STABILITY OF CONFIGURATION SPACES OF POSITIVE AND NEGATIVE PARTICLES

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§1. Introduction.

The main purpose of the present paper is to prove a result concerning certain "particle" or "multi-configuration" spaces. Such spaces have become in recent years objects of much interest because of their connection with spaces of holomorphic mappings from Riemann surfaces to complex manifolds, and with mathematical physics. However, they are also of interest for another reason: they often provide convenient models for various mapping spaces, particularly loop spaces (see e.g. [Ma], [Mc], [Se1], [Se2], [GKY1], [GKY2], [GKY3], [Gu2], [K], [H]). The known results relating particle spaces and mapping spaces fall into two basic types.

The first consists of "stable results", which assert that an "infinite particle space" is homotopy (or homology) equivalent to a certain function space. There are general methods for proving such results. An account of one such method, based on the so called "scanning map" of Segal, has been given by Kallel ([K]).

The results of the second type are "unstable". Infinite particle spaces have a natural filtration by "finite" subspaces containing less than a given number of particles. In all known cases homotopy (or homology) groups of these subspaces stabilize. More precisely, let $\mathcal{C}$ denote an "infinite particle space" and let

$$\mathcal{C}_0 \subset \mathcal{C}_1 \subset \ldots \subset \mathcal{C}_i \subset \ldots \subset \mathcal{C}$$

be a filtration by finite particle spaces. Results of the second type state that the inclusion maps $\mathcal{C}_i \to \mathcal{C}_{i+1}$ are homotopy (or homology) equivalences up to some dimension $k = k(i)$ such that $k(i) \to \infty$ as $i \to \infty$. Together with the corresponding stable result, this implies that the spaces $\mathcal{C}_i$ can be viewed as finite dimensional approximations to the function space of the stable theorem. A large number of such unstable results is known, but there is no really general method for proving them (though probably the one that comes closest is due to Vassiliev, [Va]).

The classical cases involve symmetric products and related spaces. As a first example, for any connected space X, consider the free abelian monoid on X, denoted by $\mathrm{SP}^\infty(X,*)$ (where $*$ is a chosen basepoint in X.) This is an "infinite particle space" which can be viewed as a direct limit of maps $\mathrm{SP}^i(X) \hookrightarrow \mathrm{SP}^{i+1}(X)$ where $\mathrm{SP}^i(X)$ denotes the i-th symmetric product of X and the inclusion map is given by adding the basepoint.

The relevant stable result in this situation asserts that the limit space $\mathrm{SP}^\infty(X,*)$ is homotopy equivalent to the infinite loop space $\prod_{i=0}^\infty K(\tilde{H}_i(X,\mathbb{Z}),i)$. The unstable result asserts that the map $\mathrm{SP}^i(X) \hookrightarrow \mathrm{SP}^{i+1}(X)$ is a homotopy equivalence up to dimension i. Hence $\mathrm{SP}^i(X)$ may be considered as a finite dimensional approximation of $\prod_{i=0}^\infty K(\tilde{H}_i(X,\mathbb{Z}),i)$.

As a second example, consider the space of all finite subsets of $\mathbb{R}^n$. This space is, of course, disconnected, in fact it is the disjoint union $\coprod_{i\geq 0} C_i(\mathbb{R}^n)$, where $C_i(X)$ denotes the space of all subsets of X of cardinality i. There are natural maps (up to homotopy) $C_i(\mathbb{R}^n) \to C_{i+1}(\mathbb{R}^n)$ given by "adding a point near infinity". By taking the direct limit of these maps we can construct a connected infinite particle space $C_\infty(\mathbb{R}^n)$. In this case the stable result is due to Segal ([Se1]) and the unstable one to Arnold ([Ar], but see also the appendix to §5 of [Se2]). The former asserts that there is a homology equivalence $C_\infty(\mathbb{R}^n) \to \Omega^n_0 S^n$. The latter says that the map $C_i(\mathbb{R}^n) \to C_{i+1}(\mathbb{R}^n)$ is a homology equivalence up to dimension $[i/2]$. Together they give a finite dimensional configuration space approximation to $\Omega^n_0 S^n$.

