SPACES OF HOLOMORPHIC MAPS BETWEEN COMPLEX PROJECTIVE SPACES

ANDRZEJ KOZLOWSKI AND KOHHEI YAMAGUCHI

Abstract. For an integer $d \geq 0$, let $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ denote the space consisting of all holomorphic maps $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. In this paper we shall study the homotopy type of the space $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$.

1. Introduction.

For each integer $d \geq 0$, we denote by $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ the space consisting of all holomorphic maps $f : \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ of degree d. The corresponding space of continuous maps is denoted by $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. We also denote by $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ the subspace of $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ consisting of all maps $f \in \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ which preserve the base-points, and the corresponding space of basepoint preserving continuous maps is denoted by $\mathrm{Map}^*_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. The space of holomorphic maps are of interest both from a classical and modern point of view (e.g. [1], [3], [7]). In this paper we shall study the topology of spaces $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ for the case $1 \leq k \leq n$ from the point of view of homotopy theory. However, since the case $k = 1$ was well studied until now until now ([3], [6], [8], [10], [11], [17], [22], [23]), we shall mainly consider the case $2 \leq k \leq n$.

2. The case $d = 1$.

In this section we study the case $d = 1$.

Proposition 2.1. *If $1 \leq k \leq n$, there is a homotopy equivalence $\beta_{k,n} : U_n/U_{n-k} \xrightarrow{\simeq} \mathrm{Hol}^*_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$.*

Proof. Fro now on, we choose the point $z_m = [1 : 0 \cdots : 0] \in \mathbb{C}\mathrm{P}^m$ as the base point of $\mathbb{C}\mathrm{P}^m$ and we identify $\mathrm{Hol}^*_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) = \{f \in \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) : f(z_k) = z_n\}$. Define the map $\beta'_{k,n} : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Hol}^*_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by the matrix multiplication

$$\beta'_{k,n}(A)([x_0 : x_1 : \cdots : x_k]) = [x_0 : x_1 \cdots : x_k : 0 : \cdots : 0] \cdot \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$$

for $([x_0 : \cdots : x_k], A) \in \mathbb{C}\mathrm{P}^k \times \mathrm{GL}_n(\mathbb{C})$.

Since the subgroup $\mathrm{GL}_{n-k}(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ is fixed by this map, this induces the map $\beta''_{k,n} : \mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-k}(\mathbb{C}) \to \mathrm{Hol}^*_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. The direct computation easily shows that $\beta''_{k,n}$ is in fact a homeomorphism. Because $U_m \subset \mathrm{GL}_m(\mathbb{C})$ is a deformation retract, it induces naturally a desired homotopy equivalence $\beta_{k,n} : U_n/U_{n-k} \xrightarrow{\simeq} \mathrm{Hol}^*_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. □

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Corollary 2.2. *Let $1 \leq k < n$ be integers.*

(i) $\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *is simply connected and* $\pi_2(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}$.

(ii) $ev^* : \mathbb{Z} = H^2(\mathbb{C}\mathrm{P}^n, \mathbb{Z}) \xrightarrow{\cong} H^2(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n), \mathbb{Z}) = \mathbb{Z}$ *is an isomorphism.*

(iii) *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \\ \iota_1 \downarrow & & \iota \downarrow \\ K(\mathbb{Z}, 2) & \xrightarrow{=} & K(\mathbb{Z}, 2) \end{array}$$

where the map $\iota_1 : \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *represents the generator of the homotopy set* $[\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n), K(\mathbb{Z}, 2)] \cong H^2(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n), \mathbb{Z}) = Z$ *and the map* $\iota : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z}, 2)$ *denotes the natural inclusion map.*

Proof. (i) Consider the homotopy exact sequence of the evaluation fibration sequence $\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{\subseteq} \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{ev} \mathbb{C}\mathrm{P}^n$, where ev is deined by $ev(f) = f(z_k)$ for $f \in \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$. Since $\pi_m(\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) \cong \pi_m(U_n/U_{n-k}) = 0$ for $m = 1$ or 2, we have the isomorphisms

$$\begin{cases} ev_* : \pi_2(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_2(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}, \\ ev_* : \pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)) \xrightarrow{\cong} \pi_1(\mathbb{C}\mathrm{P}^n) = 0. \end{cases}$$

(ii), (iii); The assertion (ii) easily follows from (i) and (iii) follows from (ii). □

Definition 2.3. Let $h_n : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ ne the standard Hopf fibering with fibre S^1 and we define the space $\tilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ by

$$\tilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) = \{(f, x) \in \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \times S^{2n+1} : ev(f) = h_n(x)\}.$$

