Spaces of algebraic and continuous maps between real algebraic varieties

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Abstract

We consider the inclusion of the space of algebraic (regular) maps between real algebraic varieties in the space of all continuous maps. For a certain class of real algebraic varieties, which include real projective spaces, it is well known that the space of real algebraic maps is a dense subset of the space of all continuous maps. Our first result shows that, for this class of varieties, the inclusion is also a homotopy equivalence. After proving this, we restrict the class of varieties to real projective spaces. In this case, the space of algebraic maps has a 'minimum degree' filtration by finite dimensional subspaces and it is natural to expect that the homotopy types of the terms of the filtration approximate closer and closer the homotopy type of the space of continuous mappings as the degree increases. We prove this and compute the lower bounds of this approximation for 'even' components of these spaces (more precisely, we prove a very similar and closely related result, and state this one as a conjecture). This result can be seen as a generalization of the results of [23], [26] and [13] on the topology of the space of real rational maps and the space of real polynomials without n-fold roots, which can be viewed as real analogues of Segal's work [25] on the space of complex rational maps.

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1 Introduction.

Let X and Y be two topological spaces with some additional structures, e.g. that of a complex or symplectic manifold or algebraic variety. The space $\mathcal{S}(X,Y)$ of continuous maps $X \to Y$ preserving the structure is a subspace of the space of all continuous maps $\mathrm{Map}(X,Y)$ and it is natural to ask if the two spaces are in some topological sense (e.g. homotopy, homology) equivalent. Early examples of this type of phenomenon can be found in [9]. In many cases of interest the infinite dimensional space $\mathcal{S}(X,Y)$ has a filtration by finite dimensional subspaces, given by some kind of "map degree", and the topology of these finite dimensional spaces approximates the topology of the entire space of continuous maps; the approximation becoming more accurate as the degree increases. In recent years a great deal of attention has been devoted to studying problems of the above kind in the case where the "additional structure" in question is the structure of a complex manifold (so that the structure preserving maps are holomorphic maps), with the space X a compact surface and the space Y a certain complex manifold. The first explicit result of this kind seems to have been the following theorem of Segal:

Theorem 1.1 (G. Segal, [25]). *If M_g is a compact closed Riemann surface of genus g, the inclusions*

$$\begin{cases} j_{d,\mathbb{C}} : \mathrm{Hol}_d(M_g, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(M_g, \mathbb{C}\mathrm{P}^n) \\ i_{d,\mathbb{C}} : \mathrm{Hol}^*_d(M_g, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}^*_d(M_g, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homology equivalences through dimension $(2n-1)(d-2g)-1$ if $g \geq 1$ and homotopy equivalences through dimension $(2n-1)d-1$ if $g=0$.

Here $\mathrm{Hol}^*_d(M_g, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Hol}_d(M_g, \mathbb{C}\mathrm{P}^n)$) and $\mathrm{Map}^*_d(M_g, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Map}_d(M_g, \mathbb{C}\mathrm{P}^n)$) denote the spaces consisting of all based (resp. free) holomorphic maps or of continuous maps $f : M_g \to \mathbb{C}\mathrm{P}^n$ of the degree d.

Remark. A map $f : X \to Y$ is called *a homology (resp. homotopy) equivalence through dimension N* if the induced homomorphism

$$f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z}) \quad (\text{resp. } f_* : \pi_k(X) \to \pi_k(Y))$$

is an isomorphism for all $k \leq N$.

Segal conjectured in [25] that this result should generalize to a much larger class of target spaces, such as complex Grassmannians and flag manifolds, and even possibly to higher dimensional source spaces. Almost all of the work inspired by Segal's results was concerned with extending the Segal results to a larger class of target spaces, while the source space was kept (complex) one dimensional (e.g. [3], [8], [10], [11], [12], [18]). Several authors (e.g. [2], [5]) have attempted to find the most general target spaces for which the stability theorem holds. There have been, however, very few attempts to investigate (as suggested by Segal) the phenomenon of topological stability for source spaces of complex dimension greater than 1. The first steps in this direction were taken by Havlicek [16] who considered the space of holomorphic maps from $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ to complex Grassmanians and Kozlowski and Yamaguchi [19] who studied the case of linear maps $\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$, where $m \leq n$. A major step appeared to have been taken when Mostovoy [24] published a proof of the following analogue of Segal's theorem for the space of holomorphic maps from $\mathbb{C}\mathrm{P}^m$ to $\mathbb{C}\mathrm{P}^n$.

Theorem 1.2 (J. Mostovoy, [24])**.** *If $2 \leq m \leq n$ and $d \geq 1$ are integers, the inclusion $j_{d,\mathbb{C}} : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ is a homotopy equivalence through dimension $D_{\mathbb{C}}(d; m, n)$ if $m < n$, and a homology equivalence through dimension $D_{\mathbb{C}}(d; m, n)$ if $m = n$, where $\lfloor x \rfloor$ denotes the integer part of a real number x and the number $D_{\mathbb{C}}(d; m, n)$ is given by $D_{\mathbb{C}}(d; m, n) = (2n - 2m + 1)\Big(\lfloor \frac{d+1}{2} \rfloor + 1\Big) - 1$.*

Unfortunately there are serious gaps in Mostovoy's proof of Theorem 1.2, which, as far as we are aware, have not been filled at the time of our writing this. So, at least at this time, Theorem 1.2 should be regarded as a conjecture. Nevertheless, in what follows we shall continue to refer to the result as "Mostovoy's theorem" and many of Mostovoy's ideas will play important roles in what follows.

A striking feature of the argument in [24] is that it uses real rather than complex methods. This suggests that the same or a similar method should be applicable to proving a real analogue of Mostovoy's theorem. In fact it turns out that the analogues of the problems that affect Mostovoy's argument in the complex case are much milder in the real one and allow us to obtain a more satisfactory, although probably not optimal, result. As explained in [25], the original motivation for Segal's work on rational functions came from the real case. The first "real analogue" of Segal's theorem was proved by Segal himself in the same work. Different real analogues (equivalent to one another) are given in [23] and [13]. It is these results that we generalize here. We begin by stating and describing them. While most of our statements will be concerned only with the real case, there will be some that apply to both the real and the complex one. To deal with such cases we will write $\mathbb{K}$ when we mean either $\mathbb{R}$ or $\mathbb{C}$.

We begin by stating our "stable" results, which we state in greater generality than we need here, but which are analogous to the already mentioned well known theorems about the density of algebraic maps in the space of continuous ones.

Let $\mathbb{G}_{n,k}(\mathbb{K})$ be the Grassmanian manifold of k dimensional $\mathbb{K}$-subspaces in $\mathbb{K}^n$. For an affine real algebraic variety X, let $\mathrm{Alg}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ denote the space consisting of all algebraic (i.e. regular) maps $f : X \to \mathbb{G}_{n,k}(\mathbb{K})$ and $\mathrm{Map}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ the spaces of continuous maps.

Theorem 1.3 (Stable Theorem). *Let X be a compact affine real algebraic variety, with the property that every topological $\mathbb{K}$-vector bundle of rank k over X is topologically isomorphic to an algebraic $\mathbb{K}$-vector bundle. Then the inclusion $i : \mathrm{Alg}(X, \mathbb{G}_{n,k}(\mathbb{K})) \to \mathrm{Map}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ is a weak homotopy equivalence.*

Note that the assumption of Theorem 2.2 is satisfied for the case $\mathbb{K} = \mathbb{R}$ and $X = \mathbb{R}\mathrm{P}^m$ or $X = \mathbb{C}\mathrm{P}^m$. For our "unstable" results we limit ourselves to the case where on both sides we have real projective spaces.

Definition. Let m and n be positive integers with $1 \leq m < n$, and let $z_0, \cdots, z_m$ denote fixed variables. We choose the point $\mathbf{e}_k = [1 : 0 : \cdots : 0] \in \mathbb{R}\mathrm{P}^k$ $(k = m, n)$ as the base point of $\mathbb{R}\mathrm{P}^k$.

(i) For $\epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_0(\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$, let $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ be the corresponding path component of $\mathrm{Map}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$, and $\mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ the subspace of $\mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ consisting of all based maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ such that $f(\mathbf{e}_m) = \mathbf{e}_n$.

(ii) Let $A_d(m, n)(\mathbb{R})$ denote the space of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in \mathbb{R}[z_0, \cdots, z_m]^{n+1}$ of homogeneous polynomials of degree d without non-trivial common real roots (but possibly with non-trivial common *complex* roots). Let $A_d(m, n) \subset A_d(m, n)(\mathbb{R})$ denote the subspace consisting of $(n+1)$-tuples $(f_0, \cdots, f_n) \in A_d(m, n)(\mathbb{R})$ such that the coefficient of z_0^d in f_0 is 1 and in the other f_k's zero.

(iii) A map $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ is called *an algebraic map of degree d* if it is represented as $f = [f_0 : \cdots : f_n]$ for some $(f_0, \cdots, f_n) \in A_d(m, n)(\mathbb{R})$. Let $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ denote the space of free algebraic maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^m$ of degree d. Let $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ denote the subspace of based algebraic maps $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$.

(iv) If $m \geq 2$ then, for a fixed map $g \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$, let $\mathrm{Alg}^*_d(m, n; g)$ and $F_d(m, n; g)$ denote the spaces defined by

$$\begin{cases} \mathrm{Alg}^*_d(m, n; g) & = \{f \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}, \\ F_d(m, n; g) & = \{f \in \mathrm{Map}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) : f|\mathbb{R}\mathrm{P}^{m-1} = g\}. \end{cases}$$

Observe that there is a homotopy equivalence $F_d(m, n; g) \simeq \Omega^m S^n$.

(v) Let $A_d(m,n;g) \subset A_d(m,n)$ denote the subspace consisting of all $(n+1)$-tuples of homogenous polynomials which represents algebraic maps in $\mathrm{Alg}_d(m,n;g)$.

Observe that if and algebraic map $f \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is represented as $f = [f_0 : \cdots : f_n]$ for some $(f_0, \cdots, f_n) \in A_d(m,n)$, then the same map is also represented as $f = [g_m f_0 : \cdots : g_m f_n]$, where $g_m = \sum_{k=0}^m z_k^2$. So there is an inclusion $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Alg}^*_{d+2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and we have *a stabilization map* $s_d : A_d(m,n) \to A_{d+2}(m,n)$ given by $s_d(f_0, \cdots, f_n) = (g_m f_0, \cdots, g_m f_n)$ compatible with the inclusion of spaces of algebraic maps. A map $f \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is called an algebraic map of *minimal degree* d if $f \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. There is a natural projection map

$$\Psi_d : A_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n). \tag{1}$$

If $i_{d,\mathbb{R}} : \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ denotes the inclusion, we also have a natural projection map

$$i_d = i_{d,\mathbb{R}} \circ \Psi_d : A_d(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \tag{2}$$

where $[d]_2 \in \mathbb{Z}/2$ denotes the integer d mod 2.

We can now state two (closely related) real analogues of Segal's theorem ($m = 1$).

Theorem 1.4 (J. Mostovoy, [23])**.** *If $n \geq 2$ and $d \geq 1$, the inclusion $i_{d,\mathbb{R}}$: $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \simeq \Omega S^n$ is a homotopy equivalence through dimension $D_1(d,n) = d(n-1) - 2$.*

Theorem 1.5 ([19], [27])**.** *If $n \geq 2$ and $d \geq 1$, the natural projection map $i_d : A_d(1,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \simeq \Omega S^n$ is a homotopy equivalence through dimension $D_2(d,n) = (d+1)(n-1) - 2$.*

Mostovoy's stability dimension $D_1(d,n)$ is much smaller than $D_2(d,n)$ in Theorem 1.5, which is, in fact, optimal. Thus theorem 1.5 is actually stronger than Theorem 1.4, but this depends on the fact that $\Psi_d : A_d(1,n) \xrightarrow{\simeq} \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence. This follows from Proposition 2.1 of [23], but the proof given in [23] does not seem to us convincing, and so we shall give a different one in section 5. This will allow us to replace $A_d(1,n)$ in Theorem 1.5 by $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n)$.

Now we consider the generalization of Theorem 1.5 for $m \geq 2$. For a connected space X, let $C_r(X)$ denote the space of r-distinct unordered points in X. Let $M(m,n)$ and $D(d;m,n)$ denote the positive integers given by

$$M(m,n)=2\lceil \frac{m+1}{n-m}\rceil +1, \quad D(d;m,n)=(n-m)\big(\lfloor \frac{d+1}{2}\rfloor +1\big)-1,$$

where $\lceil x\rceil=\min\{N\in\mathbb{Z}:N\geq x\}$.

We now state our two main results, which generalizeTheorem 1.5 to $m\geq 2$ in two ways.

Theorem 1.6. *Let $2\leq m<n$ be integers and $g\in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1},\mathbb{R}\mathrm{P}^n)$ be a fixed map of minimal degree d.*

(i) *The inclusion $i_d^\prime:\mathrm{Alg}^*_d(m,n;g)\to F_d(m,n;g)\simeq \Omega^mS^n$ induces a split epimorphism on $H_k(\ ,\mathbb{Z})$ for any $1\leq k\leq D(d;m,n)$.*

(ii) *For any $k\geq 1$, $H_k(\mathrm{Alg}^*_d(m,n;g),\mathbb{Z})$ contains the subgroup*

$$G^d_{m,n}=\bigoplus_{r=1}^{\lfloor \frac{d+1}{2}\rfloor} H_{k-(n-m)r}(C_r(\mathbb{R}^m),(\pm \mathbb{Z})^{\otimes (n-m)})$$

as a direct summand.

