STABLE SPLITTING OF THE SPACE OF POLYNOMIALS WITH ROOTS OF BOUNDED MULTIPLICITY

M. A. Guest, A. Kozlowski, and K. Yamaguchi

§1. Introduction.

The motivation for this paper derives from the work of F. Cohen, R. Cohen, B. Mann and R. Milgram ([5], [6]) and that of V. Vassiliev ([15]). The former gives a description of the stable homotopy type of the space of basepoint preserving holomorphic maps of degree d from the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ to the complex projective space $\mathbb{C}P^m$. We denote this space by $\mathrm{Hol}_d^*(S^2, \mathbb{C}P^m)$. Let $D_j = F(\mathbb{C}, j)_+ \wedge_{\Sigma_j} S^j$ be the j-th subquotient of the May-Milgram model for $\Omega^2 S^3$ ([11], [14]), where $F(X, j)$ denotes the configuration space of j disjoint points in X,

$$F(X, j) = \{(x_1, \cdots, x_j) \in X^j : x_i \neq x_j \text{ if } i \neq j\},$$

$F(X, j)_+ = F(X, j) \cup \{*\}$ ($*$ is a disjoint base point) and Σ_j is the symmetric group on j letters which acts on both $F(X, j)$ and the j-sphere $S^j = S^1 \wedge S^1 \cdots \wedge S^1$ by permuting coordinates.

Cohen, Cohen, Mann and Milgram proved

Theorem ([5], [6]). *There is a stable homotopy equivalence*

$$\mathrm{Hol}_d^*(S^2, \mathbb{C}P^{n-1}) \simeq_s \vee_{j=1}^d \Sigma^{2(n-2)j} D_j,$$

where Σ^k denotes the k fold reduced suspension.

On the other hand, Vassiliev studied the space $\mathrm{SP}_n^d(\mathbb{C})$ consisting of all monic complex polynomials $g(z) = z^d + a_1 z^{d-1} + \cdots + a_{d-1} z + a_d$ $(a_j \in \mathbb{C})$ of degree d without roots of multiplicity $\geq n$ and proved

Theorem ([15]). *There is a stable homotopy equivalence*

$$\mathrm{Hol}_d^*(S^2, \mathbb{C}P^{n-1}) \simeq_s \mathrm{SP}_n^{dn}(\mathbb{C}).$$

Remark. Let $C_d(X)$ denote the quotient space $C_d(X) = F(X, d)/\Sigma_d$. Then since $\mathrm{SP}_2^d(\mathbb{C}) = C_d(\mathbb{C})$ and there is a stable homotopy equivalence $C_{2d}(\mathbb{C}) \simeq_s \vee_{j=1}^d D_j$ ([3]), the above two results coincide when $n = 2$. However, it is easy to see that they do not coincide when $n \geq 3$.

Combining these two theorems we see that $\mathrm{SP}_n^{dn}(\mathbb{C})$ and $\vee_{j=1}^d \Sigma^{2(n-2)j} D_j$ are stable homotopy equivalent. This raises the problem of establishing this equivalence directly. The first aim of this paper is to do just that. In other words, in this paper we shall prove, without using the above results, the following:

Theorem 1. *There is a stable homotopy equivalence*

$$f_d : \vee_{j=1}^{d} \Sigma^{2(n-2)j} D_j \xrightarrow{\simeq_s} \mathrm{SP}_n^{dn}(\mathbb{C}).$$

We prove this basically by imitating the method of [5] with $\mathrm{Hol}_d^*(S^2, \mathbb{C}P^{n-1})$ replaced by $\mathrm{SP}_n^{dn}(\mathbb{C})$. One virtue of this approach is that we can then apply the method of R. Cohen and D. Shimamoto ([7]), to obtain immediately the following stronger version of Vassiliev's theorem:[1]

Theorem 2. *If $n \geq 3$, there is a homotopy equivalence*

$$\mathrm{SP}_n^d(\mathbb{C}) \simeq \mathrm{Hol}_{[d/n]}^*(S^2, \mathbb{C}P^{n-1}),$$

where $[x]$ denotes the integer part of x.