There is also another "classical" example of this kind, invented by McDuff ([Mc]). One considers the space $C^\pm(X)$ of configurations of "positive and negative particles" on X. Its elements are equivalence classes of pairs of finite subsets (S,T) of X, such that whenever the same point of X is an element of both S and T it can be "cancelled". In other words, a positive particle and a negative particle can "collide and disappear". The space of positive and negative particles is an infinite particle space, and, as usual, is the limit of finite spaces of positive and negative particles. McDuff's result asserts that the identity components of $C^\pm(\mathbb{C})$ and $\Omega^2\big(S^2\times S^2/\Delta\big)$ are homotopy equivalent. Probably because the space $C^\pm(\mathbb{C})$ has, until now, lacked connections with other well known spaces of topology or geometry (we describe one such connection below) McDuff's result has attracted little attention. In particular, the natural question concerning the validity of an unstable version of McDuff's theorem does not appear to have been discussed. The purpose of this paper is to provide such an unstable theorem:

Theorem 1.1. *Let $C^\pm_d(\mathbb{C})$ denote the space of positive and negative particles in $\mathbb{C}$, with no more than d particles of either kind. Then the inclusion map*

$$C^\pm_d(\mathbb{C}) \to C^\pm(\mathbb{C})$$

is a homology equivalence up to dimension $[d/2]$ and is a homotopy equivalence up to dimension $[(d-1)/3]$.

To say that the map $f: X\to Y$ is a *homotopy* (or *homology*) *equivalence up to dimension* N means that the induced map $f_*:\pi_i(X)\to\pi_i(Y)$ (or $f_*:H_i(X,\mathbb{Z})\to H_i(Y,\mathbb{Z})$) is bijective when $i<N$ and surjective when $i=N$.

One reason why this result seems interesting to us is the rather large difference between the homology and homotopy stabilization dimensions. Generally, in those cases where both a homology and homotopy stabilization can be proved, the stabilization dimensions turn out to be very close (e.g. see [Se2], [GKY1], [Gu1], [Gu2], [H]) and there are no reasons to believe that they should be different. However, in this case there are reasons to believe that the difference between the homology and homotopy stabilization dimensions may be genuine and not just a consequence of the method of proof (cf. [GKY3]). We also feel that the technique of proof, and in particular the passage from a homology stabilization theorem to a homotopy stabilization theorem, may have a wider applicability and is thus interesting in its own right.

Another reason for possible interest in this result is that McDuff's space $C^{\pm}(\mathbb{C})$ is in fact related to a space with a more geometrical character. Let M_2 denote the subvariety of $\mathbb{C}P^3$ defined by the equation $z_1^2 = z_2 z_3$ (a quadric cone); this is homeomorphic to the space $\big(S^2 \times S^2/\Delta\big)$. Let $\mathrm{Hol}_d(S^2, M_2)$ be the space of based holomorphic maps $S^2 \to M_2$ of degree d. Then it follows from [Gu1] that the limit space $\lim_{d\to\infty} \mathrm{Hol}_d(S^2, M_2)$ is homotopy equivalent to $\Omega_0^2\big(S^2 \times S^2/\Delta\big)$. Moreover, $\mathrm{Hol}(S^2, X)$ can be identified with a subspace of the moduli space of framed Yang-Mills instantons on S^4 ([At]) and it may be worthwhile investigating the unstable result for the space $C^{\pm}(\mathbb{C})$ from these alternative points of view.

§2. The stable result.

Our main results concern the configuration space of "positive and negative particles" ([Mc]). This space is obtained from the usual configuration space of distance particles by a "Grothendieck construction", i.e. by "adjoining inverses". However, its nature becomes clearer when we view it as a special case of a more general construction, and we shall give some results in this additional generality.

Let $\mathrm{AG}(X)$ denote the free abelian topological group generated by a based space X ([DT]). We refer to the elements of $\mathrm{AG}(X)$ as divisors on X. A *positive divisor* is one of the form $\sum_i n_i x_i$, with $n_i \geq 0$ for all i. If $Y \subset X$ is a closed subspace let $\mathrm{AG}(X,Y)$ denote the group

$$\mathrm{AG}(X,Y) = \mathrm{AG}(X/Y, *) \cong \mathrm{AG}(X)/\,\mathrm{AG}(Y).$$

As is pointed out in [DT], in the case $(X,Y) = (S^2, \infty)$ the elements of $\mathrm{AG}(X,Y)$ can be thought as "rational functions" on $\mathbb{C}$, where we consider S^2 as the space $\mathbb{C} \cup \infty$. In other words, a finite formal sum

$$\textstyle\sum_{j=1}^{t} \alpha_j x_j - \sum_{i=1}^{l} \beta_i y_i \in \mathrm{AG}(S^2, \infty)$$

(where $\alpha_j, \beta_i \geq 1$ are integers and $x_j, y_i \in \mathbb{C}$)

may be identified with the rational function

$$f(z) = \frac{(z-x_1)^{\alpha_1}(z-x_2)^{\alpha_2}\ldots(z-x_t)^{\alpha_t}}{(z-y_1)^{\beta_1}(z-y_2)^{\beta_2}\ldots(z-y_l)^{\beta_l}}.$$

This should not be confused with the better known spaces of rational functions studied by Segal in [Se2]. In Segal's case the zeros and the poles of a rational function are not allowed to coincide. In our case the zeros and poles *can* coincide, and if this happens the function is identified with the one obtained by cancellation of common factors of the numerator and the denominator. In particular our space of "rational functions" is not topologized as a subspace of the space of continuous mappings $\Omega^2 S^2$.