(iii) *If $d\geq M(m,n)$, the inclusion $i_d^\prime:\mathrm{Alg}^*_d(m,n;g)\to F_d(m,n;g)\simeq \Omega^mS^n$ is a homotopy equivalence through dimension $D(d;m,n)$ if $m+2\leq n$ and a homology equivalence through dimension $D(d;m,n)$ if $m+1=n$.*

Theorem 1.7. *Let $2\leq m<n$ and $d=2d^*\equiv 0 \pmod 2$ be positive integers.*

(i) *The map $i_d:A_d(m,n)\to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)$ induces an epimorphism on $H_k(\ ,\mathbb{Z})$ for any $1\leq k\leq D(d;m,n)$.*

(ii) *If $d=2d^*\geq M(m,n)$, the map $i_d:A_d(m,n)\to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m,\mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence through dimension $D(d;m,n)$ if $m+2\leq n$ and a homology equivalence through dimension $D(d;m,n)$ if $m+1=n$.*

Corollary 1.8. *If $2\leq m<n$ and $d\geq M(m,n)$ is an even integer, the stabilization map $s_d:A_d(m,n)\to A_{d+2}(m,n)$ is a homotopy equivalence through dimension $D(d;m,n)$ for $m+2\leq n$ and a homology equivalence through dimension $D(d;m,n)$ for $m+1=n$.*

The stabilization bound in our more general Theorem 1.6 or Theorem 1.7 in the case $m=1$ is not as good as that in Theorem 1.5, but still better than that in Theorem 1.4. On the other hand there are two respects in which Theorem 1.7 is not fully satisfactory.

Firstly, in both cases the source and target spaces have two components which we call the odd and the even one (in the case of algebraic maps elements of the even or odd component are represented by tuples of polynomials of even or respectively odd degree without non-trivial common roots). When $m = 1$, the odd and the even components are easily seen to be homotopy equivalent, so it is enough to consider only the even components to obtain a proof of homotopy equivalence for both even and odd ones (this is, indeed, how the argument in [23] works). However, this is no longer the case when $m > 1^*$, in which case our argument applies only to the even components. We believe a similar result to be true for the odd components but at this stage are unable to prove it.

The second respect in which Theorem 1.7 is less satisfactory than Theorem 1.4 is that for $m > 1$ we are no longer able to prove that the spaces $A_d(m, n)$ and $\mathrm{Alg}^*_d(\mathbb{RP}^m, \mathbb{RP}^n)$ are homotopy equivalent. We state this instead as a conjecture:

Conjecture 1.9. $\Psi_d : A_d(m, n) \to \mathrm{Alg}^*_d(\mathbb{RP}^m, \mathbb{RP}^n)$ *is a homotopy equivalence.*

For $m = 1$ the conjecture is actually a theorem. It follows from [[23], Proposition 2.4], which we will prove below. Unfortunately even the formulation (not to mention a proof) of an analogous result for $m \geq 2$ seems far from clear to us. We shall consider this issue in section 5.3 below. If Conjecture 1.9 is true then the following improved version of Theorem 1.7 holds.

Conjecture 1.10. *If* $2 \leq m < n$ *and* $d \geq M(m, n)$ *is an even integer, the inclusion maps*

$$\begin{cases} i_{d,\mathbb{R}} : \mathrm{Alg}^*_d(\mathbb{RP}^m, \mathbb{RP}^n) \to \mathrm{Map}^*_0(\mathbb{RP}^m, \mathbb{RP}^n) \\ j_{d,\mathbb{R}} : \mathrm{Alg}_d(\mathbb{RP}^m, \mathbb{RP}^n) \to \mathrm{Map}_0(\mathbb{RP}^m, \mathbb{RP}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d; m, n)$ *if* $m + 2 \leq n$ *and homology equivalences through dimension* $D(d; m, n)$ *if* $m + 1 = n$.

It is easy to see that the maps Ψ_d have contractible fibers but this alone does not suffice to conclude that they are homotopy equivalences, as will be discussed in detail later.

This paper is organized as follows. In section 2, we study the space of regular maps and give the proof of Theorem 1.3, and in section 3 we review

*For example, if $2 \leq m < n$ and n is even, it is known that $\mathrm{Map}^*_0(\mathbb{RP}^m, \mathbb{RP}^n)$ and $\mathrm{Map}^*_1(\mathbb{RP}^m, \mathbb{RP}^n)$ are not homotopy equivalent ([6]).

the case of the spaces of maps between real projective spaces. In section 4, we study spectral sequences of the Vassiliev type and prove Theorem 1.6 and Theorem 1.7. Finally, in section 5 we investigate Conjecture 1.9 and related topics.

2 Stable real results.

We start by defining what we mean by an "algebraic map" , usually referred to in algebraic geometry as a "regular" map.

Definition. Let $V \subset \mathbb{K}^n$ be an algebraic subset and U be a (Zariski) open subset of V. We say that a function $f : U \to \mathbb{K}$ is *an regular function on* U if it can be written as the quotient of two polynomials $f = g/h$, with $h^{-1}(0) \cap U = \emptyset$. For a subset $W \subset \mathbb{K}^p$, a map $\varphi : U \to W$ is called *an algebraic map* if its coordinate functions are regular functions.

Clearly regular functions on an algebraic set form a sheaf. An regular map between two algebraic sets is one that induces a map of sheaves or regular functions (is a morphism of ringed spaces). These definitions extend in a natural way to abstract algebraic varieties, and in particular to projective varieties. An algebraic map between real or complex algebraic varieties is a continuous map (with respect to the complex or real topology) and a C^∞ map in the case of smooth varieties.

To state our stable results in the most general form we need the concept of an algebraic $\mathbb{K}$ vector bundle over a real algebraic variety .

Recall ([4]) that *a pre-algebraic vector bundle* over a real algebraic variety X is a triple $\xi = (E, p, X)$, such that E is a real algebraic variety, $p : E \to X$ is an algebraic map, the fiber over each point is a $\mathbb{K}$-vector space and there is a covering of the base X by Zariski open sets over which the vector bundle E is biregularly isomorphic to the trivial bundle. *An algebraic vector bundle* over X is a pre-algebraic vector bundle which is algebraically isomorphic to a pre-algebraic vector sub-bundle of a trivial bundle. Most of the usual vector bundle constructions when performed on algebraic vector bundles give rise to algebraic vector bundles; in particular the Whitney sum, and the tensor product of two algebraic bundles is algebraic, so is the pull-back of an algebraic bundle by a regular map between real algebraic varieties. Important examples of algebraic vector bundles are the universal vector bundles over the Grassmanian manifold $\mathbb{G}_{n,k}(\mathbb{K})$ of all k dimensional $\mathbb{K}$-subspace in $\mathbb{K}^n$ and their complementary bundles.

Lemma 2.1 ([4]). *A topological vector bundle over a real algebraic variety X, which is stably isomorphic to an algebraic vector bundle, is topologically* *isomorphic to an algebraic vector bundle.*

The above lemma and the Stone-Weierstrass theorem are the main ingredients in proving the following result:

Theorem 2.2 ([4]). *Let X be a compact affine real algebraic variety, with the property that every topological $\mathbb{K}$-vector bundle of rank k over X is topologically isomorphic to an algebraic $\mathbb{K}$-vector bundle. Then the space of algebraic mappings* $\mathrm{Alg}(X, \mathbb{G}_{n,k}(\mathbb{K}))$ *is dense in* $\mathrm{Map}(X, \mathbb{G}_{n,k}(\mathbb{K}))$.

Note that we are considering $\mathbb{G}_{n,k}(\mathbb{C})$ as a real affine algebraic variety (see [4]). Using essentially the same method as in [4] and an idea from [24] we will prove our Theorem 1.3, which asserts that, under the assumptions of Theorem 2.2, the space of algebraic maps is not only dense in the space of continuous maps but is also homotopy equivalent to it.

Note that both spaces $\mathbb{R}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^m$ satisfy the assumption of Theorem 2.2. The real case, that is the most important for us here, follows from Lemma 2.1 and from the fact the $\tilde{KO}(\mathbb{R}\mathrm{P}^m)$ is generated by the class of a unique non-trivial line bundle ([17] Theorem 12.7). The other cases follow form known results about $\tilde{KO}(\mathbb{C}\mathrm{P}^m)$, $\tilde{K}(\mathbb{C}\mathrm{P}^m)$ and $\tilde{K}(\mathbb{R}\mathrm{P}^m)$.

The assumption of Theorem 1.3 is not satisfied in general, but it is known that for every compact smooth manifold M, there exists a non-singular real algebraic variety X diffeomorphic to M such that every topological vector bundle over X is isomorphic to a real algebraic one ([4]).

To prove Theorem 1.3 we use the following Proposition:

Proposition 2.3. *Let X be as in Theorem 2.2 and let Y be a finite CW-complex. Let $F : Y \times X \to \mathbb{G}_{n,k}(\mathbb{K})$ be a continuous map. Then F can be approximated uniformly by maps $G : Y \times X \to \mathbb{G}_{n,k}(\mathbb{K})$, such that the restriction $G_y : \{y\} \times X \to \mathbb{G}_{n,k}(\mathbb{K})$ is a algebraic map for each $y \in Y$.*

Proof. We imitate the method of proof of Lemma 4 of [24]. Elements of $\mathrm{Map}(Z, \mathbb{G}_{n,k}(\mathbb{K}))$ are in one to one correspondence with with objects of the form: k-dimensional $\mathbb{K}$-bundle E on Z together with n sections, which, at each point z of Z span the fiber E_z of the bundle. Let $f : Y \times X \to \mathbb{G}_{n,k}(\mathbb{K})$ be a continuous map and denote by E_f and $s_1, \ldots, s_n$ the the corresponding bundle on $Y \times X$ and sections spanning the fiber at each point. Since Y is a finite complex, we can choose an open contractible cover $\{U_k\}_{k=1}^{l}$ of Y and and associated partition of unity $\{\rho_k : Y \to \mathbb{R}\}_{k=1}^{l}$. Since U_k is contractible, because of the assumption on X, for each k there is an algebraic bundle P_k on X such that the restriction of E_f to $U_k \times X$ is isomorphic to a bundle induced from P_k by the projection map. We can thus identify each such restriction with P_k. Let Σ denote the family of sections of E_f over $Y \times X$ generated by all sections of the form $\rho_k s$ in U_k, where s is an algebraic section of P_k, extended to the whole of $Y \times X$ by 0. It is easy to see that the family of sections Σ satisfies the conditions of the Stone-Weierstrass theorem for vector bundles [4] so each section s_i can be uniformly approximated by algebraic sections. By choosing sections close enough to the s_i we can choose approximations consisting of n algebraic sections spanning the fiber. These sections define the required algebraic map $Y \times X \to \mathbb{G}_{n,k}(\mathbb{K})$. $\square$

Proof of Theorem 1.3. Let N be any positive integer. It suffices to show that $i_* : \pi_N(\mathrm{Alg}(X, \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K}))) \to \pi_N(\mathrm{Map}(X, \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K})))$ is an isomorphism. First, we will show that i_* is injective. Let $\alpha = [F] \in \pi_N(\mathrm{Map}(X, \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K})))$ be any element and let $\tilde{F} : S^N \times X \to \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K})$ be the adjoint of the representative map F. By Proposition 2.3 there is a uniform approximation to $\tilde{F}$, $\tilde{G} : S^N \times X \to \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K})$, such that $\tilde{G}|X \times \{s\}$ is a algebraic map for each $s \in S^N$. Let $G : S^N \to \mathrm{Alg}(X, \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K}))$ denote the adjoint of $\tilde{G}$. Clearly, $i_*([G]) = \alpha$. Hence i_* is surjective.

Next, we show that i_* is injective. Suppose that $i_*(\beta_0) = i_*(\beta_1)$ for $\beta_l \in \pi_N(\mathrm{Alg}(X, \mathbb{G}\mathrm{r}_{n,k}(\mathbb{K})))$ $(l = 0, 1)$. Let $\beta_l = [F_l]$ and let $\tilde{F}_l$ denote the adjoint of F_l for $l \in \{0, 1\}$. There is a homotopy $\Phi : X \times S^N \times [0, 1] \to X$ such that $\Phi|X \times S^N \times \{l\} = \tilde{F}_k$ for $l \in \{0, 1\}$.

Then by Proposition 2.3 there is a uniform approximation $\Psi : X \times S^N \times [0, 1] \to X$ to Φ such that $\Psi|X \times \{(s, t)\}$ is a algebraic map for any $(s, t) \in S^N \times [0, 1]$. Hence, $\beta_0 = \beta_1$ and i_* is injective. $\square$

3 Maps between real projective spaces.

From now on, we shall consider only algebraic and continuous maps between real projective spaces. In other words, we consider the case $(\mathbb{K}, k) = (\mathbb{R}, 1)$ in Theorem 1.3. We intend to return to the other cases elsewhere.