Corollary 3 **([10]).** *Let $n \geq 3$. Then there is a map*

$$\mathrm{SP}_n^d(\mathbb{C}) \to \Omega_0^2 \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$$

which is a homotopy equivalence up to dimension $(2n-3)[d/n]$.

First we recall a few definitions and results. Let $\mathrm{SP}_n^d(|z| < d)$ denote the subspace of $\mathrm{SP}_n^d(\mathbb{C})$ consisting of all polynomials $g(z)$ all of whose roots are contained in $\{|z| < d\}$. We may identify $\mathrm{SP}_n^d(\mathbb{C}) \cong \mathrm{SP}_n^d(|z| < d)$ in a natural way. Let $\alpha \in \mathbb{C}$ be any fixed number such that $|\alpha| > d$. Define the stabilization map $\mathrm{SP}_n^d(\mathbb{C}) \to \mathrm{SP}_n^{d+1}(\mathbb{C})$ by

$$\begin{array}{ccccc} \mathrm{SP}_n^d(\mathbb{C}) & \xrightarrow{\cong} & \mathrm{SP}_n^d(|z| < d) & \longrightarrow & \mathrm{SP}_n^{d+1}(\mathbb{C}) \\ & & g(z) & \longrightarrow & g(z) \cdot (z - \alpha) \end{array}$$

Although the definition of of this map depends on the choice of the number α, we only need its homotopy class, which does not. Similarly we can define the stabilization map (homotopy class) $\mathrm{SP}_n^d(\mathbb{C}) \to \mathrm{SP}_n^{d+j}(\mathbb{C})$ as the composite

$$\mathrm{SP}_n^d(\mathbb{C}) \to \mathrm{SP}_n^{d+1}(\mathbb{C}) \to \cdots \to \mathrm{SP}_n^{d+j-1}(\mathbb{C}) \to \mathrm{SP}_n^{d+j}(\mathbb{C})$$

and let $\mathrm{SP}^{d+j}(\mathbb{C}) / \mathrm{SP}^d(\mathbb{C})$ be the mapping cone of the stabilization map $\mathrm{SP}^d(\mathbb{C}) \to \mathrm{SP}^{d+j}(\mathbb{C})$.

Let $T_d : \mathrm{SP}_n^d(\mathbb{C}) \to \Omega_d^2 \mathbb{C}P^{n-1}$ be the *jet* map given by

$$T_d(g)(z) = [g(z) : g'(z) : g''(z) : \cdots : g^{(n-1)}(z)] \quad \text{for } z \in \mathbb{C} \cup \infty = S^2.$$

We shall make use of the following two results of [10]:

Theorem 4 **([10]).** *If $n \geq 3$, the jet embedding induces a homotopy equivalence*

$$T = \lim_{d \to \infty} T_d : \varinjlim_{d} \mathrm{SP}_n^d(\mathbb{C}) \to \Omega_0^2 \mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}.$$

Here the limit is taken over the stabilization maps $\mathrm{SP}_n^d(\mathbb{C}) \to \mathrm{SP}_n^{d+1}(\mathbb{C})$.

Theorem 5 **([1], [10]).** *If $n \geq 3$ and $1 \leq j < n$, then the stabilization map $\mathrm{SP}_n^{dn}(\mathbb{C}) \to \mathrm{SP}_n^{dn+j}(\mathbb{C})$ is a homotopy equivalence.*

[1]The same result is stated in a recent pre-print of S. Kallel

§2. C_2-structures.

In this section we show how to deduce our main results from theorems 4 and 5.

The first step of our argument is to define a C_2-structure on $\mathrm{SP}_n(\mathbb{C})$ in the manner of [2] and [11], where $\mathrm{SP}^0_n(\mathbb{C}) = \{*\}$ and $\mathrm{SP}_n(\mathbb{C}) = \coprod_{d\geq 0} \mathrm{SP}^d_n(\mathbb{C})$.