If n, m are non-negative integers, we denote by $\mathrm{AG}_{n,m}(X,Y) \subset \mathrm{AG}(X,Y)$ the subset of elements of the form

$$\sum_{j=1}^{n} \alpha_j x_j - \sum_{j=1}^{m} \beta_j y_j \in \mathrm{AG}(X,Y),$$

$$\text{where and } x_j \neq x_i, y_j \neq y_i \text{ if } j \neq i.$$

$$1 \leq \alpha_j \leq n, 1 \leq \beta_j \leq m \text{ are integers, } x_j, y_j \in X - Y,$$

The space $\mathrm{AG}_{n,m}(S^2, \infty)$ can be identified with the space of "rational functions" whose numerators are monic polynomials with roots of multiplicity $\leq n$ and whose denominators are monic polynomials with roots of multiplicity $\leq m$. When $n = m$ we shall write $\mathrm{AG}_n(X,Y) = \mathrm{AG}_{n,n}(X,Y)$ and $\mathrm{AG}_n(X) = \mathrm{AG}_n(X,\emptyset)$.

Let $\mathrm{AG}^{n,m}(X,Y) \subset \mathrm{AG}(X,Y)$ denote the subset of elements of the form

$$\sum_j \alpha_j x_j - \sum_j \beta_j y_j,$$

$$\text{where } x_i \neq x_j, y_i \neq y_j \in X - Y \text{ if } i \neq j,$$

$$\alpha_j, \beta_j \geq 1 \text{ are integers and } \sum_j \alpha_j \leq n, \sum_j \beta_j \leq m.$$

In particular, if $n = m$, we shall write $\mathrm{AG}^n(X,Y) = \mathrm{AG}^{n,n}(X,Y)$ and $\mathrm{AG}^n(X) = \mathrm{AG}^n(X,\emptyset)$. We can view the space $\mathrm{AG}^n(S^2,\infty)$ as the space of "rational functions" which can be represented as quotients of monic polynomials with numerators of degrees $\leq n$ and denominators with degrees $\leq m$.

Note that in the first case above we consider pairs of divisors of arbitrary degrees but with bounded multiplicity while in the second case the divisors have bounded degrees (hence of course also bounded multiplicity). Thus $\mathrm{AG}^{n,m}(X,Y) \subset \mathrm{AG}_{n,m}(X,Y)$

The following theorem generalizes theorem 1.3 of [Mc] and can be proved in the same way.

Theorem 2.1. *There is a homotopy equivalence*

$$S : \mathrm{AG}_n(\mathbb{C}) \xrightarrow{\simeq} \Omega_0^2 \,\mathrm{AG}^n(S^2, \infty).$$

Sketch of Proof. The proof consists of two parts. First one considers the "scanning" map

$$S : \mathrm{AG}_n(\mathbb{C}) \to \mathrm{Map}^*(\mathbb{C} \cup \infty, \mathrm{AG}_n(S^2,\infty)) = \Omega_0^2 \,\mathrm{AG}_n(S^2,\infty)$$

and shows that it is a homotopy equivalence, using the technique introduced by Segal in [Se2]. One than completes the argument by showing that there is a homotopy equivalence

$$\mathrm{AG}_n(S^2,\infty) \simeq \mathrm{AG}^n(S^2,\infty).$$

using essentially the same argument as in [Mc].

For $n = 1$ the space $\mathrm{AG}_1(X)$ is the space of finite sums

$$\sum_i x_i - \sum_k y_k \quad (x_i \neq x_j, y_k \neq y_l \text{ if } i \neq j, k \neq l)$$

This is just the space of configurations positive and negative particles denoted by $C^{\pm}(X)$ in ([Mc]). On the other hand, the space $\mathrm{AG}^1(S^2,\infty)$ is the space of elements of $\mathrm{AG}(S^2,\infty)$ of the form $a - b$, i.e.

$$\mathrm{AG}^1(S^2,\infty) \cong (S^2 \times S^2)/\Delta, \qquad \text{where } \Delta \text{ is the diagonal.}$$

Thus, for $n = 1$, theorem 1 indeed coincides with theorem 1.3 of [Mc] (with $M = \mathbb{C}$).

In general, the space $\mathrm{AG}^n(S^2,\infty)$ is just given by the quotient

$$\mathrm{SP}^n(S^2) \times \mathrm{SP}^n(S^2)/\sim$$

where the equivalence relation $\sim$ is generated by the following relation:

$$(\delta + \eta, \gamma + \eta) \sim (\delta + \nu, \gamma + \nu)$$

for divisors $\delta, \gamma, \eta, \nu$ on S^2 such that $\deg(\delta + \eta) = \deg(\gamma + \eta) = \deg(\delta + \nu) = \deg(\gamma + \nu) = n$.