A real algebraic (regular) map $f : \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ can always be represented as $f = [f_0 : f_1 : \cdots : f_n]$, where $f_0, \cdots, f_n \in \mathbb{R}[z_0, z_1, \cdots, z_m]$ are homogenous polynomials of the same degree d with no common *real* root other than $\mathbf{0}_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$ (but possibly with common *non-real* roots). We will refer to a algebraic map represented in this way as *an algebraic map of degree d*. Of course a representation of a algebraic map as an algebraic map of degree d is not unique; by multiplying all the components by a nowhere-vanishing polynomial we obtain a a higher degree algebraic representation of the same map. For an algebraic map f, we shall refer to the smallest possible degree of the polynomials in such a representation as the *minimal degree* of f. Note that unlike degree, which depends on a representation of an algebraic map, the minimal degree depends only on the (algebraic) map itself.

Clearly, an element of $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ can always be represented in the form $f = [f_0 : f_1 : \cdots : f_n]$, such that the coefficient of z_0^d in f_0 is 1 and in the other polynomials f_i $(i \neq 0)$ 0. In general, such a representation is also not unique. For example, if we multiply all polynomials f_i by two different homogeneous polynomials in $z_0, z_1, \cdots, z_m$, which contain a power of z_0 with coefficient 1 and are always positive on $\mathbb{R}\mathrm{P}^m$, we will obtain two distinct representations of the same algebraic map.

Recall that we always have $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Alg}^*_{d+2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and $\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Alg}_{d+2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$, because $[f_0 : f_1 : \cdots : f_n] = [g_m f_0 : g_m f_1 : \cdots : g_m f_n]$, where $g_m = \sum_{k=0}^m z_k^2$.

Definition. For $\epsilon = 0$ or 1, define subspaces $\mathrm{Alg}^*_\epsilon(m,n) \subset \mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and $\mathrm{Alg}_\epsilon(m,n) \subset \mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ by

$$\begin{cases} \mathrm{Alg}^*_\epsilon(m,n) & = \bigcup_{k=1}^\infty \mathrm{Alg}^*_{\epsilon+2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n), \\ \mathrm{Alg}_\epsilon(m,n) & = \bigcup_{k=1}^\infty \mathrm{Alg}_{\epsilon+2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n). \end{cases}$$

Corollary 3.1. *If $1 \leq m < n$ and $\epsilon = 0$ or 1, the inclusion maps $i : \mathrm{Alg}^*_\epsilon(m,n) \xrightarrow{\simeq} \mathrm{Map}^*_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and $j : \mathrm{Alg}_\epsilon(m,n) \xrightarrow{\simeq} \mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ are homotopy equivalences.*

Proof. It follows from Theorem 1.3 that j is a homotopy equivalence. The statements and proofs of Proposition 2.3 and Theorem 1.3 are valid also for spaces of based maps, thus proving that i is also a homotopy equivalence. □

4 Spectral sequences of the Vassiliev type.

In this section, we construct two spectral sequences converging to the homologies $H_*(A_d(m,n;g), \mathbb{Z})$ and $H_*(A_d(m,n), \mathbb{Z})$, and give proofs of Theorem 1.6 and Theorem 1.7.

From now on, we assume $2 \leq m < n$ and let $g \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ be a fixed algebraic map of minimal degree d, such that $g = [g_0 : \cdots : g_n]$ with $(g_0, \cdots, g_n) \in A_d(m-1,n)$. Note that $(g_0, \cdots, g_n)$ is uniquely determined by g (because of the minimal degree condition).

4.1 Discriminants Σ^*_d and Σ_d.

Let $\mathcal{H}_d \subset \mathbb{R}[z_0, \cdots, z_m]$ denote the subspace consisting of all homogenous polynomials of degree d. For $\epsilon \in \{0,1\}$, let $\mathcal{H}^\epsilon_d \subset \mathcal{H}_d$ be the subspace consisting of all homogenous polynomials $f \in \mathcal{H}_d$ such that the coefficient of $(z_0)^d$ of f is ϵ. Since $A_d(m,n)$ is the space consisting of all $(n+1)$-tuples $(f_0,\cdots,f_n) \in A_d(m,n)(\mathbb{R})$ such that the coefficient of z_0^d in f_0 is 1 and those of other f_k's are all zero, $A_d(m,n) \subset A_d := \mathcal{H}_d^0 \times (\mathcal{H}_d^1)^n$. Note that A_d is an affine space of dimension of $N_d = (n+1)\Big(\binom{m+d}{m} - 1\Big)$.

Next, we set $B_k = \{g_k + z_m h : h \in \mathcal{H}_{d-1}\}$ $(k = 0, 1, \cdots, n)$ and define the subspace $A_d^* \subset A_d$ by $A_d^* = B_0 \times B_1 \times \cdots \times B_n$. Note that A_d^* is an affine space of dimension $N_d^* = (n+1)\binom{m+d-1}{m}$.

Definition. Let $A_d(m,n;g) \subset A_d^*$ be the subspace $A_d(m,n;g) = A_d^* \cap A_d(m,n)$. Let $\Sigma_d^* \subset A_d^*$ and $\Sigma_d \subset A_d$ denote the discriminants of $A_d(m,n;g)$ in A_d^* or of $A_d(m,n)$ in A_d defined as the complements

$$\Sigma_d^* = A_d^* \setminus A_d(m,n;g), \quad \Sigma_d = A_d \setminus A_d(m,n).$$

Since $g \in \mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ has minimal degree d, it is clear that the restriction

$$\Psi_d|_{A_d(m,n;g)} : A_d(m,n;g) \xrightarrow{\cong} \mathrm{Alg}_d^*(m,n;g) \tag{3}$$

is a homeomorphism.

Lemma 4.1. (i) *If $(f_0,\cdots,f_n) \in \Sigma_d^*$ and $\mathbf{x} = (x_0,\cdots,x_m) \in \mathbb{R}^{m+1}$ is a non-trivial common root of $f_0,\cdots,f_n$, then $x_m \neq 0$.*

(ii) *If $(f_0,\cdots,f_n) \in \Sigma_d^*$, then the number of the distinct common real roots of $\{f_0,\cdots,f_n\}$ is at most d^m.*

(iii) *If $m+2 \leq n$, then $A_d(m,n;g)$ and $A_d(m,n)$ are simply connected.*

Proof. (i) If $x_m = 0$ then, since $f_k|_{z_m=0} = g_k$, $(x_0,\cdots,x_{m-1}) \neq (0,\cdots,0)$ is a non-trivial common root of $g_0,\cdots,g_n$, which is a contradiction.

(ii) For each $0 \leq k \leq n$, let $L_k \subset \mathbb{R}\mathrm{P}^m$ and $L_k' \subset \mathbb{R}\mathrm{P}^m$ be the hypersurfaces defined by $L_k = \{[x_0 : \cdots : x_m] \in \mathbb{R}\mathrm{P}^m : f_k(x_0,\cdots,x_m) = 0\}$ and $L_k' = \{[x_0 : \cdots : x_m] \in \mathbb{R}\mathrm{P}^m : g_k(x_0,\cdots,x_m) = 0\}$. Since

$$\{[x_0 : \cdots : x_m] \in \mathbb{R}\mathrm{P}^m : x_m = 0\} \cap \big(\cap_{k=0}^n L_k\big) = \cap_{k=0}^n L_k' = \emptyset,$$

it follows from the Projective Dimension Theorem ([15]) that $V = \cap_{k=0}^n L_k$ is a finite set. Applying Bêzout's Theorem [[7], page 145 (3)] we deduce that V has at most d^m points.

(iii) Since the codimension of $A_d(m,n;g)$ in A_d^* is $n-m+1$, $A_d(m,n;g)$ is simply connected if $m+2 \leq n$. Similarly, the fact that the codimension of $A_d(m,n)$ in A_d is also $n-m+1$ implies that $A_d(m,n)$ is simply connected if $m+2 \leq n$. $\square$

Definition. (i) For a finite set $\mathbf{x} = \{x_0, x_1, \cdots, x_k\} \subset \mathbb{R}^N$, let $\sigma(\mathbf{x})$ denote the convex hull spanned by $\mathbf{x}$. Note that $\sigma(\mathbf{x})$ is an k-dimensional simplex if and only if vectors $\{x_j - x_0\}_{j=1}^k$ are linearly independent. In particular, $\sigma(\mathbf{x})$ is a k-dimensional simplex if $x_0, \cdots, x_k$ are linearly independent over $\mathbb{R}$.

(ii) Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding. Let $\mathcal{X}^\Delta$ and $h^\Delta : \mathcal{X}^\Delta \to Y$ denote the space and the map defined by

$$\mathcal{X}^\Delta = \big\{(y, w) \in Y \times \mathbb{R}^N : w \in \sigma(i(h^{-1}(y)))\big\} \subset Y \times \mathbb{R}^N, \ h^\Delta(y, w) = y.$$

The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called *a simplicial resolution* of (h, i). If for each $y \in Y$ any k points of $i(h^{-1}(y))$ span $(k-1)$-dimensional affine subspace of $\mathbb{R}^N$, $(\mathcal{X}^\Delta, h^\Delta)$ is called *a non-degenerate simplicial resolution*, otherwise the simplicial resolution is said to be *degenerate.*

(iii) For each $k \geq 0$, let $\mathcal{X}_k^\Delta \subset \mathcal{X}^\Delta$ be the subspace given by

$$\mathcal{X}_k^\Delta = \big\{(y, \omega) \in \mathcal{X}^\Delta : \omega \in \sigma(\{u_1, \cdots, u_k\}), \{u_1, \cdots, u_k\} \subset i(h^{-1}(y))\big\}.$$

We make the identification $X = \mathcal{X}_1^\Delta$ by identifying the point x with $(h(x), i(x))$, and obtain the following increasing filtration:

$$\emptyset = \mathcal{X}_0^\Delta \subset X = \mathcal{X}_1^\Delta \subset \mathcal{X}_2^\Delta \subset \cdots \subset \mathcal{X}_k^\Delta \subset \mathcal{X}_{k+1}^\Delta \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^\Delta = \mathcal{X}^\Delta.$$

Remark. Non-degenerate simplicial resolutions have a long history (see e.g. [26]), while degenerate ones appear to have been introduced for the first time in [24].

Lemma 4.2 ([24], [26])**.** *Let $h : X \to Y$ be a surjective map such that, for any $y \in Y$, $h^{-1}(y)$ is a finite set and let $i : X \to \mathbb{R}^N$ be an embedding.*

(i) *If X and Y are closed semi-algebraic spaces and the two maps h, i are polynomial maps, then $h^\Delta : \mathcal{X}^\Delta \xrightarrow{\simeq} Y$ is a homotopy equivalence.*

(ii) *There is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$ of (h, j) is non-degenerate, and the space $\tilde{\mathcal{X}}^\Delta$ is uniquely determined up to homeomorphism. Moreover, there is a filtration preserving homotopy equivalence $q^\Delta : \tilde{\mathcal{X}}^\Delta \xrightarrow{\simeq} \mathcal{X}^\Delta$ such that $q^\Delta | X = id_X$.* $\square$

Remark. When h is not finite to one, every simplicial resolution is degenerate. In this case it is also possible to define the associated non-degerate resolution with properties analogous to the above (for details see [24]).

4.2 The space $A_d(m,n;g)$.

The definitions and lemmas that begin this section are "real analogues" of "complex" definitions and lemmas in section 4 of [24].

Definition. Let $Z_d^* \subset \Sigma_d^* \times \mathbb{R}^m$ denote *the tautological normalization* of the discriminant Σ_d^* consisting of all pairs $(\mathbf{f},\mathbf{x}) = ((f_0,\cdots,f_n),(x_0,\cdots,x_{m-1})) \in \Sigma_d^* \times \mathbb{R}^m$ such that the polynomials $f_0,\cdots,f_n$ have a non-trivial common real root $(\mathbf{x},1) = (x_0,\cdots,x_{m-1},1)$. Projection on the first factor gives a surjective map $\pi'_d : Z_d^* \to \Sigma_d^*$.