Definition 2.1. (1) Let $\alpha : \mathbb{C} \xrightarrow{\cong} D_+$ and $\beta : \mathbb{C} \xrightarrow{\cong} D_-$ be fixed homeomorphisms, where:

$$D_+ = \{z \in \mathbb{C} : |z - 2\sqrt{-1}| < 1\} \quad \text{and} \quad D_- = \{z \in \mathbb{C} : |z + 2\sqrt{-1}| < 1\}.$$

For a monic polynomial $f = f(z) = \prod_j (z - \gamma_j) \in \mathbb{C}[z]$, let $\alpha(f)$ and $\beta(f)$ denote the polynomials:

$$\alpha(f) = \prod_j (z - \alpha(\gamma_j)), \quad \beta(f) = \prod_j (z - \beta(\gamma_j)).$$

Define the map $* : \mathrm{SP}^k_n(\mathbb{C}) \times \mathrm{SP}^l_n(\mathbb{C}) \to \mathrm{SP}^{k+l}_n(\mathbb{C})$ by $f(z) * g(z) = \alpha(f) \cdot \beta(g)$.

(2) Let $J^2 = J \times J = (0,1) \times (0,1)$ be an open unit cube in $\mathbb{C} = \mathbb{R}^2$. An open *little* 2-*cube* is an affine embedding $c : J^2 \to J^2$ with parallel axes.

Let $C_2(j)$ be the space of j-tuples $(c_1, \cdots, c_j)$ of open little 2-cubes with mutually disjoint images, i.e.

$$C_2(j) = \{(c_1, \cdots, c_j) : c_i\text{'s are open little 2-cubes, } c_i(J^2) \cap c_k(J^2) = \emptyset \text{ if } i \neq k\}.$$

Define the C_2-structure map $\mathcal{I} : C_2(j) \times_{\Sigma_j} (\mathrm{SP}^d_n(\mathbb{C}))^j \to \mathrm{SP}^{jd}_n(\mathbb{C})$ by

$$((c_1, \cdots, c_j), (f_1, \cdots, f_j)) \mapsto c_1(f_1) * (c_2(f_2) * (c_3(f_3) * (\cdots * (c_j(f_j))) \cdots)$$

where for $f(z) = \prod_i (z - z_i) \in \mathbb{C}[z]$ and an open little 2-cube σ, we let

$$\sigma(f) = \prod_i (z - \sigma(z_i)).$$

Lemma 2.2. *The maps* $\{\mathcal{I} : C_2(j) \times_{\Sigma_j} (\mathrm{SP}^d_n(\mathbb{C}))^j \to \mathrm{SP}^{jd}_n(\mathbb{C})\}$ *induce a (homotopy associative)* C_2*-operad structure on* $\mathrm{SP}_n(\mathbb{C}) = \coprod_{d\geq 0} \mathrm{SP}^d_n(\mathbb{C})$.

Proof. Analogous to (4.12) of [2]. □

Corollary 2.3. *If* $n \geq 3$*, there is a homotopy equivalence*

$$\Omega B(\mathrm{SP}_n(\mathbb{C})) \simeq \Omega^2 \mathbb{C}P^{n-1}.$$

Proof. This follows from the group-completion theorem and theorem 4. □

Definition 2.4. Define the *jet* map $T_d : \mathrm{SP}^d_n(\mathbb{C}) \to \mathrm{Hol}^*_d(S^2, \mathbb{C}P^{n-1}) \subset \Omega^2_d \mathbb{C}P^{n-1}$ by $T_d(f) = (f(z), f'(z), f''(z), \cdots, f^{(n-1)}(z))$.

Let $*' : \mathrm{Hol}^*_{d_1}(S^2, \mathbb{C}P^{n-1}) \times \mathrm{Hol}^*_{d_2}(S^2, \mathbb{C}P^{n-1}) \to \mathrm{Hol}^*_{d_1+d_2}(S^2, \mathbb{C}P^{n-1})$ be the product defined in (4.8) of [2].