From now on we shall consider only the case $n = 1$ and $X = \mathbb{C}$. Analogous results hold for the general case but the statements and proofs are considerably more complicated. We shall however retain our notation $\mathrm{AG}_1(X)$ instead of McDuff's $C^{\pm}(X)$.

For each configuration

$$\xi = \sum_{j=1}^{d_1} x_j - \sum_{i=1}^{d_2} y_i \in AG_1(X),$$

we define the (total) *charge* of ξ by

$$\mathrm{charge}(\xi) = d_1 - d_2.$$

Since $\pi_0(AG_1(\mathbb{C})) \cong \pi_2((S^2 \times S^2)/\Delta) = \mathbb{Z}$, there is a decomposition into path-components:

$$AG_1(\mathbb{C}) = \coprod_{d \in \mathbb{Z}} AG_{1,k}(\mathbb{C})$$

where we take

$$AG_{1,k}(\mathbb{C}) = \{\xi \in AG_1(\mathbb{C}) : \text{charge}(\xi) = k\}.$$

It is easy to see that $AG_{1,k}(\mathbb{C}) \simeq AG_{1,l}(\mathbb{C})$ for any k, l. We consider the case $k = 0$ only. Let $AG^d_{1,0}(\mathbb{C})$ denote the finite dimensional subspace of $AG_{1,0}(\mathbb{C})$ defined by

$$AG^d_{1,0}(\mathbb{C}) = \{\sum_{j=1}^{d} x_j - \sum_{j=1}^{d} y_j \in AG_{1,0}(\mathbb{C})\} \subset AG_{1,0}(\mathbb{C}).$$

This is the "finite particle space" that we shall investigate.

§3. The unstable result.

Recall that the space $AG_{1,0}(\mathbb{C})$ is the union of finite dimensional subspaces $AG^d_{1,0}(\mathbb{C})$'s,

$$AG_{1,0}(\mathbb{C}) = \lim_{d\to\infty} AG^d_{1,0}(\mathbb{C}).$$

To prove theorem 1, it is sufficient to show that following two assertions hold:

Theorem 3.1. *Let $i_d : AG^d_{1,0}(\mathbb{C}) \to AG^{d+1}_{1,0}(\mathbb{C})$ be the inclusion map.*

(1) The map i_d is a homology equivalence up to dimension $[d/2]$, and

(2) it is a homotopy equivalence up to dimension $[(d-1)/3]$.

In this section we shall prove the first of the above two statements.

Let $U \subset \mathbb{C}$ be an open set. Choose an open subset $V \subset U \subset \mathbb{C}$ homeomorphic to U and $\neq U$. Next choose any point $z \in U - V$ and fix it. Now define the stabilization map $C_d(U) \to C_{d+1}(U)$ by adding the point z "from the edge ":

$$\begin{array}{ccccc} C_d(U) & \xrightarrow{\cong} & C_d(V) & \longrightarrow & C_{d+1}(U) \\ & & \{x_1, \cdots, x_j\} & \longrightarrow & \{x_1, \cdots, x_j, z\} \end{array}$$

Of course, this definition of depends on the choices of V and the point z but the homotopy class of the stabilization map is independent of the choices made.

Next, recall the following result ([Se2]):

Lemma 3.2. *For any open subset $U \subset \mathbb{C}$, the stabilization map $C_d(U) \to C_{d+1}(U)$ is a homology equivalence up to dimension $[d/2]$.* □

For a subset $U = \mathbb{C} - \{\text{finite points}\}$ and each finite subset $\xi \subset U$, let $C^\xi_d(U)$ denote the subspace

$$C^\xi_d(U) = \{\alpha \in C_d(U) : \alpha \cap \xi \neq \emptyset\}.$$

Lemma 3.3. *The stabilization map $C^{\xi}_d(U) \to C^{\xi}_{d+1}(U)$ is a homology equivalence up to dimension $[(d-1)/2]$.*

Proof. Let $\xi = \{z_1, \cdots, z_m\}$. The proof is based on the induction on m.

If $m = 0$, this follows from (3.2). If $m = 1$, since $C^{\xi}_d(U) \cong C_{d-1}(U - \{z_1\})$ the assertion follows. For the inductive step we use the fact that

$$C_d^{\{z_1,\dots,z_{m+1}\}}(U) = C_d^{\{z_1,\dots,z_m\}}(U) \cup C_d^{\{z_{m+1}\}}(U),$$

with

$$C_d^{\{z_{m+1}\}}(U) \cong C_{d-1}(U - \{z_{m+1}\}),$$

and $C_d^{\{z_1,\dots,z_m\}}(U) \cap C_d^{\{z_{m+1}\}}(U) \cong C_{d-1}^{\{z_1,\dots,z_m\}}(U - \{z_{m+1}\})$.