Let $\phi_d^* : A_d^* \xrightarrow{\cong} \mathbb{R}^{N_d^*}$ be any fixed homeomorphism, and let H_d be the set consisting of all monomials $\varphi_I = z^I = z_0^{i_0} z_1^{i_1} \cdots z_m^{i_m}$ of degree d ($I = (i_0,i_1,\cdots,i_m) \in \mathbb{Z}_{\geq 0}^{m+1}$, $|I| = \sum_{k=0}^m i_k = d$). Next, following Mostovoy ([24]) we define the Veronese embedding, which will play a key role in our argument. The reason for using this embedding is that it defines a degenerate simplicial resolution, which gives rise to a spectral sequence with E_1 terms equal to zero outside a certain range (see Lemma 4.5 below.) Let $\psi_d^* : \mathbb{R}^m \to \mathbb{R}^{M_d}$ be the map given by $\psi_d^*(x_0,\cdots,x_{m-1}) = \Big(\varphi_I(x_0,\cdots,x_{m-1},1)\Big)_{\varphi_I \in \mathrm{H}_d}$, where $M_d := \binom{d+m}{m}$. We define the embedding $\Phi_d^* : Z_d^* \to \mathbb{R}^{N_d^* + M_d}$ by

$$\Phi_d^*((f_0,\cdots,f_n),\mathbf{x}) = (\phi_d^*(f_0,\cdots,f_n),\psi_d^*(\mathbf{x})).$$

Lemma 4.3. (i) *If $\{y_1,\cdots,y_r\} \in C_r(\mathbb{R}^m)$ is any set of r distinct points in $\mathbb{R}^m$ and $r \leq d+1$, then the r vectors $\{\psi_d^*(y_k) : 1 \leq k \leq r\}$ are linearly independent over $\mathbb{R}$ and span an $(r-1)$-dimensional simplex in $\mathbb{R}^{M_d}$.*

(ii) *If $1 \leq r \leq d+1$, there is a homeomorphism $\mathcal{Z}^\Delta(d)_r \setminus \mathcal{Z}^\Delta(d)_{r-1} \cong \tilde{\mathcal{Z}}^\Delta(d)_r \setminus \tilde{\mathcal{Z}}^\Delta(d)_{r-1}$.*

Proof. Writing $y_k = (y_{0,k},\cdots,y_{m-1,k})$ for $1 \leq k \leq r$, for each $i \neq j$ we can find a number l ($0 \leq l \leq m-1$) such that $y_{l,i} \neq y_{l,j}$. By a linear change of coordinates we can ensure that $l = 0$ for all i,j. The assertion (i) follows form the fact that the Vandermonde matrix constructed from the powers $y_{0,i}$ is non-singular, and (ii) easily follows from (i). $\square$

Definition. Let $(\mathcal{Z}^\Delta(d),\ {}'\pi_d{}^\Delta : \mathcal{Z}^\Delta(d) \to \Sigma_d^*)$ and $(\tilde{\mathcal{Z}^\Delta}(d),\ {}'\tilde{\pi}_d^\Delta : \mathcal{Z}^\Delta(d) \to \Sigma_d^*)$ denote the simplicial resolution of (π'_d,Φ_d^*) and the corresponding non-degenerate simplicial resolution with the natural increasing filtrations

$$\begin{cases} \mathcal{Z}^\Delta(d)_0 = \emptyset \subset \mathcal{Z}^\Delta(d)_1 \subset \mathcal{Z}^\Delta(d)_2 \subset \cdots \subset \mathcal{Z}^\Delta(d) = \bigcup_{k=0}^\infty \mathcal{Z}^\Delta(d)_k, \\ \tilde{\mathcal{Z}}^\Delta(d)_0 = \emptyset \subset \tilde{\mathcal{Z}}^\Delta(d)_1 \subset \tilde{\mathcal{Z}}^\Delta(d)_2 \subset \cdots \subset \tilde{\mathcal{Z}}^\Delta(d) = \bigcup_{k=0}^\infty \tilde{\mathcal{Z}}^\Delta(d)_k. \end{cases}$$

By Lemma 4.2 the map $'\pi_d^{\Delta}:\mathcal{Z}^{\Delta}(d)\xrightarrow{\simeq}\Sigma_d^*$ is a homotopy equivalence, and it extends to a homotopy equivalence $'\pi_{d+}^{\Delta}:\mathcal{Z}^{\Delta}(d)_+\xrightarrow{\simeq}\Sigma_{d+}^*$, where X_+ denotes the one-point compactification of a locally compact space X.

Since $\mathcal{Z}^{\Delta}(d)_{r+}/\mathcal{Z}^{\Delta}(d)_{r-1+}\cong(\mathcal{Z}^{\Delta}(d)_r\setminus\mathcal{Z}^{\Delta}(d)_{r-1})_+$, we have the spectral sequence $\left\{E_t^{r,s}(d),d_t:E_t^{r,s}(d)\to E_t^{r+t,s+1-t}(d)\right\}\Rightarrow H_c^{r+s}(\Sigma_d^*,\mathbb{Z})$, where $H_c^k(X,\mathbb{Z})$ denotes the cohomology group with compact supports given by $H_c^k(X,\mathbb{Z}):=H^k(X_+,\mathbb{Z})$ and $E_1^{r,s}(d):=H_c^{r+s}(\mathcal{Z}^{\Delta}(d)_r\setminus\mathcal{Z}^{\Delta}(d)_{r-1},\mathbb{Z})$.

It follows from the Alexander duality that there is a natural isomorphism

$$H_k(A_d(m,n;g),\mathbb{Z})\cong H_c^{N_d^*-k-1}(\Sigma_d^*,\mathbb{Z})\quad\text{for }1\le k\le N_d^*-2. \tag{4}$$

Using (4) and reindexing we obtain a spectral sequence

$$\left\{\tilde{E}^t_{r,s}(d),\tilde{d}^t:\tilde{E}^t_{r,s}(d)\to\tilde{E}^t_{r+t,s+t-1}(d)\right\}\Rightarrow H_{s-r}(A_d(m,n;g),\mathbb{Z}) \tag{5}$$

if $s-r\le N_d^*-2$, where $\tilde{E}^1_{r,s}(d)=H_c^{N_d^*+r-s-1}(\mathcal{Z}^{\Delta}(d)_r\setminus\mathcal{Z}^{\Delta}(d)_{r-1},\mathbb{Z})$.

Lemma 4.4. *If $1\le r\le\lfloor\frac{d+1}{2}\rfloor$, $\mathcal{Z}^{\Delta}(d)_r\setminus\mathcal{Z}^{\Delta}(d)_{r-1}$ is homeomorphic to the total space of a real vector bundle $\xi_{d,r}$ over $C_r(\mathbb{R}^m)$ with rank $l_{d,r}^*:=N_d^*-nr-1$.*

Proof. For $\mathbf{y}=\{y_1,\cdots,y_r\}\in C_r(\mathbb{R}^m)$, let $\sigma_\psi(\mathbf{y})$ denote the convex hull spanned by $\{\psi_d^*(y_1),\cdots,\psi_d^*(y_r)\}$, and let $\mathrm{int}(\sigma_\psi(\mathbf{y}))$ be its interior. By Lemma 4.3 , if $1\le r\le\lfloor\frac{d+1}{2}\rfloor$ and σ_1 and σ_2 are any two $(r-1)$-simplices in $\mathbb{R}^{M_d}$ with $\sigma_1\neq\sigma_2$ whose vertices are in the image of ψ_d^*, then either $\sigma_1\cap\sigma_2=\emptyset$ or $\sigma_1\cap\sigma_2$ is their common face of lower dimension. Therefore, if $1\le r\le\lfloor\frac{d+1}{2}\rfloor$ and $y\in\mathbb{R}^{M_d}$ is contained in the interior of some $(r-1)$-simplex σ whose vertices are in the image of ψ_d^*, all vertices of this σ are uniquely determined up to order. Hence, if $1\le r\le\lfloor\frac{d+1}{2}\rfloor$, one can define a map $\pi:\mathcal{Z}^{\Delta}(d)_r\setminus\mathcal{Z}^{\Delta}(d)_{r-1}\to C_r(\mathbb{R}^m)$ by

$$((f_0,\cdots,f_n),y)\mapsto\left\{\{y_1,\cdots,y_r\}\;\middle|\;\begin{array}{l}(y_k,1)\in\mathbb{R}^{m+1}\text{ is a common root}\\ \text{of }(f_0,\cdots,f_n)\text{ for }1\le k\le r,\\ y\in\mathrm{int}(\sigma_\psi(\{y_1,\cdots,y_r\})),\\ y_i\neq y_j\text{ if }i\neq j\end{array}\right\}.$$

In general, the condition that a polynomial in $B_k=\{g_k+z_mh:h\in\mathcal{H}_{d-1}\}$ vanishes at a given non-zero point gives one linear condition on its coefficients, and determines an affine hyperplane in B_k. By Lemma 4.3, if $r\le d+1$, then the condition that a polynomial in B_k vanishes at r distinct points $\mathbf{x}=\{x_k\}_{k=1}^r$ produces exactly r independent conditions on its coefficients if and only if the corresponding convex hull $\sigma_\psi(\mathbf{x})$ is an $(r-1)$-simplex. Hence, if $1 \leq r \leq d$, the space of polynomials in B_k which vanish at r distinct points is the intersection of r affine hyperplanes in general position and thus has codimension r in B_k.

Thus, the fiber of π is homeomorphic to the product of open $(r-1)$-simplex with the real vector space of dimension $(n+1)\big(\binom{d+m-1}{m} - r\big) = N_d^* - (n+1)r$. One can also easily see that the map π is the projection map of a locally trivial fiber bundle. Hence, π is a real vector bundle over $C_r(\mathbb{R}^m)$ with the rank $l_{d,r}^*$, where $l_{d,r}^* := N_d^* - (n+1)r + r - 1 = N_d^* - nr - 1$. $\square$

Lemma 4.5. (i) *If $1 \leq r \leq d+1$, all non-zero entries of $\tilde{E}^1_{r,s}(d)$ are situated in the range $s \geq r\big(n+1-m\big)$.*

(ii) *If $d+1 < r \leq d^m$, all non-zero entries of $\tilde{E}^1_{r,s}(d)$ are situated in the range $s \geq r(n+1-m) - n - 1$.*

(iii) *If $r > d^m$, $\tilde{E}^1_{r,*}(d) = 0$.*

Proof. (i) If $r \leq d+1$, it follows from Lemma 4.3 that, for any r distinct points $\{y_1, \cdots, y_r\} \in C_r(\mathbb{R}^m)$, the vectors $\{\psi_d^*(y_1), \cdots, \psi_d^*(y_r)\}$ are linearly independent. Hence, we obtain the inequality

$$\begin{aligned} \dim\big(\mathcal{Z}^\Delta(d)_r \setminus \mathcal{Z}^\Delta(d)_{r-1}\big) &\leq \big(N_d^* - r(n+1)\big) + rm + (r-1) \\ &= N_d^* - r(n-m) - 1. \end{aligned}$$

Since $N_d^* - r(n-m) - 1 \geq N_d^* + r - s - 1$ if and only if $s \geq r(n+1-m)$, the result follows.

(ii), (iii): The assertion easily follows from Lemma 4.1 and it remains to show (ii). If $d+1 < r \leq d^m$, then, even if the r vectors $\{\psi_d^*(y_1), \cdots, \psi_d^*(y_r)\}$ span an $(r-1)$-dimensional simplex, some $(r-1)$ of them are linearly independent and we have the inequality

$$\begin{aligned} \dim\big(\mathcal{Z}^\Delta(d)_r \setminus \mathcal{Z}^\Delta(d)_{r-1}\big) &\leq \big(N_d^* - (r-1)(n+1)\big) + rm + (r-1) \\ &= N_d^* - r(n-m) + n. \end{aligned}$$

Thus, since $N_d^* - r(n-m) + n \geq N_d^* + r - s - 1$ if and only if $s \geq r(n+1-m) - n - 1$, (ii) follows. $\square$

Lemma 4.6. *If $1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor$, there is a natural isomorphism*

$$\tilde{E}^1_{r,s}(d) \cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}).$$

Proof. Suppose that $1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor$. Then by Lemma 4.4, there is a homeomorphism $(\mathcal{Z}^\Delta(d)_r \setminus \mathcal{Z}^\Delta(d)_{r-1})_+ \cong T(\xi_{d,r})$, where $T(\xi)$ denotes the Thom complex of a vector bundle ξ. Since $N_d^* + r - s - 1 - l_{d,r}^* = (n+1)r - s$

and $rm-\{(n+1)r-s\}=s-(n-m+1)r$, by using the Thom isomorphism and Poincaré duality, we obtain a natural isomorphism

$$\tilde{E}^1_{r,s}(d)\cong H^{N^*_d+r-s-1}(T(\xi_{d,r}),\mathbb{Z})\cong H_{s-(n-m+1)r}(C_r(\mathbb{R}^m),(\pm\mathbb{Z})^{\otimes(n-m)}).\quad\square$$

Next we recall the spectral sequence constructed by V. Vassiliev [[26], page 109–115]. From now on, we will assume that $m<n$ and X is a finite m dimensional simplicial complex C^∞-imbedded in $\mathbb{R}^L$. Considering S^n and X as subspaces $S^n\subset\mathbb{R}^{n+1}, X\subset\mathbb{R}^L$, we identify $\mathrm{Map}(X,S^n)$ with the space $\mathrm{Map}(X,\mathbb{R}^{n+1}\setminus\{\mathbf{0}_{n+1}\})$. We also choose a map $\varphi: X\to\mathbb{R}^{n+1}\setminus\{\mathbf{0}_{n+1}\}$ and fix it. Observe that $\mathrm{Map}(X,\mathbb{R}^{n+1})$ is a linear space and consider the complements $\mathfrak{A}^n_m(X)=\mathrm{Map}(X,\mathbb{R}^{n+1})\setminus\mathrm{Map}(X,S^n)$ and $\tilde{\mathfrak{A}}^n_m(X)=\mathrm{Map}^*(X,\mathbb{R}^{n+1})\setminus\mathrm{Map}^*(X,S^n)$.