Then it follows from [[5], (2.1)] and (iii) of corollary 2.2 that there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{j}_d} & \tilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{ev}} & S^{2n+1} \\ = \downarrow & & p_n \downarrow & & h_n \downarrow \\ \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{j_d} & \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \\ \downarrow & & \iota_1 \downarrow & & \iota \downarrow \\ \{*\} & \longrightarrow & K(\mathbb{Z}, 2) & \xrightarrow{=} & K(\mathbb{Z}, 2) \end{array}$$

where $\tilde{ev}$ and p_n are the first and he second projections, and all horizontal and vertical sequences are fibration sequences.

Lemma 2.4. *If* $1 \leq k < n$, *the space* $\tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *is a* 2*-conective covering of* $\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *and there is a fibration sequence (up to homotopy)*

$$(*)_{k,n} \qquad \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{\tilde{j}_d} \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{\tilde{ev}} S^{2n+1}.$$

Proof. The second assertion is clear and the first assertion can be easily obtained by the diagram chasing of the exact sequences induced from the above diagram. □

Definition 2.5. Let $\Delta_m \subset U_m$ denote the subgroup consisting of all multiples of E_m contained in U_m defined by $\Delta_m = \{\alpha E_m : \alpha \in \mathbb{C}^*, |\alpha| = 1\}$, where E_m denotes the $(m \times m)$-identity matrix. Define the homogenous space $X_{n+1,k+1}$ by $X_{n+1.k+1} = U_{n+1}/(\Delta_{k+1} \times U_{n-k})$. Similarly, let $W_{n+1,k+1}$ be the complex Stiefel manifold of orthogonal $(k+1)$-frames in $\mathbb{C}^{n+1}$ given by $W_{n+1,k+1} = U_{n+1}/U_{n-k}$.

Consider U_{n+1} action on $\mathbb{C}\mathrm{P}^n$ induced by the matrix multiplication in a usual way. This action naturally induces map $\phi'_{k,n} : U_{n+1} \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ which is defined by

$$\phi'_{k,n}(A)([x_0 : x_1 : \cdots : x_k]) = [x_0 : x_1 \cdots : x_k : 0 : \cdots : 0] \cdot A$$

for $([x_0 : \cdots : x_k], A) \in \mathbb{C}\mathrm{P}^k \times U_{n+1}$.

Because the subgroups $U_{n-k} \subset \Delta_{k+1} \times U_{n-k} \subset U_{n+1}$ are fixed by this map, the map $\phi'_{k,n}$ induces the maps

$$\begin{cases} \phi''_{k,n} : W_{n+1,k+1} = U_{n+1}/U_{n-k} \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \phi_{k,n} : X_{n+1,k+1} = U_{n+1}/(\Delta_k \times U_{n-k}) \to \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases}$$

such that the diagram

$$\begin{array}{ccc} U_{n+1} & \xrightarrow{\phi'_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \downarrow & & =\downarrow \\ W_{n+1,k+1} & \xrightarrow{\phi''_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ \downarrow & & =\downarrow \\ X_{n+1,k+1} & \xrightarrow{\phi_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{array}$$

is commutative, where the left vertical maps are natural projections.

From now on, we assume $1 \leq k < n$ and consider the composite of maps $W_{n+1,k+1} \xrightarrow{\phi''_{k,n}} \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{\iota_1} K(\mathbb{Z}, 2)$. Since $\iota_1 \circ \phi''_{k,n} \in [W_{n+1,k+1}, K(\mathbb{Z}, 2)] \cong H^2(W_{n+1,k+1}, \mathbb{Z}) = 0$, the map $\iota_1 \circ \phi''_{k,n}$ is null-homotopic. Hence using the fibration sequence $(*)_{k,n}$, there is a map

$$\tilde{\phi}_{k,n} : W_{n+1,k+1} \to \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$$

such that the following diagram is homotopy commutative:

$$\begin{array}{ccc} W_{n+1,k+1} & \xrightarrow{\tilde{\phi}_{k,n}} & \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ =\downarrow & & p_n\downarrow \\ W_{n+1,k+1} & \xrightarrow{\phi''_{k,n}} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{array}$$

Remark 2.6. Since $H^2(X_{n+1,k+1}, \mathbb{Z}) \neq 0$, the map $\phi_{k,n}$ does not necessarily lift to $\tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$.

