Algebraic Geometry: preparation to the exam

Below I give some exercises that should be useful during the exam. They are often quite easy (with some exceptions) and need only understanding of some notions that you should know. Note that many of the exercises were done during exercise sessions so they are not new. I give their numbers only to stress that you should know their solutions.

In the exam please expect one exercise with a dimension count analogous to the last lecture (proofs of Bertini's theorem and existence of lines on cubic surfaces) or as in Exercise 4 below.

R. Hartshorne, Algebraic Geometry

After the AG course all exercises from Chapter I, Sections 1-5 and Chapter II, Sections 1-2 are within your range. I give numbers of some important exercises.

Chapter I, Exercises 1.3, 1.12, 2.6, 2.9, 2.10, 2.12, 2.13, 2.14, 2.15, 3.11, 3.14, 3.17, 4.4, 4.5, 4.7, 4.10, 5.2, 5.3, 5.4 (a) is non-trivial so you might prefer to do special cases only; see also Shafarevich Basic algebraic geometry 1, Chapter IV, Section 1, Examples 3 and 4), 5.6, 5.7, 5.11, 5.12

Chapter II, Exercises 1.21, 2.1, 2.2, 2.3, 2.4, 2.5, 2.7, 2.8, 2.11, 2.14, 2.15, 2.19

Also exercises from Shafarevich Basic algebraic geometry I are useful. In particular, all exercises in Chapter 1, Section 6 are interesting.

Some other exercises:

Exercise 1.

Let $f, g \in \mathbb{C}[x_0, ..., x_n]$ be relatively prime homogeneous polynomials. Assume that $\deg f = k$ and $\deg g = k-1$ for some $k \geq 2$. Show that

$$X = \{[x_0 : \cdots : x_n] : f(x_1, \ldots, x_n) + x_0 g(x_1, \ldots, x_n) = 0\}$$

in $\mathbb{CP}^n$ is not rational.

Exercise 2.

Show that two general cubic surfaces in $\mathbb{CP}^3$ are not projectively equivalent.

Exercise 3.

Let $C \subset \mathbb{CP}^2$ be the zero set of a homogeneous irreducible polynomial f of degree d. Show that there exists a non-empty Zariski open subset $U \subset (\mathbb{CP}^2)^*$ such that each line $L \in U$ intersects C in precisely d points and each line $L \notin U$ intersects C in a smaller number of points.

Exercise 4.

Prove that for each $d \geq 4$ there exists a surface of degree d in $\mathbb{P}^3$ which does not contain a projective line.