Note that $\mathfrak{A}^n_m(X)$ consists of all continuous maps $f:X\to\mathbb{R}^{n+1}$ passing through $\mathbf{0}_{n+1}$. We will denote by $\Theta^k_\varphi(X)\subset\mathrm{Map}(X,\mathbb{R}^{n+1})$ the subspace consisting of all maps f of the forms $f=\varphi+p$, where p is the restriction to X of a polynomial map $\mathbb{R}^L\to\mathbb{R}^{n+1}$. Let $\Theta^k_X\subset\Theta^k_\varphi(X)$ denote the subspace consisting of all $f\in\Theta^k_\varphi(X)$ passing through $\mathbf{0}_{n+1}$. In [[26], page 111-112] Vassiliev uses the space $\Theta^k(X)$ as a finite dimensional approximation of $\mathfrak{A}^n_m(X)$.[†]

Let $\tilde{\Theta}^k_X$ denote the subspace of Θ^k_X consisting of all maps $f\in\Theta^k_X$ which preserve the base points. By a variation of the preceding argument, Vassiliev also shows that $\tilde{\Theta}^k_X$ can be used as a finite dimensional approximation of $\tilde{\mathfrak{A}}^n_m(X)$ [[26], page 112].

Let $\mathcal{X}_k\subset\tilde{\Theta}^k_X\times\mathbb{R}^L$ denote the subspace consisting of all pairs $(f,\alpha)\in\tilde{\Theta}^k_X\times\mathbb{R}^L$ such that $f(\alpha)=\mathbf{0}_{n+1}$, and let $p_k:\mathcal{X}_k\to\tilde{\Theta}^k_X$ be the projection onto the first factor. Then, by making use of (non-degenerate) simplicial resolutions of the surjective maps $\{p_k:k\geq 1\}$, one can construct a geometric resolution $\{\tilde{\mathfrak{A}}^n_m(X)\}$ of $\tilde{\mathfrak{A}}^n_m$, whose cohomology is naturally isomorphic to the homology of $\mathrm{Map}^*(X,S^n)$. From the natural filtration $F_1\subset F_2\subset F_3\subset\cdots\subset\bigcup_{k=1}^\infty F_k=\{\tilde{\mathfrak{A}}^n_m(X)\}$, we obtain the associated spectral sequence:

$$\{E^t_{r,s},d^t:E^t_{r,s}\to E^t_{r+t,s+t-1}\}\Rightarrow H_{s-r}(\mathrm{Map}^*(X,S^n),\mathbb{Z}).\tag{6}$$

The following result follows easily from [[26], Theorem 2 (page 112) and (32) (page 114)].

Lemma 4.7 ([26]). *Let $2\leq m<n$ be integers and let X be a finite m-dimensional simplicial complex with a fixed base point $x_0\in X$.*

[†]Note that the proof of this fact given by Vassiliev, makes use of the Stone-Weierstrass theorem, so, although we are now not using the stable result of Section 2, something like it is also implicitly involved here.

(i) *If* $r \geq 1$, $E^1_{r,s} = H_{s-(n-m+1)r}(C_r(X \setminus \{x_0\}), (\pm\mathbb{Z})^{\otimes(n-m)})$.

(ii) *If* $r < 0$ *or* $s < 0$ *or* $s < (n-m+1)r$, *then* $E^1_{r,s} = 0$.

(iii) *If* $X = S^m$, *then, for any* $t \geq 1$, $d^t = 0 : E^t_{r,s} \to E^t_{r+t,s+t-1}$ *for all* (r,s), *and* $E^1_{r,s} = E^\infty_{r,s}$. *Moreover, for any* $k \geq 1$, *the extension problem for* $Gr(H_k(\Omega^m S^n, \mathbb{Z})) = \bigoplus_{r=1}^\infty E^\infty_{r,r+k}$ *is trivial and there is an isomorphism*

$$H_k(\Omega^m S^n, \mathbb{Z}) \cong \bigoplus_{r=1}^\infty E^1_{r,k+r} = \bigoplus_{r=1}^\infty H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}). \quad \square$$

Although we are really interested in the inclusion map i'_d of Theorem 1.6, we will first define another map j'_d, for which the analogous result is easier to prove. We will then deduce the result for i'_d from the one for j'_d.

Definition. Let $j'_d : A_d(m,n;g) \to \Omega^m(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}) \simeq \Omega^m S^n$ by given by

$$j'_d(\mathbf{f})(\mathbf{x}) = \big(f_0(x_0, \cdots, x_m), \cdots\cdots, f_n(x_0, \cdots, x_m)\big)$$

for $(\mathbf{f}, \mathbf{x}) = ((f_0, \cdots, f_n), (x_0, \cdots, x_m)) \in A_d(m,n;g) \times S^m$.

We will prove the analogue of Theorem 1.6 for the map j'_d by applying the spectral sequence (6) to the case $X = S^m$.

Theorem 4.8. *Let* $2 \leq m < n$ *be integers and* $g \in \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ *be a fixed map of minimal degree* d.

(i) *The map* $j'_d : A_d(m,n;g) \to \Omega^m S^n$ *induces a split epimorphism on* $H_k(\ ,\mathbb{Z})$ *for any* $1 \leq k \leq D(d;m,n)$.

(ii) *For any* $k \geq 1$, $H_k(A_d(m,n;g),\mathbb{Z})$ *contains the subgroup*

$$G^d_{m,n} = \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-(n-m)r}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)})$$

as a direct summand.

(iii) *If* $d \geq M(m,n)$, *the map* $j'_d : A_d(m,n;g) \to \Omega^m S^n$ *is a homotopy equivalence through dimension* $D(d;m,n)$ *if* $m+2 \leq n$ *and a homology equivalence through dimension* $D(d;m,n)$ *if* $m+1 = n$.

Proof. From now on, we identify $\Omega^m S^n = \Omega^m(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})$ and consider the spectral sequence (6) for $X = S^m$. Note that the image of the map j'_d lies in a space of mappings that arise from restrictions of polynomial mappings

$\mathbb{R}^m \to \mathbb{R}^n$. Since $\tilde{\mathcal{Z}}^\Delta(d)$ is a non-degerate simplicial resolution, the map j'_d naturally extends to a filtration preserving map ${}'\tilde{\pi} : \tilde{\mathcal{Z}}^\Delta(d) \to \{\tilde{\mathfrak{A}}^n_m(S^n)\}$ between resolutions. By Lemma 4.2 there is a filtration preserving homotopy equivalence $q^\Delta : \tilde{\mathcal{Z}}^\Delta(d) \xrightarrow{\simeq} \mathcal{Z}^\Delta(d)$. Then the filtration preserving maps $\mathcal{Z}^\Delta(d) \xleftarrow[\simeq]{q^\Delta} \tilde{\mathcal{Z}}^\Delta(d) \xrightarrow{{}'\tilde{\pi}} \{\tilde{\mathfrak{A}}^n_m(S^n)\}$ induce a homomorphism of spectral sequences $\{\tilde{\theta}^t_{r,s} : \tilde{E}^t_{r,s}(d) \to E^t_{r,s}\}$, where $\{\tilde{E}^t_{r,s}(d), \tilde{d}^t\} \Rightarrow H_{s-r}(A_d(m,n;g), \mathbb{Z})$ and $\{E^t_{r,s}, d^t\} \Rightarrow H_{s-r}(\Omega^m S^n, \mathbb{Z})$.

Note now that, by Lemma 4.3, Lemma 4.6, Lemma 4.7 and the naturality of Thom isomorphism, for $r \leq \lfloor \frac{d+1}{2} \rfloor$ there is a commutative diagram

$$\begin{array}{ccc} \tilde{E}^1_{r,s}(d) & \xrightarrow[\cong]{T} & H_{s-r(n-m+1)}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}) \\ {\scriptstyle \tilde{\theta}^1_{r,s}}\downarrow & & \| \\ E^1_{r,s} & \xrightarrow[\cong]{T} & H_{s-r(n-m+1)}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}) \end{array} \tag{7}$$

Hence, if $r \leq \lfloor \frac{d+1}{2} \rfloor$, $\tilde{\theta}^1_{r,s} : \tilde{E}^1_{r,s}(d) \xrightarrow{\cong} E^1_{r,s}$ and thus so is $\tilde{\theta}^\infty_{r,s} : \tilde{E}^\infty_{r,s}(d) \xrightarrow{\cong} E^\infty_{r,s}$.

Next, consider the number

$$D_{min} = \min\{N |\ N \geq s-r,\ s \geq (n+1-m)r,\ 1 \leq r < \lfloor \frac{d+1}{2} \rfloor + 1\}.$$

Clearly D_{min} is the largest integer N which satisfies the inequality $(n+1-m)r > r+N$ for $r = \lfloor \frac{d+1}{2} \rfloor + 1$, hence

$$D_{min} = (n-m)(\lfloor \frac{d+1}{2} \rfloor + 1) - 1 = D(d;m,n). \tag{8}$$

We note that, for dimensional reasons, $\tilde{\theta}^\infty_{r,s} : \tilde{E}^\infty_{r,s}(d) \xrightarrow{\cong} E^\infty_{r,s}$ is always an isomorphism when $r \leq \lfloor \frac{d+1}{2} \rfloor$ and $s - r \leq D(d;m,n)$.

Moreover, from Lemma 4.5 it easily follows that $\tilde{E}^1_{r,s}(d) = E^1_{r,s} = 0$ when $s - r \leq D(d;m,n)$ and $\lfloor \frac{d+1}{2} \rfloor < r \leq d+1$. Hence, we have:

(8.1) If $s \leq r + D(d;m,n)$ and $r \leq d+1$, then $\tilde{\theta}^\infty_{r,s} : \tilde{E}^\infty_{r,s}(d) \xrightarrow{\cong} E^\infty_{r,s}$ is an isomorphism.

On the other hand, with the help of Lemma 4.5, we can show that $E^1_{r,s} = 0$ if $s \leq r + D(d;m,n)$ and $r > d+1$. Hence, by (8.1) we have:

(8.2) The map j'_d induces an epimorphism on $H_k(\ ,\mathbb{Z})$ for any $1 \leq k \leq D(d;m,n)$.

Next, since $d^t = 0$ for any $t \geq 1$, from the equality $d^t \circ \tilde{\theta}^t_{r,s} = \tilde{\theta}^t_{r+t,s+t-1} \circ \tilde{d}^t$ and some diagram chasing, we obtain $\tilde{E}^1_{r,s}(d) = \tilde{E}^\infty_{r,s}(d)$ for all $r \leq \lfloor \frac{d+1}{2} \rfloor$. Moreover, since the extension problem for $Gr(H_k(\Omega^m S^n, \mathbb{Z})) = \bigoplus_{r=1}^\infty E^\infty_{r,k+r} = \bigoplus_{r=1}^\infty E^1_{r,k+r}$ is trivial, we deduce from(8.1) that the associated graded group $Gr(H_k(A_d(m,n;g),\mathbb{Z})) = \bigoplus_{r=1}^\infty \tilde{E}^\infty_{r,k+r}(d)$ is also trivial until the $\lfloor \frac{d+1}{2} \rfloor$-th term of the filtration. Hence, $H_k(A_d(m,n;g),\mathbb{Z})$ contains the subgroup

$$\bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} \tilde{E}^1_{r,k+r}(d) = \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} \tilde{E}^\infty_{r,k+r}(d) \cong \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-r(n-m)}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)})$$

as a direct summand, which proves the assertion (ii).

If $1 \leq k \leq D(d;m,n)$, then, by Lemma 4.7 and dimensional reasons, there is an isomorphism

$$H_k(\Omega^m S^n, \mathbb{Z}) \cong \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-r(n-m)}(C_r(\mathbb{R}^m), (\pm\mathbb{Z})^{\otimes(n-m)}) = G^d_{m,n}.$$

Hence, from (8.2) and (ii), we obtain that j'_d induces a split epimorphism on $H_k(\ ,\mathbb{Z})$ for any $1 \leq k \leq D(d;m,n)$. Thus we have proved (i).

Now it remains to prove the assertion (iii). From Lemma 4.5 we see that $\tilde{E}^1_{r,s}(d) = E^1_{r,s} = 0$ for $r > d+1$ if $s < g(r) := (n-m+1)r - n - 1$. Hence, setting $f(r) := r + D(d;m,n)$, we have

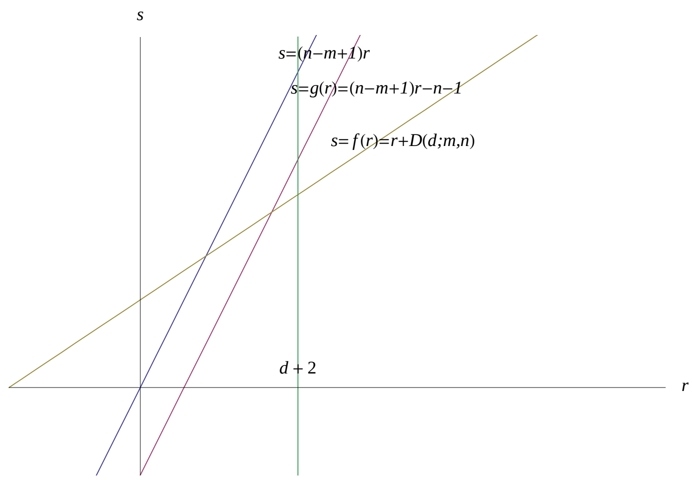
$$\begin{aligned} & g(d+2) - f(d+2) = (n-m)(d+1-\lfloor \frac{d+1}{2} \rfloor) - n \geq 1 \\ \Leftrightarrow \quad & d - \lfloor \frac{d+1}{2} \rfloor \geq \lceil \frac{m+1}{n-m} \rceil \Leftrightarrow \lfloor \frac{d}{2} \rfloor \geq \lceil \frac{m+1}{n-m} \rceil. \end{aligned}$$

Thus, if $d \geq M(m,n) = 2\lceil \frac{m+1}{n-m} \rceil + 1$, $g(d+2) \geq f(d+2) + 1$. Hence, by Lemma 4.5, we obtain

(8.3) If $d \geq M(m,n)$, $\tilde{E}^1_{r,s}(d) = \tilde{E}^\infty_{r,s}(d) = 0$ for any (r,s) as long as $s - r \leq D(d;m,n)$ and $r > d+1$.