Assuming the statement for m, we obtain the statement for $m+1$ from the Mayer-Vietoris sequence and the 5-Lemma. However, since $C_d^{\{z_1,\dots,z_m\}}(U) \cap C_d^{\{z_{m+1}\}}(U)$ is not an open subset of $C_d^{\{z_1,\dots,z_{m+1}\}}(U)$ the above argument cannot be applied directly. Instead, let $\epsilon > 0$ be a sufficiently small fixed number. We replace the set $C_d^{\{z_1,\dots,z_{m+1}\}}(U)$ by the set

$U^{z_1,\dots,z_m} = \{\{x_1,\dots,x_d\}$: such that $|x_i - z_j| < \epsilon/100$ for some $1 \le i \le d, 1 \le j \le m\}$.

Similarly we replace $C_d^{\{z_{m+1}\}}(U)$ by the open subset

$V^{z_{m+1}} = \{\{x_1,\dots,x_d\}$: such that $|x_i - z_{m+1}| < \epsilon/100$ for some $1 \le i \le d\}$. Without loss of generality we can assume that all the x_i are far away, say $|x_i - x_j| > 1$ for $i \ne j$, and that all the z_i are also far apart, say $|z_i - z_j| > 1$ for $i \ne j$. Thus there can be no more than one z_k very close to a given x_j. We can now see that

$$C_d^{\{z_1,\dots,z_{m+1}\}}(U) \simeq U^{z_1,\dots,z_m}, \quad C_n^{\{z_{m+1}\}} \simeq V^{z_{m+1}}$$

and $U^{z_1,\dots,z_m} \cap V^{z_{m+1}} \simeq C_{d-1}^{\{z_1,\dots,z_m\}}(U - \{z_{m+1}\})$. $\square$

Proof of Theorem 3.1 (1). From now on we shall write $C_d = C_d(\mathbb{C})$. Note that there is a homeomorphism

$$AG_1^{d+1,d+1}(\mathbb{C}) \cong AG_1^{d,d}(\mathbb{C}) \cup_f (C_{d+1} \times C_{d+1})$$

where the identification map f is given by

$$\begin{array}{ccccc} C_{d+1} \times C_{d+1} & \stackrel{\supset}{\longleftarrow} & D_{d+1,d+1} & \stackrel{f}{\longrightarrow} & AG_1^{d,d} \\ & & (\{x_j\},\{y_i\}) & \longrightarrow & \sum_j x_j - \sum_i y_i \end{array}$$

and we take

$$D_{d_1,d_2} = \{(\alpha,\beta) \in C_{d_1} \times C_{d_2} : \alpha \cap \beta \ne \emptyset\}.$$

There is a homeomorphism

$$AG_1^{d+1,d+1}(\mathbb{C})/A_1^{d.d}(\mathbb{C}) \cong (C_{d+1} \times C_{d+1})/D_{d+1,d+1}.$$

Hence it suffices to show that the inclusion map $j : D_{d+1,d+1} \to C_{d+1} \times C_{d+1}$ is a homology equivalence up to dimension $[d/2]$.

To do so, we need:

Lemma 3.4. *(1) The stabilization map* $s_1 : D_{d_1,d_2} \to D_{d_1+1,d_2}$ *is a homology equivalence up to dimension* $[(d_1-1)/2]$.

(2) The stabilization map $s_2 : D_{d_1,d_2} \to D_{d_1,d_2+1}$ *is a homology equivalence up to dimension* $[(d_2-1)/2]$.

We postpone the proof of (3.4) and first prove part (1) of theorem 3.1

Consider the homotopy commutative diagram

$$\begin{array}{ccc} D_{d+1,d+1} & \xrightarrow{s_1} & D_{d+2,d+1} \\ j \downarrow \cap & & s_2 \downarrow \\ C_{d+1} \times C_{d+1} & \xrightarrow{s'} & D_{d+2,d+2}, \end{array}$$

where s' is the stabilization map given by adding points at infinity.

By (3.4), the map $s_2 \circ s_1$ is a homology equivalence up to dimension $[d/2]$. Hence the induced homomorphism

$$j_* : H_k(D_{d+1,d+1}) \to H_k(C_{d+1} \times C_{d+1})$$

is injective when $k \leq [d/2] - 1$.