Theorem 2.7. *If* $1 \leq k < n$, $\tilde{\phi}_{k,n} : W_{n+1,k+1} \xrightarrow{\simeq} \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ *is a homotopy equivalence.*

Proof. It follows from the definitions of the maps $\beta_{k,n}$, and $\tilde{\phi}_{k,n}$ that the following diagram is homotopy commutative

$$\begin{array}{ccccc}
U_n/U_{n-k} & \xrightarrow{q_1} & U_{n+1}/U_{n-k} = W_{n+1,k+1} & \xrightarrow{q_2} & U_{n+1}/U_n \\
\beta_{k,n}\downarrow \simeq & & \tilde{\phi}_{k,n}\downarrow & & \beta \downarrow \cong \\
\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{j}_1} & \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\tilde{ev}} & S^{2n+1}
\end{array}$$

where two horizontal sequences are fibration sequences, β is a natural homeomorphism and q_m $(m = 1, 2)$ are natural projections. Then it follows from the homotopy exact sequences of the above fibrations that $\tilde{\phi}_{k,n}$ is a homotopy equivalence. □

Theorem 2.8. *If $1 \leq k \leq n$, the map $\phi_{k,n} : X_{n+1,k+1} \xrightarrow{\simeq} \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)$ is a homotopy equivalence.*

Proof. If $k = n = 1$, $X_{2,2} = U_2/\Delta_2 \cong \mathbb{R}\mathrm{P}^3 \cong \mathrm{PSL}_2(\mathbb{C})$ and the assertion is clear. Next consider the case $1 \leq k < n$. Then there is a homotopy commutative diagram

$$\begin{array}{ccccc}
\Delta_{k+1} & \xrightarrow{q_1'} & W_{n+1,k+1} & \xrightarrow{q_2'} & U_{n+1}/(\Delta_{k+1} \times U_{n-k}) = X_{n+1,k+1} \\
\beta' \downarrow \cong & & \tilde{\phi}_{k,n} \downarrow \simeq & & \phi_{k,n} \downarrow \\
S^1 & \longrightarrow & \tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) & \xrightarrow{p_n} & \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n)
\end{array}$$

where two horizontal sequences are fibration sequences, β' is a natural homeomorphism and q'_m $(m = 1, 2)$ are natural projections. Then it follows from the homotopy exact sequences of the above fibrations that $\phi_{k,n}$ is a homotopy equivalence.

Finally we prove the assertion for the case $k = n \geq 2$.

[**I suppose that the assertion is true for the case $k = n \geq 2$. But I do not know how to prove this in this case. If $k = n \geq 2$, probably $\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ is not simply connected and it seems useless to consider the space $\tilde{\mathrm{Hol}}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$. Moreover, if so $\pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)\mathbb{Z}$. (See corollary 2.9.)**

More generally, I suppose that $\pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)d^n\mathbb{Z}$ for any $d \geq 1$!] □

Corollary 2.9. *If $k = n$, then $\pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \mathbb{Z}/(n+1)\mathbb{Z}$.*

Proof. $\pi_1(\mathrm{Hol}_1(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) \cong \pi_1(U_{n+1}/\Delta_{n+1}) = \mathbb{Z}/(n+1)\mathbb{Z}$. □

Corollary 2.10. *Let $1 \leq k \leq n$ and let*

$$\begin{cases} i_{k,n} : \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \\ j_{k,n} : \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \end{cases}$$

be inclusion maps. Then $i_{k,n}$ and $j_{k,n}$ are homotopy equivalences up to dimension $D(n;k) = 4n - 4k + 1$.

Proof. Consider the composite of maps $j_{k,n} \circ \phi_{k,n}$:

$$X_{n+1,k+1} \underset{\simeq}{\xrightarrow{\phi_{k,n}}} \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n) \xrightarrow{j_{k,n}} \mathrm{Map}_1(\mathbb{C}\mathrm{P}^k, \mathbb{C}\mathrm{P}^n).$$

We remark that Sasao ([16], (1.1)) proved that the map $j_{k,n} \circ \phi_{k,n}$ is a homotopy equivalence up to dimension $D(n;k)$. Because $\phi_{k,n}$ is a homotopy equivalence, $j_{k,n}$ is a homotopy equivalence up to dimension $D(n;k)$.