Then by (8.1) and (8.3), we see that $\tilde{\theta}^\infty_{r,s} : \tilde{E}^\infty_{r,s}(d) \xrightarrow{\cong} E^\infty_{r,s}$ is an isomorphism for any (r,s) as long as $s - r \leq D(d;m,n)$ and $d \geq M(m,n)$ hold.

Hence, we have that j'_d is a homology equivalence through dimension $D(d;m,n)$ if $d \geq M(m,n)$. If $m+2 \leq n$ and $d \geq M(m,n)$, since $A_d(m,n;g)$ and $\Omega^m S^n$ are simply connected, j'_d is a homotopy equivalence through dimension $D(d;m,n)$. Thus, (iii) is also proved. $\square$



Corollary 4.9. *Let $2 \leq m < n$ be integers, $g \in \mathrm{Alg}^*_d(\mathbb{RP}^{m-1}, \mathbb{RP}^n)$ be a fixed map of minimal degree d, and let $\mathbb{F} = \mathbb{Z}/p$ (p: prime) or $\mathbb{F} = \mathbb{Q}$. Then the map $j'_d : A_d(m,n;g) \to \Omega^m S^n$ induces a split epimorphism on the homology group $H_k(\ ,\mathbb{F})$ for any $1 \leq k \leq D(d;m,n)$.*

Proof. If we apply the argument in the proof of Theorem 4.8 above, with the homology groups $H_k(\ ,\mathbb{Z})$ and $H_k(\ ,(\pm\mathbb{Z})^{\otimes k})$ replaced by $H_k(\ ,\mathbb{F})$ and $H_k(\ ,(\pm\mathbb{F})^{\otimes k})$, the assertion follows. $\square$

Let $\gamma_k : S^k \to \mathbb{RP}^k$ denote the usual double covering map and let the map $\gamma^\#_m : \mathrm{Map}^*_\epsilon(\mathbb{RP}^m, \mathbb{RP}^n) \to \Omega^m\mathbb{RP}^n$ be given by $\gamma^\#_m(h) = h \circ \gamma_m$. It s easy to see that the diagram

$$\begin{tikzcd}
A_d(m,n;g) \arrow[r, "\Psi'_d", "\cong"'] \arrow[d, "j'_d"'] & \mathrm{Alg}^*_d(m,n;g) \arrow[r, "i'_d", "\subset"'] & F_d(m,n;g) \arrow[d, "i'"', "\cap"] \\
\Omega^m S^n \arrow[r, "\Omega^m\gamma_n", "\simeq"'] & \Omega^m\mathbb{RP}^n & \mathrm{Map}^*_{[d]_2}(\mathbb{RP}^m, \mathbb{RP}^n) \arrow[l, "\gamma^\#_m"']
\end{tikzcd} \tag{9}$$

is commutative, where $i' : F_d(m,n;g) \xrightarrow{\subset} \mathrm{Map}^*_{[d]_2}(\mathbb{RP}^m, \mathbb{RP}^n)$ and $\Psi'_d = \Psi_d | A_d(m,n;g)$ denote the inclusion and restriction, respectively.

Lemma 4.10. *If $2 \leq m < n$ and $g \in \mathrm{Alg}^*_d(\mathbb{RP}^{m-1}, \mathbb{RP}^n)$ is a fixed map of minimal degree d, the map $\gamma^\#_m \circ i' : F_d(m,n;g) \to \Omega^m\mathbb{RP}^n$ is a homotopy equivalence through dimension $D(d;m,n)$.*

Proof. Since there is a homotopy equivalence $F_d(m,n;g) \simeq \Omega^m\mathbb{RP}^n$, the spaces $F_d(m,n;g)$ and $\Omega^m\mathbb{RP}^n$ are simple. Hence it suffices to show that the map $\gamma^\#_m \circ i'$ is a homology equivalence through dimension $D(d;m,n)$.

Let $\mathbb{F} = \mathbb{Z}/p$ (p: prime) or $\mathbb{F} = \mathbb{Q}$, and consider the induced homomorphism $(\gamma_m^{\#} \circ i')_* = H_k(\gamma_m^{\#} \circ i', \mathbb{F}) : H_k(F_d(m,n;g), \mathbb{F}) \to H_k(\Omega^m \mathbb{R}\mathrm{P}^n, \mathbb{F})$. Since $\Omega^m \gamma_n$ is a homotopy equivalence, by Corollary 4.9 and the diagram (9) $(\gamma_m^{\#} \circ i')_*$ is an epimorphism for any $1 \leq k \leq D(d;m,n)$. However, since there is a homotopy equivalence $F_d(m,n;g) \simeq \Omega^m \mathbb{R}\mathrm{P}^n$, $\dim_{\mathbb{F}} H_k(F_d(m,n;g), \mathbb{F}) = \dim_{\mathbb{F}} H_k(\Omega^m \mathbb{R}\mathrm{P}^n, \mathbb{F}) < \infty$ for any k. Hence, $H_k(\gamma_m^{\#} \circ i', \mathbb{F})$ is an isomorphism for any $1 \leq k \leq D(d;m,n)$. Then by using the universal coefficient Theorem, we see that $\gamma_m^{\#} \circ i'$ induces an isomorphism on $H_k(\ ,\mathbb{Z})$ for any $1 \leq k \leq D(d;m,n)$, too. $\square$

Proof of Theorem 1.6. (ii) Since $\Psi_d' : A_d(m,n;g) \xrightarrow{\cong} \mathrm{Alg}_d^*(m,n;g)$ is a homeomorphism, the assertion (ii) follows from (ii) of Theorem 4.8.

(i), (iii): Since the map $\gamma_m^{\#} \circ i'$ a homotopy equivalence through dimension $D(d;m,n)$, the assertions (i) and (iii) follow from (9) and Theorem 4.16. $\square$

4.3 The space $A_d(m,n)$.

In this section consider the unstable problem for the space $A_d(m,n)$, where $d = 2d^*$ is even. For this purpose, throughout in section 4.3, we always assume that $d = 2d^*$ is an even positive integer.

Let $f_0, \cdots, f_n \in \mathbb{R}[z_0, \cdots, z_m]$ be homogenous polynomials of the same degree d. If $\mathbf{x} = (x_0, \cdots, x_m) \neq \mathbf{0}_{m+1} \in \mathbb{R}^{m+1}$ is their non-trivial common root, $\alpha\mathbf{x} = (\alpha x_0, \cdots, \alpha x_m)$ is also their common real root for any $\alpha \neq 0 \in \mathbb{R}$. We will refer to the the element $\mathbf{y} = [\mathbf{x}] \in \mathbb{R}\mathrm{P}^m$ as *a common real root* of $f_0, \cdots, f_n$.

Definition. Let $Z_d \subset \Sigma_d \times \mathbb{R}\mathrm{P}^m$ denote *the tautological normalization* of Σ_d consisting of all pairs $(\mathbf{f}, \mathbf{y}) = ((f_0, \cdots, f_n), [x_0 : \cdots : x_m]) \in \Sigma_d \times \mathbb{R}\mathrm{P}^m$ such that $\mathbf{y}$ is a common real root of $f_n, \cdots, f_n$. Projection onto the first factor gives a surjective map $\pi_d : Z_d \to \Sigma_d$. Let $\phi_d : A_d \xrightarrow{\cong} \mathbb{R}^{N_d}$ be any fixed homeomorphism. We now define an embedding (resembling the Veronese embedding) as follows. Let the map $\psi_d : \mathbb{R}\mathrm{P}^m \to \mathbb{R}^{M_d}$ be given by

$$\psi_d([x_0 : \cdots : x_m]) = \Big[\frac{\varphi_I(x_0, \cdots, x_m)}{\sum_{k=0}^m x_k^d}\Big]_{\varphi_I \in \mathrm{H}_d}$$

where $M_d = \binom{d+m}{m}$. Define the embedding $\Phi_d : Z_d \to \mathbb{R}^{N_d + M_d}$ by

$$\Phi_d((f_0, \cdots, f_n), \mathbf{x}) = (\phi_d(f_0, \cdots, f_n), \psi_d(\mathbf{x})).$$

Lemma 4.11. (i) *Let $\{y_1, \cdots, y_r\} \in C_r(\mathbb{R}\mathrm{P}^m)$ be any r distinct points in $\mathbb{R}\mathrm{P}^m$ where $r \leq d+1$. Then the vectors $\{\psi_d(y_k) : 1 \leq k \leq r\}$ are linearly independent over $\mathbb{R}$ and span an $(r-1)$-dimensional simplex in $\mathbb{R}^{M_d}$.*

(ii) *If* $1 \leq r \leq d+1$, *there is a homeomorphism* $\mathcal{W}^{\Delta}(d)_r \setminus \mathcal{W}^{\Delta}(d)_{r-1} \cong \tilde{\mathcal{W}}^{\Delta}(d)_r \setminus \tilde{\mathcal{W}}^{\Delta}(d)_{r-1}$.

Proof. Since the assertion (ii) follows from (i), it remains to show (i). By using a suitable linear transformation and by homogeneity we can assume that the last coordinate of each y_i is 1. The result follows from the proof of Lemma 4.3. $\square$

Since $A_d \cong \mathbb{R}^{N_d}$ and $\Sigma_d = A_d \setminus A_d(m,n)$, we obtain, by Alexander duality, a natural isomorphism

$$H_k(A_d(m,n),\mathbb{Z}) \xrightarrow{\cong} H_c^{N_d-k-1}(\Sigma_d,\mathbb{Z}) \quad \text{for } 1 \leq k \leq N_d-2. \tag{10}$$

Definition. Let $(\mathcal{W}^{\Delta}(d), \pi_d^{\Delta} : \mathcal{W}^{\Delta}(d) \to \Sigma_d)$ and $(\tilde{\mathcal{W}}^{\Delta}(d), \tilde{\pi}_d^{\Delta} : \mathcal{W}^{\Delta}(d) \to \Sigma_d)$ denote the simplicial resolution of (π_d, Φ_d) and its associated non-degerate simplicial resolution with increasing filtrations

$$\begin{cases} \mathcal{W}^{\Delta}(d)_0 = \emptyset \subset \mathcal{W}^{\Delta}(d)_1 \subset \mathcal{W}^{\Delta}(d)_2 \subset \cdots \subset \mathcal{W}^{\Delta}(d) = \bigcup_{k=0}^{\infty} \mathcal{W}^{\Delta}(d)_k, \\ \tilde{\mathcal{W}}^{\Delta}(d)_0 = \emptyset \subset \tilde{\mathcal{W}}^{\Delta}(d)_1 \subset \tilde{\mathcal{W}}^{\Delta}(d)_2 \subset \cdots \subset \tilde{\mathcal{W}}^{\Delta}(d) = \bigcup_{k=0}^{\infty} \tilde{\mathcal{W}}^{\Delta}(d)_k. \end{cases}$$

Note that by Lemma 4.2 the map $\pi_d^{\Delta} : \mathcal{W}^{\Delta}(d) \xrightarrow{\simeq} \Sigma_d$ is a homotopy equivalence, and it extends to a homotopy equivalence $\tilde{\pi}_{d+}^{\Delta} : \mathcal{W}^{\Delta}(d)_+ \xrightarrow{\simeq} \Sigma_{d+}$.

Since $\mathcal{W}^{\Delta}(d)_{r+}/\mathcal{W}^{\Delta}(d)_{r-1+} \cong (\mathcal{W}^{\Delta}(d)_r \setminus \mathcal{W}^{\Delta}(d)_{r-1})_+$, by applying Alexander duality (10) and reindexing we obtain a spectral sequence

$$\left\{E_{r,s}^t(d), d^t : E_{r,s}^t(d) \to E_{r+t,s+t-1}^t(d)\right\} \Rightarrow H_{s-r}(A_d(m,n),\mathbb{Z}) \tag{11}$$

if $s-r \leq N_d-2$, where $E_{r,s}^1(d) = H_c^{N_d+r-s-1}(\mathcal{W}^{\Delta}(d)_r \setminus \mathcal{W}^{\Delta}(d)_{r-1},\mathbb{Z})$.

Lemma 4.12. *If* $1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor$, $\mathcal{W}^{\Delta}(d)_r \setminus \mathcal{W}^{\Delta}(d)_{r-1}$ *is homeomorphic to the total space of a real vector bundle* $\xi_{d,r}$ *over* $C_r(\Gamma_m)$ *with rank* $l_{d,r} := N_d - nr - 1$, *where we set* $\Gamma_m = \mathbb{R}\mathrm{P}^m \setminus \{\mathbf{e}_m\}$.

