Algebraic Geometry, Fall 2013

Homework, set 2, for November 18th

All varieties are defined over an algebraically closed field k.

1. Let $(X, \mathcal{O}_X)$ be an affine variety. Take $f \in \mathcal{O}_X(X)$ and consider the open set $U_f = X \setminus V(f)$. Prove that that U_f with induced topology and the induced sheaf is an affine variety.

2. Prove the following properties of prevarieties:

   (a) Every prevariety is irreducible.

   (b) Every prevariety is a noetherian space.

3. Show that the set $U = \mathbb{A}^2(k) \backslash (0,0)$ with Zariski topology and structural sheaf of regular functions is not an affine variety. Is it a prevariety?

4. Let $(X, \mathcal{O}_X)$ be an affine k-variety. For a point $x \in X$ we consider the stalk $\mathcal{O}_{X,x}$ with maximal ideal $\mathfrak{m}_x$ and the residue field $k(x)$. Prove that:

   (a) $k(x) \simeq k$

   (b) $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a $k(x)$-module.

   (c) If $X \subset \mathbb{A}^n(k)$ then the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ over k is $\leq n$.

   (d) Let X be the affine variety associated to the affine algebraic set $V(x_1^2 - x_2^3) \subset \mathbb{A}^2(k)$. Calculate $\mathfrak{m}_x/\mathfrak{m}_x^2$ at $x = (0,0)$.

5. *Frobenius morphism.* Suppose characteristic of k is $p > 0$. Take $\mathbb{A}^n(k)$ with coordinates $(x_1, \ldots, x_n)$. Consider a morphism $\phi : \mathbb{A}^n(k) \to \mathbb{A}^n(k)$ given by the formula $\phi(x_1, \ldots, x_n) = (x_1^p, \ldots, x_n^p)$. Show that ϕ is bijective, continuous and open but it is not an isomorphism.

6. Let $(X, \mathcal{O}_X)$ be a locally ringed space.

   (a) Let $U \subset X$ be an open and closed subset. Show that there exists a unique section $e_U \in \mathcal{O}_X(X)$ such that $e_U|_V = 1$ for all open subsets $V \subset U$ and $e_U|_V = 0$ for all open subsets $V \subset X - U$.

   (b) Show that $U \to e_U$ yields a bijection between the set S of open and closed subsets of X and the set of idempotent elements of $\mathcal{O}_X(X)$.

   (c) Show that $e_U e_V = e_{U \cap V}$ for $U, V \in S$.

   (d) Prove that the following conditions are equivalent:

      - X is connected.
      - There exists no idempotent element $e \in \mathcal{O}_X(X)$ with $e \neq 0, 1$.
      - There exists no decomposition of $\mathcal{O}_X(X)$ as a product of two non-zero rings.

7. Let R be a local ring. Prove that $Spec\, R$ is connected.

8. Give an example of a morphism of ringed spaces between affine schemes which is not a morphism of locally ringed spaces.