If $s : C_d \to C_{d+1}$ is a stabilization map, then by (4.2), the map $s \times s : C_d \times C_d \to C_{d+1} \times C_{d+1}$ is a homology equivalence up to dimension $[d/2]$. Since $s \times s$ is homotopic to the composite of maps

$$C_d \times C_d \xrightarrow{s'} D_{d+1,d+1} \xrightarrow{j} C_{d+1} \times C_{d+1}$$

the homomorphism

$$j_* : H_k(D_{d+1,d+1}) \to H_k(C_{d+1} \times C_{d+1})$$

is surjective when $k \leq [d/2]$.

Thus j is a homology equivalence up to dimension $[d/2]$. This completes the proof of (1) of theorem 3.1. $\square$

Proof of Lemma 3.4. Since (1) and (2) are completely analogous we only prove the former.

Let $p_2 : D_{d_1,d_2} \to C_{d_2}$ (respectively. $q_2 : D_{d_1+1,d_2} \to C_{d_2}$) be the second projection. Consider the commutative diagram

$$\begin{array}{ccc} C^{\xi}_{d_1} & \longrightarrow & C^{\xi}_{d_1+1} \\ \downarrow & & \downarrow \\ D_{d_1,d_2} & \xrightarrow{s_1} & D_{d_1+1,d_2} \\ {\scriptstyle p_2}\downarrow & & {\scriptstyle q_2}\downarrow \\ C_{d_2} & \xrightarrow{=} & C_{d_2} \end{array}$$

where $\xi \in C_{d_2}$ is a fixed configuration and vertical sequences are fibrations.

Since the stabilization map $C^{\xi}_{d_1} \to C^{\xi}_{d_1+1}$ is a homology equivalence up to dimension $[(d_1-1)/2]$ (from (3.3)), s_1 is also a homology equivalence up to dimension $[(d_1-1)/2]$. □

§4. Universal covering spaces.

In this section we consider the universal covering of $A^d_{1,0}(\mathbb{C})$ and give the proof of (2) of theorem 3.1.

Let $F(X,m)$ denote the ordered configuration space $\{(x_1,\cdots,x_m) \in X^m : x_i \neq x_j \text{ if } i \neq j\}$.

Lemma 4.1. *If* $d \geq 2$, $\pi_1(A^d_{1,0}(\mathbb{C})) \cong \mathbb{Z}/2$.

Proof. It follows from the method given in appendix of [GKY1] that $\pi_1 = \pi_1(A^d_{1,0}(\mathbb{C}))$ is abelian. So it suffices to show that

$$(*) \qquad H_1(A^d_{1,0}(\mathbb{C}),\mathbb{Z}) \cong \mathbb{Z}/2 \quad \text{for } d \geq 2.$$

Let us write

$$C_m = C_m(\mathbb{C}) \text{ and } A^d_{1,0} = A^d_{1,0}(\mathbb{C}).$$

First we deal with the case $d = 2$. recall that

$$A^2_{1,0}(\mathbb{C}) = A^2_{1,0} \simeq A^1_{1,0} \cup_f C_2 \times C_2,$$

where $f : D_2 \to A^1_{1,0}$ is the identification map. Since $A_{1,0}(\mathbb{C})$ is contractible, we have $A^2_{1,0} \simeq (C_2 \times C_2)/D_2$. We begin by showing that $H_1(D_2) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Let A be the closed subspace of D_2 consisting of configurations $(\alpha, \beta) \in C_2 \times C_2$ such that $\alpha = \beta$. Thus, $A = C_2 \simeq S^1$, and $H_1(A) \cong \mathbb{Z}\gamma$, where γ is (the Hurewicz image of) a loop which interchanges the two distinct points. Let A' be the open neighbourhood of A in D_2 consisting of configurations $(\alpha, \beta) \in C_2 \times C_2$ such that α and β have at least one common point, z say, and the remaining two points z_1, z_2 are such that $|z_1 - z_2| < \frac{1}{10} \max\{|z - z_1|, |z - z_2|\}$. It is clear that A,A' are homotopy equivalent. We consider the Mayer-Vietoris sequence for the open covering $D_2 = A' \cup (D_2 - A)$ of D_2. We have $D_2 - A \cong F(\mathbb{C}, 3)$, since an element of $D_2 - A$ corresponds to three distinct points, one positive, one negative, and one "neutral". It follows from the known homology of $F(\mathbb{C}, n)$ that $H_1(D_2 - A) \cong \mathbb{Z}\gamma_{+-} \oplus \mathbb{Z}\gamma_{+0} \oplus \mathbb{Z}\gamma_{-0}$, where γ_{+-} is a loop which moves the positive point once around the negative point, and similarly for γ_{+0}, γ_{-0}. A similar discussion for $A' \cap (D_2 - A) \simeq S^1 \times S^1$ shows that $H_1(A' \cap (D_2 - A)) \cong \mathbb{Z}\delta_{+-} \oplus \mathbb{Z}\delta_{+0}$ (here $\delta_{-0} \simeq \delta_{+0}$). It follows now from the Mayer-Vietoris sequence that $H_1(D_2) \cong \mathbb{Z}\gamma \oplus \mathbb{Z}\gamma_0 \cong \mathbb{Z}\gamma \oplus \mathbb{Z}\gamma_{+0}$.

