Algebraic Geometry, Fall 2013

Homework, set 3, for December 16th

All varieties are defined over an algebraically closed field k.

1. Let $X \subset \mathbb{P}^2(k)$ be a k-subvariety defined by $y^2z = x^3$, where $[x, y, z]$ are homogeneous coordinates on $\mathbb{P}^2(k)$. Show that X is rational but it is not isomorphic to $\mathbb{P}^1(k)$.

2. Let $X \subset \mathbb{P}^2(k)$ be a k-subvariety defined by $y^2z = x^2(x + z)$, where $[x, y, z]$ are homogeneous coordinates on $\mathbb{P}^2(k)$. Show that X is rational but it is not isomorphic to $\mathbb{P}^1(k)$.

3. Show that the quadric surface $xy - zw = 0$ in $\mathbb{P}^3$ is rational but it is not isomorphic to $\mathbb{P}^2$.

4.
   - Show that intersection of two varieties does not need to be a variety.
   - Find irreducible components of the intersection of two quadric surfaces in $\mathbb{P}^3(k)$ given by $x^2 - yw = 0$ and $xy - zw = 0$.

5. Let C be a conic given by $x^2 - yz = 0$ in $\mathbb{P}^2(k)$ and let L be a line given by $y = 0$. Show that $C \cap L$ is set-theoretically a point P but $I(P) \neq I(C) + I(L)$. What is a scheme-theoretic explanation of this example?

6. Fix $n \geq 2$. Let H_i be a hyperplane in $\mathbb{P}^n$ given by $x_i = 0$. Let $U = \mathbb{P}^n - (H_i \cap H_j)$ for some $i \neq j$. Show that $\mathcal{O}_{\mathbb{P}^n}(U) = k$.

7. Let C be the curve $y^2 = x^3$ in $\mathbb{A}^2$. Let $f : X \to \mathbb{A}^2$ be the blow up at the point $O = (0, 0)$. Let $E := f^{-1}(O)$ and let $\tilde{C}$ be the closure of $f^{-1}(C - O)$ in X. Show that $\tilde{C} \cap E$ is one point and $\tilde{C} \simeq \mathbb{A}^1$. Show that $g = f|_{\tilde{C}} : \tilde{C} \to C$ is a homeomorphism but it is not an isomorphism.

8. Is the map g from the previous exercise finite?