Next, consider the comutative diagram

$$\begin{array}{ccccc} \mathrm{Hol}_1^*(\mathbb{CP}^k,\mathbb{CP}^n) & \longrightarrow & \mathrm{Hol}_1(\mathbb{CP}^k,\mathbb{CP}^n) & \xrightarrow{ev} & \mathbb{CP}^n \\ {\scriptstyle i_{k,n}}\big\downarrow & & {\scriptstyle j_{k,n}}\big\downarrow & & {\scriptstyle =}\big\downarrow \\ \mathrm{Map}_1^*(\mathbb{CP}^k,\mathbb{CP}^n) & \longrightarrow & \mathrm{Map}_1(\mathbb{CP}^k,\mathbb{CP}^n) & \xrightarrow{ev} & \mathbb{CP}^n \end{array}$$

where two horizontal sequences are evaluation fibrations. Then because $j_{k,n}$ is a homotopy equivalence up to dimension $D(n;k)$, $i_{k,n}$ is also a homotopy equivalence up to dimension $D(n;k)$. □

References

1. M. F. Atiyah and N. J. Hitchin, *The geometry and dynamics of magnetic monoples*, Princeton Univ. Press, 1988.
2. A. L. Blakers and W. S. Massay, *The homotopy groups of triads* (III), Ann. of Math. **5**8 (1953), 409–417.
3. F. R. Cohen, R. L. Cohen, B. M. Mann and R. J. Milgram, *The topology of rational functions and divisors of surfaces*, Acta Math. **166** (1991), 163–221.
4. F. R. Cohen, T. I. Lada and J. P. May, *The homology of iterated loop spaces*, Lecture Notes in Math. (Springer-Verlag) **533**, 1976.
5. F. R. Cohen, J. C. Moore and J. A. Neisendorfer, *The double suspension and exponents of the homotopy groups of spheres*, Ann. of Math. **110** (1979) 549–565.
6. R. L. Cohen and D. Shimamoto, *Rational functions, labelled configurations and Hilbert schemes*, J. London Math. Soc. **43** (1991), 509–528.
7. S. K. Donaldson, *Nahm's equations and the classification of monopoles*, Commun. Math. Phys. **96** (1984), 387–407.
8. M. A. Guest, A. Kozlowski, M. Murayama and K. Yamaguchi, *The homotopy type of spaces of rational functions*, J. Math. Kyoto Univ. **35** (1995), 631–638.
9. M. A. Guest, A. Kozlowski, and K. Yamaguchi, *The topology of spaces of coprime polynomials*, Math. Z. **217** (1994), 435–446.
10. M. A. Guest, A. Kozlowski, and K. Yamaguchi, *Spaces of polynomials with roots of bounded multiplicity*, Fund. Math. **116** (1999), 93–117.
11. M. A. Guest, A. Kozlowski, and K. Yamaguchi, *Stable splitting of the space of polynomials with roots of bounded multiplicity*, J. Math. Kyoto Univ. **38** (1998), 351–366.
12. M. A. Guest, *The topology of the space of rational curves on a toric variety*, Acta Math. **174** (1995), 119–145.
13. I. M. James, *On the homotopy groups of certain pairs and triads*, Quart. J. Math. **5** (1954), 260–270.
14. A. Kozlowski and K. Yamaguchi, *Topology of complements of discriminants and resultants*, J. Math. Soc. Japan **52** (2000), 949–959.
15. J. Mostovoy, *Spaces of rational loops on a real projective space*, Trans. Amer. Math. Soc. **353** (2001), 1959–1970.
16. S. Sasao, *The homotopy of* $\mathrm{Map}(\mathbb{CP}^m,\mathbb{CP}^n)$, J. London Math. Soc. **8** (1974), 193–197.
17. G. B. Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), 39–72.
18. H. Toda, *Composition methods in homotopy groups of spheres*, Princeton Univ. Press **49**, 1962.
19. G. W. Whitehead, *On products in homotopy groups*, Ann. of Math. **47** (1946), 460–475.
20. K. Yamaguchi, *Complements of resultants and homotopy types*, J. Math. Kyoto Univ. **39** (1999), 675–684.

21. K. Yamaguchi, *Spaces of holomorphic maps with bounded multiplicity*, Quart. J. Math. **52** (2001), 249–259.
22. K. Yamaguchi, *Universal coverings of spaces of holomorphic maps*, (to appear) Kyushu J. Math.
23. K. Yamaguchi, *Connective coverings of spaces of holomorphic maps*, preprint.

Department of Mathematics, Toyama International University, Kaminikawa, Toyama, Tokyo 930-1262, Japan
E-mail address: andrzej@tuins.ac.jp

Department of Information Mathematics, University of Electro-Communications, Chofugaoka, Chofu, Tokyo 182-8585, Japan
E-mail address: kohhei@im.uec.ac.jp