Let us now consider the homology exact sequence of the pair $(C_2 \times C_2, D_2)$. We have just identified $H_1(D_2)$, and we have $H_1(C_2 \times C_2) \cong \mathbb{Z}\eta \oplus \mathbb{Z}\xi$, where η is a loop which interchanges two positive points and ξ is a loop which interchanges two negative points. The map $H_1(D_2) \to H_1(C_2 \times C_2)$ is given by $\gamma \mapsto \eta + \xi$, $\gamma_{-0} \mapsto 2\xi$. Hence $H_1(C_2^{\pm,0}) \cong \mathbb{Z}/2$ as required; it is generated by a loop which interchanges two points of the same type.

The case $d \geq 4$ follows from the results for the case $d = 2$. In fact, from (1) of theorem 3.1, if $d \geq 4$, then the induced map

$$H_1(A^d_{1,0}) \xrightarrow{\cong} H_1(A_{1,0}) \cong H_1((S^2 \times S^2)/\Delta) = \mathbb{Z}/2$$

is an isomorphism and the case $d \geq 4$ is clear.

Finally we deal with the case $d = 3$. From (1) of theorem 3.1, we know that the induced homomorphisms

(a) $$\mathbb{Z}/2 = H_1(A^2_{1,0}) \to H_1(A^3_{1,0})$$

and

(b) $$H_1(A^3_{1,0}) \xrightarrow{i_*} H_1(A^4_{1,0}) = \mathbb{Z}/2$$

are surjective. Hence from (a), $H_1(A^3_{1,0}) = 0$ or $\mathbb{Z}/2$. However, in the former case, since i_* is surjective we deduce from (b) that $H_1(A^4_{1,0}) = 0$. This is a contradiction; hence we have shown that $H_1(A^3_{1,0}) = \mathbb{Z}/2$. $\square$

Definition 4.2. (1) Let $A_m \subset \Sigma_m$ denote the alternating subgroup of the m-th symmetric group of m letters $\{1, 2, \cdots, m\}$. Define the space $\tilde{C}_m(X)$ by

$$\tilde{C}_m(X) = \begin{cases} \{\pm 1\} & m = 0 \\ X \times \{\pm 1\} & m = 1 \\ F(X, m)/A_m & m \geq 2 \end{cases}$$

where the group $A_m \subset \Sigma_m$ acts on the space $F(X,m)$ by the permutations of coordinates. Let $\tilde{C}(X) = \coprod_{m\geq 0} \tilde{C}_m(X)$ (disjoint union). There is a non-trivial double covering $\tilde{C}(X) \to C(X) = \coprod_{m\geq 0} C_m(X)$. Each element $\xi = [x_1, \cdots, x_m] \in \tilde{C}_m(X)$ can be viewed as a usual configuration $\{x_1, \cdots, x_m\}$ with fixed orientation. We call $\tilde{C}(X)$ "the oriented configuration space" on X. The group $\mathbb{Z}/2$ acts freely on $\tilde{C}(X)$ by the change of orientation.

(2) Let $\approx$ denote the equivalence relation on $\tilde{C}(X) \times \tilde{C}(X)$ generated by the relation

$$([\xi], [\eta]) \approx ([\xi - \{x\}], [\eta - \{x\}])$$

if $x \in \xi \cap \eta$ $(\xi, \eta \in C(X))$.

This induces an equivalence relation $\approx$ on $\tilde{C}(X) \times_{\mathbb{Z}/2} \tilde{C}(X)$ and we write

$$A\tilde{G}_1(X) = (\tilde{C}(X) \times_{\mathbb{Z}/2} \tilde{C}(X))/ \approx .$$

It is easy to see that there is a non-trivial double covering

$$p : A\tilde{G}_1(X) \to AG_1(X).$$

Let

$$A\tilde{G}_{1,0}{}^d(X) = p^{-1}(AG_{1,0}^d(X)).$$

If $d \geq 3$, there is a non-trivial double covering

$$p : A\tilde{G}_{1,0}{}^d(X) \to AG_{1,0}^d(X).$$

The following result follows from (4.1):

Lemma 4.3. *The map $p : A\tilde{G}_{1,0}{}^d(\mathbb{C}) \to AG_{1,0}^d(\mathbb{C})$ is a universal covering.* $\square$

Recall the following result, proved in [GKY3]:

Proposition 4.4 ([GKY3]**).** *Let $U = \mathbb{C} - \{\text{finite points}\}$. The stabilization map*

$$\tilde{C}_d(U) \to \tilde{C}_{d+1}(U)$$

is a homology equivalence up to dimension $[(d-1)/3]$. $\square$

By means of methods analogous to the ones we used to prove (3.3) and (3.4) we can obtain from 4.4 the following results, whose proofs we omit.