Proof. The proof is completely analogous to that of Lemma 4.4. $\square$

Corollary 4.13. *If* $1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor$, *then there are a natural isomorphism*

$$E_{r,s}^1(d) \cong H_{s-(n-m+1)r}(C_r(\Gamma_m), (\pm\mathbb{Z})^{\otimes(n-m)}).$$

Proof. If $1 \leq r \leq \lfloor \frac{d+1}{2} \rfloor$, by Lemma 4.12, there is a homeomorphism $(\mathcal{W}^{\Delta}(d)_r \setminus \mathcal{W}^{\Delta}(d)_{r-1})_+ \cong T(\xi_{d,r})$. Then the above isomorphism follows from the Thom isomorphism theorem and Poincaré duality. $\square$

Lemma 4.14. (i) *If $1 \leq r \leq d+1$, then all non-zero entries of $E^1_{r,s}(d)$ are situated in the range $s \geq r\big(n+1-m\big)$.*

(ii) *If $r > d+1$, then all non-zero entries of $E^1_{r,s}(d)$ are situated in the range $s \geq r(n+1-m)-n-1$.*

Proof. The proof is completely analogous to that of Lemma 4.5. $\square$

Definition. We take $\tilde{\mathbf{e}}_n = (1,0,\cdots,0) \in \mathbb{R}^{n+1}$ and let $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n)$ be the space consisting of all based maps $f : (\mathbb{R}\mathrm{P}^m, \mathbf{e}_m) \to (S^n, \tilde{\mathbf{e}}_n)$.

For an even integer $d = 2d^*$, let $j_d : A_d(m,n) \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}^{n+1}\setminus\{\mathbf{0}\}) \simeq \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n)$ be the map defined by

$$j_d(f_0,\cdots,f_n)([x_0:\cdots:x_m]) = \Big(\frac{f_0(x_0,\cdots,x_m)}{(\sum_{k=0}^m x_k^2)^{d^*}},\cdots,\frac{f_n(x_0,\cdots,x_m)}{(\sum_{k=0}^m x_k^2)^{d^*}}\Big)$$

for $([x_0:\cdots:x_m],(f_0,\cdots,f_n)) \in \mathbb{R}\mathrm{P}^m \times A_d(m,n)$. Note that the map j_d is well defined only if d is even.

Lemma 4.15. *If $m+2 \leq n$, $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n)$ is simply connected.*

Proof. The assertion follows by considering the restriction fibration

$$\Omega^m S^n \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, S^n).$$

and applying induction on m. $\square$

Theorem 4.16. *Let $2 \leq m < n$ and $d = 2d^* \equiv 0 \pmod 2$ be positive integers.*

(i) *The map $j_d : A_d(m,n) \to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, S^n)$ induces an epimorphism on $H_k(\ ,\mathbb{Z})$ for any $1 \leq k \leq D(d;m,n)$.*

(ii) *If $d = 2d^* \geq M(m,n)$, the map $j_d : A_d(m,n) \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n)$ is a homotopy equivalence through dimension $D(d;m,n)$ for $m+2 \leq n$ and a homology equivalence through dimension $D(d;m,n)$ for $m+1 = n$.*

Proof. Since the proof is analogous to that of Theorem 4.8, we only sketch it. We apply the spectral sequence (6) to the case $X = \mathbb{R}\mathrm{P}^m$. Consider the two spectral sequences

$$\{E^t_{r,s}(d), d^t\} \Rightarrow H_*(A_d(m,n),\mathbb{Z}), \quad \{E^t_{r,s}, d^t\} \Rightarrow H_*(\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, S^n),\mathbb{Z}).$$

By using Lemma 4.2 there is a filtration preserving homotopy equivalence $q^{\Delta}_1 : \tilde{\mathcal{W}}^{\Delta}(d) \xrightarrow{\simeq} \mathcal{W}^{\Delta}(d)$. Since the map $j_d(f_0,\ldots,f_n)$ is a regular map, the image of j_d lies in the space of polynomials mappings used in the construction of the Vassiliev spectral sequence $\{E^t_{r,s}, d^t\}$. So the map j_d induces a filtration preserving map $\hat{\pi} : \tilde{\mathcal{W}}^{\Delta}(d) \to \{\tilde{\mathfrak{A}}^n_m(\mathbb{R}\mathrm{P}^m)\}$ between resolutions. Then the filtration preserving maps $\mathcal{W}^{\Delta}(d) \xleftarrow[\simeq]{q_1^{\Delta}} \tilde{\mathcal{W}}^{\Delta}(d) \xrightarrow{\hat{\pi}} \{\tilde{\mathfrak{A}}^n_m(\mathbb{R}\mathrm{P}^m)\}$ induce a homomorphism of spectral sequences $\{\theta^t_{r,s} : E^t_{r,s}(d) \to E^t_{r,s}\}$.

By Lemma 4.7, Lemma 4.11, Corollary 4.13 and the naturality of the Thom isomorphism, there is a commutative diagram

$$\begin{array}{ccc}
E^1_{r,s}(d) & \xrightarrow[\cong]{T} & H_{s-r(n-m+1)}(C_r(\Gamma_m), (\pm\mathbb{Z})^{\otimes(n-m)}) \\
\theta^1_{r,s}\big\downarrow & & \| \\
E^1_{r,s} & \xrightarrow[\cong]{T} & H_{s-r(n-m+1)}(C_r(\Gamma_m), (\pm\mathbb{Z})^{\otimes(n-m)})
\end{array}$$

for $r \leq \lfloor \frac{d+1}{2} \rfloor$. Hence, by the Comparison Theorem for spectral sequences, we see that $\theta^{\infty}_{r,s} : E^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s}$ is an isomorphism if $r \leq \lfloor \frac{d+1}{2} \rfloor$. An observation analogous to the one used in (8.1) and (8.2) together with Lemma 4.14, shows that $\theta^{\infty}_{r,s} : E^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s}$ is an isomorphism for any (r, s) as long as $s - r \leq D(d; m, n)$ and $r \leq d + 1$. Hence, j_d induces an epimorphism on $H_k(\ , \mathbb{Z})$ for any $1 \leq k \leq D(d; m, n)$, and (i) is proved.

From now on, we assume that $d \geq M(m, n)$ is even. An argument similar to the one used in (8.3), shows that $\theta^{\infty}_{r,s} : E^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s}$ is an isomorphism for any (r, s) as long as $s - r \leq D(d; m, n)$. Hence j_d is a homology equivalence through dimension $D(d; m, n)$.

If $m + 2 \leq n$, $A_d(m, n)$ and $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, S^n)$ is simply connected, j_d is a homotopy equivalence through dimension $D(d; m, n)$. Hence, (ii) is also proved. $\square$

Consider the map $\gamma_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \to \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ given by $\gamma_{n\#}(h) = \gamma_n \circ h$. Since $\mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n)$ contains the subspace of constant maps, the image of $\gamma_{n\#}$ is contained in $\mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. Thus we obtain a map $\gamma_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \to \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$.

Lemma 4.17. $\gamma_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) \xrightarrow{\simeq} \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *is a homotopy equivalence.*

Proof. We prove this by induction on m. If $m = 1$, $\gamma_{n\#}$ can be identified with the map $\Omega\gamma_n : \Omega S^n \to \Omega_0\mathbb{R}\mathrm{P}^n$ and it is clearly a homotopy equivalence. Suppose that $\gamma'_{n\#} : \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, S^n) \xrightarrow{\simeq} \mathrm{Map}^*_0(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence for some $m \geq 2$ and consider the following commutative diagram of fibration sequences

$$
\begin{array}{ccccc}
\Omega^m S^n & \longrightarrow & \mathrm{Map}^*(\mathbb{R}\mathrm{P}^m, S^n) & \xrightarrow{res} & \mathrm{Map}^*(\mathbb{R}\mathrm{P}^{m-1}, S^n) \\
{\scriptstyle \Omega^m\gamma_n}\downarrow \simeq & & {\scriptstyle \gamma_{n\#}}\downarrow & & {\scriptstyle \gamma_n{}'_{\#}}\downarrow \simeq \\
\Omega^m \mathbb{R}\mathrm{P}^n & \longrightarrow & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) & \xrightarrow{res} & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^{m-1}, \mathbb{R}\mathrm{P}^n)
\end{array}
$$

Because $\Omega^m\gamma_n$ and $\gamma_n{}'_{\#}$ are homotopy equivalences, so is $\gamma_{n\#}$. $\square$

Now we can prove Theorem 1.7 and Corollary 1.8.

Proof of Theorem 1.7. The assertion easily follows from Theorem 4.16, Lemma 4.17 and the following commutative diagram

$$
\begin{array}{ccc}
A_d(m,n) & \xrightarrow{j_d} & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, S^n) \\
\| & & {\scriptstyle \gamma_{n\#}}\downarrow \simeq \\
A_d(m,n) & \xrightarrow{i_d} & \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)
\end{array} \quad \square
$$

Proof of Corollary 1.8. Since $i_{d+2} \circ s_d = i_d : A_d(m,n) \to \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$, the assertion easily follows from Theorem 1.7. $\square$

We will discuss Conjecture 1.9 in a separate section.

Proof of Conjecture 1.10 assuming Conjecture 1.9.
By combining Theorem 1.7 with Conjecture 1.10, we obtain the assertion for the inclusion $i_{d,\mathbb{R}}$ and it remains to deal with the case $j_{d,\mathbb{R}}$. For this purpose, recall the evaluation fibration sequence

$$\mathrm{Map}_\epsilon^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{\subset} \mathrm{Map}_\epsilon(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{ev} \mathbb{R}\mathrm{P}^n$$

given by $ev(f) = f(\mathbf{e}_m)$. Let $ev_d : \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathbb{R}\mathrm{P}^n$ be the restriction $ev_d = ev|\mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$. It is easy to show that

Lemma 4.18. *The map $ev_d : \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \to \mathbb{R}\mathrm{P}^n$ is a locally trivial fiber bundle with fiber $\mathrm{Alg}_d^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$.* $\square$

With the help of Lemma 4.18, we easily deduce our assertion for $j_{d,\mathbb{R}}$ from the one for $i_{d,\mathbb{R}}$. $\square$

5 Conjecture 1.9.

5.1 Stabilized version of Conjecture 1.9.

Although at present we are unable prove Conjecture 1.9, we can prove a stabilized version of it for the 0-component.

For $\epsilon = 0$ or 1, let $A^\epsilon_\infty(m,n)$ denote the stabilized space $A^\epsilon_\infty(m,n) = \lim_{k\to\infty} A_{2k+\epsilon}(m,n)$, where the limit is taken over the stabilization maps

$$A_\epsilon(m,n) \xrightarrow{s_\epsilon} A_{2+\epsilon}(m,n) \xrightarrow{s_{2+\epsilon}} A_{4+\epsilon}(m,n) \xrightarrow{s_{4+\epsilon}} A_{6+\epsilon}(m,n) \xrightarrow{s_{6+\epsilon}} \cdots$$

From the commutative diagram

$$\begin{tikzcd}
{} \arrow[r] & A_{2k+\epsilon}(m,n) \arrow[r, "s_{2k+\epsilon}"] \arrow[d, "\Psi_{2k+\epsilon}"'] & A_{2k+2+\epsilon}(m,n) \arrow[r, "s_{2k+2+\epsilon}"] \arrow[d, "\Psi_{2k+2+\epsilon}"'] & \cdots \\
{} \arrow[r] & \mathrm{Alg}^*_{2k+\epsilon}(\mathbb{RP}^m,\mathbb{RP}^n) \arrow[r, "\subset"] & \mathrm{Alg}^*_{2k+2+\epsilon}(\mathbb{RP}^m,\mathbb{RP}^n) \arrow[r, "\subset"] & \cdots
\end{tikzcd}$$

we obtain a stabilized map $\Psi^\epsilon_\infty = \lim_{k\to\infty} \Psi_{2k+\epsilon} : A^\epsilon_\infty(m,n) \to \mathrm{Alg}^*_\epsilon(m,n)$.

Proposition 5.1. *If $2 \le m < n$ and $\epsilon = 0$, $\Psi^0_\infty : A^0_\infty(m,n) \xrightarrow{\simeq} \mathrm{Alg}^*_0(m,n)$ is a homotopy equivalence for $m+2 \le n$ and a homology equivalence for $m+1 = n$.*

Proof. Note that we have a commutative diagram

$$\begin{tikzcd}
A^0_\infty(m,n) \arrow[r, "\lim_k i_{2k}", "\simeq"'] \arrow[d, "\Psi^0_\infty"'] & \mathrm{Map}^*_0(\mathbb{RP}^m,\mathbb{RP}^n) \arrow[d, equal] \\
\mathrm{Alg}^*_0(m,n) \arrow[r, "i", "\simeq"'] & \mathrm{Map}^*_0(\mathbb{RP}^m,\mathbb{RP}^n)
\end{tikzcd}$$

Now the assertion follows from Corollary 3.1 and Theorem 1.7. $\square$

Remark. We conjecture that the analogous statement also holds for $\epsilon = 1$.

5.2 The general case.

As we have seen, if we could prove Conjecture 1.9 we could replace our Theorem 1.6 by the more interesting Conjecture 1.10, which would then of course become a theorem. Unfortunately, we have not been able to give a full proof of Conjecture 1.9, so in this section we would like to present what we have been able to accomplish in this respect, i.e. to give a proof of the conjecture for the simplest case $m = 1$ and explain the difficulties we have encountered with the case of general $m \ge 2$.

