14}} + \mathrm{res}(c^{sm}(\Sigma_1))_{p_{23}} + \mathrm{res}(c^{sm}(\Sigma_1))_{p_{24}} \end{aligned} \quad (11)$$

[3] To avoid intersections, edges in the second picture do not have right directions, i.e. the second picture is not the image of the moment map.

The first summand is unknown, while the remaining points are smooth and for example

$$\mathrm{res}(c^{sm}(\Sigma_1))_{p_{13}} = \frac{(x_4-x_1)(1+x_2-x_1)(1+x_2-x_3)(1+x_4-x_3)}{(x_4-x_1)(x_2-x_1)(x_2-x_3)(x_4-x_3)} =$$

$$= \left(1+\frac{1}{x_2-x_1}\right)\left(1+\frac{1}{x_2-x_3}\right)\left(1+\frac{1}{x_4-x_3}\right)$$

Simplifying the sum (11) we find the formulas for the restrictions of the CSM classes

$$\begin{array}{ll}
0 & deg=0\\
x_3+x_4-x_1-x_2 & deg=1\\
(x_3+x_4-x_1-x_2)^2 & deg=2\\
(x_3+x_4-x_1-x_2)(2x_1x_2-x_3x_2-x_4x_2-x_1x_3-x_1x_4+2x_3x_4) & deg=3\\
(x_3-x_1)(x_3-x_2)(x_4-x_1)(x_4-x_2) & deg=4
\end{array}$$

The computation for $mC_{\mathbb{T}}(\Sigma_1 \to Gr(2,4))$ is similar:

$$\mathrm{res}(mC(\Sigma_1))_{p_{13}} = \frac{\left(1+y\frac{t_1}{t_3}\right)\left(1+y\frac{t_2}{t_3}\right)\left(1+y\frac{t_2}{t_4}\right)}{\left(1-\frac{t_1}{t_3}\right)\left(1-\frac{t_2}{t_3}\right)\left(1-\frac{t_2}{t_4}\right)}$$

and $\chi_y(\Sigma_1) = \chi_y(Gr(2,4)) - y^4 = 1-y+2y^2-y^3$. We obtain using some software

$$\begin{aligned}
\mathrm{res}(mC(\Sigma_1))_{p_{12}} &= y^3\,\frac{t_1t_2}{t_3t_4}\left(1-\frac{t_1t_2}{t_3t_4}\right)\\
&+y^2\left(\frac{t_1^2t_2^2}{t_3^2t_4^2}+\frac{t_1^2}{t_3t_4}+\frac{t_2^2}{t_3t_4}+\frac{t_1t_2}{t_3^2}+\frac{t_1t_2}{t_4^2}+3\frac{t_1t_2}{t_3t_4}-2\frac{t_1t_2}{t_3t_4}\left(\frac{t_1}{t_3}+\frac{t_1}{t_4}+\frac{t_2}{t_3}+\frac{t_2}{t_4}\right)\right)\\
&+y\left(1-\frac{t_1t_2}{t_3t_4}\right)\left(\frac{t_1}{t_3}+\frac{t_1}{t_4}+\frac{t_2}{t_3}+\frac{t_2}{t_4}-\frac{t_1t_2}{t_3t_4}\right)\\
&+1-\frac{t_1t_2}{t_3t_4}\,.
\end{aligned}$$

The coefficient at y^0 is equal to $1-t^{-w}$, where $w=(-1,-1,1,1)$ is the weight of the defining equation $ad-bc=0$, since the weight of the variable a,b,c,d are $(-1,0,1,0)$, $(-1,0,0,1)$, $(0,-1,1,0)$, $(0,-1,0,1)$. That is so because Σ_1 has rational singularities (as any Schubert variety, by the work of Ramanathan), hence $mC(\Sigma_1)_{y=0}$ is equal to $[\mathcal{O}_{\Sigma_1}]$.

Our goal will be not to compute the particular classes of Schubert varieties, but to find some structure governing these wild formulas. The solution for Schubert cells in G/B will be presented in final section in the language of Hecke algebra action.

12.5 Local classes, application of localization theorem. Assume that $G=\mathbb{T}$. To compute the characteristic classes of open possibly singular variety $U\subset M$ we apply the formula (8). Suppose both $M^{\mathbb{T}}$ and $Y^{\mathbb{T}}$ are finite. Let $p\in M^{\mathbb{T}}$ be a fixed point. Then

$$\frac{mC_{\mathbb{T}}(U\to M)_{|p}}{eu^K(p)} = \sum_{q\in f^{-1}(p)} \frac{mC_{\mathbb{T}}(U'\to Y)_{|q}}{eu^K(q)}\,.$$

The quotients $\frac{mC_{\mathbb{T}}(U'\to Y)_{|q}}{eu^K(q)}$ are easy to compute. If locally $D=\{z_1=z_2=\dots=z_k=0\}$, then

$$\frac{mC_{\mathbb{T}}(U'\to Y)_{|q}}{eu^K(q)} = \prod_{i=1}^{k}\frac{(1+y)t^{-w_i}}{1-t^{-w_i}}\cdot\prod_{i=k+1}^{\dim Y}\frac{1+y\,t^{-w_i}}{1-t^{-w_i}}\,,$$

by, §12.2 where w_i's are the weights of torus actions on coordinates at q.

This strategy is be applied to compute the characteristic classes of Schubert varieties in homogeneous spaces G/B.