Lemma 4.5. *Let $U = \mathbb{C} - \{$finite points$\}$. Let*

$$\tilde{C}_d^{\xi}(U) = \{[\eta] \in \tilde{C}_d(U) : \eta \cap \xi \neq \emptyset\}.$$

Then the stabilization map

$$\tilde{C}_d^{\xi}(U) \to \tilde{C}_{d+1}^{\xi}(U)$$

is a homology equivalence up to dimension $[(d-2)/3]$. $\square$

Lemma 4.6. *Let*

$$\tilde{D}_{d_1,d_2} = \{[\xi_1],[\xi_2]) \in \tilde{C}_{d_1}(\mathbb{C}) \times \tilde{C}_{d_2}(\mathbb{C}) : \xi_1 \cap \xi_2 \neq \emptyset\}.$$

(1) The stabilization map

$$\tilde{D}_{d_1,d_2} \to \tilde{D}_{d_1+1,d_2}$$

is a homology equivalence up to dimension $[(d_1-2)/3]$.

(1) The stabilization map

$$\tilde{D}_{d_1,d_2} \to \tilde{D}_{d_1,d_2+1}$$

is a homology equivalence up to dimension $[(d_2-2)/3]$. $\square$

Now we can prove (2) of theorem 3.1.

Proof of Theorem 3.1 (2).

Since $A\tilde{G}_{1,0}{}^d(\mathbb{C}) \to AG_{1,0}^d(\mathbb{C})$ is a universal covering, it suffices to show that the inclusion map $A\tilde{G}_{1,0}{}^d(\mathbb{C}) \to A\tilde{G}_{1,0}{}^{d+1}(\mathbb{C})$ is a homology equivalence up to dimension $[(d-1)/3]$.

Recall that there is a homeomorphism

$$A\tilde{G}_{1,0}{}^d(\mathbb{C}) \cong A\tilde{G}_{1,0}{}^d(\mathbb{C}) \cup_f (\tilde{C}_{d+1}(\mathbb{C}) \times_{\mathbb{Z}/2} \tilde{C}_{d+1}(\mathbb{C})).$$

Hence

$$A\tilde{G}_{1,0}{}^{d+1}(\mathbb{C})/A\tilde{G}_{1,0}{}^d(\mathbb{C}) \cong (\tilde{C}_{d+1}(\mathbb{C}) \times_{\mathbb{Z}/2} \tilde{C}_{d+1}(\mathbb{C}))/\hat{D}_{d+1,d+1}$$

where we take

$$\hat{D}_{d_1,d_2} = \{([\xi_1],[\xi_2]) \in \tilde{C}_{d_1}(\mathbb{C}) \times_{\mathbb{Z}/2} \tilde{C}_{d_2}(\mathbb{C}) : \xi_1 \cap \xi_2 \neq \emptyset\}.$$

Hence it also suffices to show that the inclusion

$$\hat{D}_{d+1,d+1} \to \tilde{C}_{d+1}(\mathbb{C}) \times_{\mathbb{Z}/2} \tilde{C}_{d+1}(\mathbb{C})$$

is a homology equivalence up to dimension $[(d-1)/3]$. By a method similar to the one given in the proof of (1) of theorem 3.1, this assertion follows from:

Lemma 4.7. *(1) The stabilization map*

$$\hat{D}_{d_1,d_2} \to \hat{D}_{d_1+1,d_2}$$

is a homology equivalence up to dimension $[(d_1-2)/3]$.

(2) The stabilization map

$$\hat{D}_{d_1,d_2} \to \hat{D}_{d_1,d_2+1}$$

is a homology equivalence up to dimension $[(d_2-2)/3]$.

Proof of (4.7). We only prove (1) since (2) can be proved in an analogous way. Consider the commutative diagram

$$\begin{array}{ccc}
\tilde{D}_{d_1,d_2} & \xrightarrow{s'} & \tilde{D}_{d_1+1,d_2} \\
\downarrow & & \downarrow \\
\hat{D}_{d_1,d_2} & \xrightarrow{s} & \hat{D}_{d_1+1,d_2} \\
\downarrow & & \downarrow \\
B\mathbb{Z}/2 & \xrightarrow{=} & B\mathbb{Z}/2
\end{array}$$

where vertical sequences are fibrations.

Since s' is a homology equivalence up to dimension $[(d_1-2)/3]$ from (4.6), s is also a homology equivalence up to dimension $[(d_1-2)/3]$. □

Hence this completes the proof of theorem 3.1. □

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