First, let us recall that the maps $\Psi_d : A_d(m,n) \to \mathrm{Alg}^*_d(\mathbb{RP}^m,\mathbb{RP}^n)$ have contractible fibers. Indeed, the elements of the fibers can be identified with positive rational functions on $\mathbb{RP}^m$ and the space of such functions is convex and hence contractible. Unfortunately, this fact alone does not suffice to conclude that Ψ_d is a homotopy equivalence unless, for instance, it is a (quasi) fibration, a simplicial map or a proper map. The map Ψ_d is not proper and it is not clear whether it is of any of the other two kinds. However, it can be viewed as a "collection" of trivial bundles with contractible fibers, as we now explain.

The space $\mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ has a natural filtration

$$\mathrm{Alg}^*_p(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \ldots \subset \mathrm{Alg}^*_{d-2}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \subset \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n),$$

where $p = 0$ or $p = 1$ depending on the parity of d. Each of the differences $\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ (for $k = 0, \ldots, \lfloor \frac{d}{2} \rfloor$) consists of maps of *minimal* degree $d-2k$. Every such map has a unique representation as a tuple of polynomials in $A^*_{d-2k}(m,n)(\mathbb{R})$ (because any two representations of minimal degree may differ at most by a constant factor, which must be 1 by the comparison of coefficients at z_0^{d-2k}). This allows to identify the space $\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ and its preimage

$$\Psi^{-1}_{d-2k}(\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)) \subset A^*_{d-2k}(m,n)(\mathbb{R}).$$

Every tuple $(f_0, \ldots, f_n) \in \Psi_d^{-1}(\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$ arises as a coordinate-wise product of the unique minimal representation $\Psi^{-1}_{d-2k}(\Psi_d(f_0, \ldots, f_n)) \in A^*_{d-2k}(m,n)(\mathbb{R})$ with some polynomial from $\mathcal{H}^1_{2k}$. Using the above identification we obtain a map:

$$l_{d,k} : \mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \times \mathcal{H}^1_{2k} \to A^*_d(m,n)(\mathbb{R})$$

which, by the last remark, is a homeomorphism onto

$$\Psi_d^{-1}(\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)).$$

Summing up, we have obtained the following:

Lemma 5.2. *The restriction of* Ψ_d *to*

$$\Psi_d^{-1}(\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n))$$

is a trivial bundle with base $\mathrm{Alg}^*_{d-2k}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \setminus \mathrm{Alg}^*_{d-2(k+1)}(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ *and fiber* $\mathcal{H}^1_{2k}$.

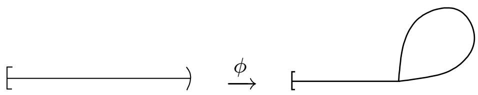
The fiber $\mathcal{H}^1_{2k}$ of this product bundle is contractible since $\mathcal{H}^1_{2k}$ is precisely the convex subset of all *positive*-valued polynomials in the affine space of all homogenous polynomials of degree $2k$ with coefficient 1 at z_0^{2k}. With this setup it seems reasonable to ask the following question:

Question. *Suppose F and G are metric spaces, with finite filtrations*

$$\begin{aligned}\emptyset &= F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_m = F, \\ \emptyset &= G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_m = G.\end{aligned}$$

Let $f : F \to G$ be a strictly[‡] *filtration-preserving map, such that each of the restrictions $f|_{F_i \setminus F_{i-1}} : F_i \setminus F_{i-1} \to G_i \setminus G_{i-1}$ is a trivial bundle with contractible fiber. Under what additional conditions is f a homotopy equivalence?*

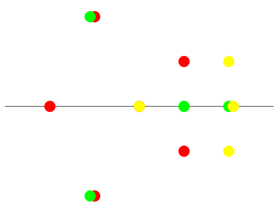
It is clear that some additional conditions are necessary to exclude examples of the following kind: let $I = [0,1]$ and $I' = [0,1)$. There is a map $I' \mapsto I \vee S^1$ that ,,winds up the loose end":



The filtration in the image of this map is $G_0 \subset G_1 \subset G_2 = \emptyset \subset I \subset I \vee S^1$, meaning that $F_0 \subset F_1 \subset F_2 = \emptyset \subset \phi^{-1}(I) \subset \phi^{-1}(I \vee S^1)$. Above each of the differences of the consecutive levels of this filtration the map ϕ is a homeomorphism.

5.3 The case $m = 1$.

It turns out that in the case $m = 1$ we can avoid these difficulties by an indirect approach. We will exploit the convenient fact that spaces of tuples of polynomials in one variable can be identified with certain configuration spaces of points or particles in the complex plane. More exactly, we identify the space $A_d(1,n)$ with the space of d particles of each of $n+1$ different colors, in the case $n = 2$, say, red, blue and yellow, located in $\mathbb{C} = \mathbb{R}^2$ symmetrically with respect to the real axis and such that no three particles of different color lie at the same point on the real axis.



Note that off the real axis the particles are completely unrestricted. The space of algebraic maps $A_d(1,n)$ can also be thought of as a configuration

[‡]This means $f(F_i) = G_i$ and $f(F_i \setminus F_{i-1}) = G_i \setminus G_{i-1}$.

space of k particles of each of $n+1$ different colors as above, where $k \leq d$, but with the additional property that when $n+1$ different particles meet (off the real line) they disappear.

Finally, we need one more configuration space, introduced in [23]. Let $T(d,n)$ denote the space of no more than $k_i \leq d$ particles of color i, where $k_i = d$ mod 2, on the real axis , with the property that any even number of particles of the same color at the same point on the real axis vanish, and of course, as before no $n+1$ particles of different colors lie at the same point. In other words, $T(d,n)$ is a configuration space modulo 2. There is a natural map $\Phi : A_d(1,n) \to T(d,n)$ which factors through $\Psi_d : A_d(1,n) \to \mathrm{Alg}_d(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n)$, $\Phi = Q_d \circ \Psi_d : A_d(1,n) \xrightarrow{\Psi_d} \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^1, \mathbb{R}\mathrm{P}^n) \xrightarrow{Q_d} T(d,n)$.

Proposition 5.3 ([23], Proposition 2.1)**.** *The maps Ψ_d and Q_d above are homotopy equivalences.*

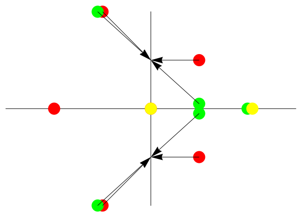
A proof of this proposition is given in [23] but as it does not seem convincing to us. So we will give another one, which we believe to be correct. More precisely, we shall prove the following:

Lemma 5.4. *The maps Q_d and $Q_d \circ \Psi_d$ are quasi-fibrations with contractible fibers.*

From this it follows at once that Ψ_d is a homotopy equivalence. Note that the proof of Proposition 2.1 in [23] is based on arguments claiming to prove that $Q_d \circ \Psi_d$ and Ψ_d are homotopy equivalences which then is used to deduce that so is Q_d. However the argument claiming to prove that Ψ_d is a homotopy equivalence makes no explicit use of any properties of the map Ψ_d, such as being a quasifibration etc. We know no way of proving directly that Ψ_d is a quasi-fibration or possesses any other properties which would make the argument in [23] valid.

Proof of Lemma 5.4. We first prove that the fiber of $Q_d \circ \Psi_d$ (and Q_d) over any point in $T(d,n)$ is contractible. Consider a configuration in $T(d,n)$.

In the fiber over this configuration, all points in the upper half plane are sent linearly to the fixed point $(1,0)$ and those in the lower half plane to $(-1,0)$. For a $2k$ or $2k+1$ fold particle lying on the real line, k of the particles are moved to $1,0$ and k are moved to to $(-1,0)$ leaving 0 or 1 particles in place. This argument shows that the fibers of both $Q_d \circ \Psi_d$ and Q_d are contractible.



Next, we will show that the maps $Q_d\circ\Psi_d$ and Q_d are both quasi-fibrations. Consider a point (configuration of particles) in $T(d,n)$. By a 'singular sub-configuration for color i' we mean a collection n particles of distinct colors other than the i-th color. Intuitively, we think of a singular configuration for the i-th color as an obstacle through which a particle of the i-th color cannot pass. By a singular sub-configuration (without specifying the color) we mean a singular sub-configuration for any color. For non-negative integers $p_1, p_2, \ldots, p_{n+1}$ with $\sum_i p_i \leq d$, let $T(d,n;p_1,\ldots,p_{n+1})$ denote the subspace of $T(d,n)$ consisting of configurations with precisely p_i particles of the i-th color. We start by proving that the restrictions of the maps $Q_d \circ \Psi_d$ and Q_d to the pre-images of $T(d,n;p_1,\ldots,p_{n+1})$ are quasi-fibrations. For any integer $k \geq 0$, let $T_k(d,n;p_1,\ldots,p_{n+1})$ denote the subspace of $T(d,n;p_1,\ldots,p_{n+1})$ consisting of configurations containing exactly k singular sub-configurations. It is easy to see that the restriction of $Q_d \circ \Psi_d$ and Q_d to the pre-image of $T_k(d,n;p_1,\ldots,p_{n+1})$ is a locally trivial fiber bundle. Now we filter the space $T(d,n;p_1,\ldots,p_{n+1})$ by closed subspaces $D_k(d,n;p_1,\ldots,p_{n+1})$ of configurations containing $\geq k$ singular sub-configurations. Set theoretic differences between these spaces are the spaces $T_k(d,n;p_1,\ldots,p_{n+1})$ over which the maps are locally trivial fiber bundles and hence quasifibrations. Now we apply the Dold-Thom criterion (Lemma 4.3 of [14]). To do so we have to construct an open neighborhood of $D_{k+1}(d,n;p_1,\ldots,p_{n+1})$ in $D_k(d,n;p_1,\ldots,p_{n+1})$, a deformation of this neighborhood onto $D_{k+1}(d,n;p_1,\ldots,p_{n+1})$, together with a corresponding covering neighborhood and a covering deformation required by the Dold-Thom criterion (since all the fibers are contractible the condition that the induced maps on the fibers are homotopy equivalences is automatically satisfied). Such neighborhoods and deformations are easy to describe intuitively. The set of points of the required neighborhood of $D_{k+1}(d,n;p_1,\ldots,p_{n+1})$ consists of the points of $D_{k+1}(d,n;p_1,\ldots,p_{n+1})$ together with those configurations in $D_k(d,n;p_1,\ldots,p_{n+1})$ with at least one non-singular sub-configuration of n particles of different colors contained in a 'sufficiently small' interval. Here 'sufficiently small' refers to the requirement that the particles in this sub-configuration be much nearer to each other than they are to any other particle and that the length of the minimal interval containing the sub-configuration be much less than the length of any interval containing a collection of $n+1$ particles of different color. The deformation can now be defined by introducing a force of attraction between particles of different color (e.g. a force field satisfying an inverse-square law). By induction on k we show that the maps $Q_d \circ \Psi_d$ and Q_d are quasifibrations over $T(d, n; p_1, \ldots, p_n)$. We now fix $p_1, p_2, \ldots, p_n$ and filter the space $T(d, n)$ according to the number of points of the $n+1$-th color. The set theoretic differences between the terms of the filtration are precisely the spaces $T(d, n; p_1, \ldots, p_{n+1})$ and we have already proved that the restriction of the maps $Q_d \circ \Psi_d$ and Q_d to the inverse images of these spaces are quasi-fibrations. We apply again the Dold-Thom criterion. For this purpose we need to construct open neighborhoods of spaces of configurations with no more than $k-2$ particles of the last color in the space of configurations of no more than k-particles of the last color. The method is again analogous to the one we used earlier. Our deformation will pull together pairs of particles of the last color which are very close by means of a gravitational force field between particles of the last color. For this purpose we must avoid hitting a "singular point" (a sub-configuration of $n-1$ particles of different colors). Again, it is easy to see that we can choose open neighborhoods and deformations with the right properties. This proves that the maps restricted to the pre-images of spaces with the number of particles of the first $n-1$ colors fixed are quasifibrations. Now we filter these spaces according to the number of particles of the last but one color. Proceeding by induction we see that $Q_d \circ \Psi_d$ and Q_d are quasi-fibrations. □

5.4 The general case revisited.

It seems now natural to ask if the above argument for the case $m = 1$ can be generalized to higher values of m. The space $T(d, n)$ have a natural generalization. Consider the space of real algebraic cycles of degree d with codimension 1 in $\mathbb{C}\mathrm{P}^n$ that are complex algebraic cycles invariant under conjugation which arise as sets of solutions of a homogeneous equation with real coefficients [22]. Let $C(d, m, n)$ denote the set of m-tuples of such cycles with disjoint support (corresponding m-tuples of roots of m-homogeneous polynomials without a common non-zero root). Elements of $C(d, m, n)$ can be identified with those of $A_d(m, n)$ up to multiplication by a constant, and so that the spaces are homotopy equivalent. Let $T(d, m, n)$ denote the space obtained from $C(d, m, n)$ by replacing real algebraic cycles by real algebraic cycles reduced mod 2 [21]. Clearly we have maps

$$A_d(m, n) \xrightarrow{\Psi_d} \mathrm{Alg}^*_d(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n) \xrightarrow{Q_d} T(d, m, n)$$

analogous to those in the case $m = 1$, and we would like to prove that they are homotopy equivalences. It is possible to show that they all have contractible fibers, but we have not been able to prove more.

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