Lemma 2.5. *The following diagram is homotopy commutative:*

$$\begin{array}{ccc} \mathrm{SP}^{d_1}_n(\mathbb{C}) \times \mathrm{SP}^{d_2}_n(\mathbb{C}) & \xrightarrow{\ *\ } & \mathrm{SP}^{d_1+d_2}_n(\mathbb{C}) \\ {\scriptstyle T_{d_1} \times T_{d_2}} \downarrow & & {\scriptstyle T_{d_1+d_2}} \downarrow \\ \mathrm{Hol}^*_{d_1}(S^2, \mathbb{C}P^{n-1}) \times \mathrm{Hol}^*_{d_2}(S^2, \mathbb{C}P^{n-1}) & \xrightarrow{\ *'\ } & \mathrm{Hol}^*_{d_1+d_2}(S^2, \mathbb{C}P^{n-1}) \end{array}$$

Proof. Analogous to (4.14) of [2]. $\square$

Lemma 2.6. *The following diagram is homotopy commutative:*

$$\begin{array}{ccc} C_2(j) \times_{\Sigma_j} (\mathrm{SP}^d_n(\mathbb{C}))^j & \xrightarrow{\ \mathcal{I}\ } & \mathrm{SP}^{jd}_n(\mathbb{C}) \\ {\scriptstyle \mathrm{id} \times_{\Sigma_j} (T_d)^j} \downarrow & & {\scriptstyle T_{jd}} \downarrow \\ C_2(j) \times_{\Sigma_j} (\mathrm{Hol}^*_d(S^2, \mathbb{C}P^{n-1}))^j & \xrightarrow{\ \mathcal{I}'\ } & \mathrm{Hol}^*_{jd}(S^2, \mathbb{C}P^{n-1}) \end{array}$$

where $\mathcal{I}'$ is the C_2 operad structure map given in [2], *(4.8).*

Proof. The proof is analogous to (4.16) of [2]. $\square$

We can now turn to the proof of theorem 1. If $n = 2$, there is nothing to prove. So, from now on, we assume that $n \geq 3$ and write $\mathrm{SP}^d_n = \mathrm{SP}^d_n(\mathbb{C})$. First, we consider the case $d = 1$.

Lemma 2.7. *There is a homotopy equivalence $S^{2n-3} \simeq \mathrm{SP}^n_n$.*

Proof. From the definition,

$$\mathrm{SP}^n_n = \{f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \in \mathbb{C}[z] : f(z) \neq (z+\alpha)^n \text{ for any } \alpha \in \mathbb{C}\}.$$

Note that $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n = (z+\alpha)^n$ if and only if

$$a_1 = n\alpha \quad \text{and} \quad a_i = \binom{n}{i}(\frac{a_1}{n})^i \quad \text{for } 2 \leq i \leq n.$$

Consider the map $\pi : \mathrm{SP}^n_n \to \mathbb{C}$ given by

$$z^n + a_1 z^{n-1} + \cdots + a_n \mapsto a_1.$$

For any $\beta \in \mathbb{C}$, taking

$$a_i = \binom{n}{i} \cdot \frac{\beta^i}{n^i} \quad \text{for } 2 \leq i \leq n,$$

defines a canonical homeomorphism

$$\pi^{-1}(\beta) \cong \mathbb{C}^{n-1} - \{(a_2, \cdots, a_n)\} \cong \mathbb{C}^{n-1} - \{0\}.$$

Hence there is a fibration $\mathbb{C}^{n-1} - \{0\} \to \mathrm{SP}^n_n \xrightarrow{\pi} \mathbb{C}$ and a homotopy equivalence $\mathrm{SP}^n_n \cong \mathbb{C} \times (\mathbb{C}^{n-1} - \{0\}) \simeq S^{2n-3}$. $\square$

Recall the following well-known result:

Lemma 2.8 ([4], [14])**.** *(1) There are stable homotopy equivalences*

$$\Omega_0^2\mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1} \simeq_s \vee_{d\geq 1} F(\mathbb{C},d)_+ \wedge (\wedge^d S^{2n-3})$$

and

$$D(n,d) = F(\mathbb{C},d)_+ \wedge_{\Sigma_d} (\wedge^d S^{2n-3}) \simeq_s \Sigma^{2(n-2)d} D_d.$$

(2) The canonical projection

$$F(\mathbb{C},d) \times_{\Sigma_d} (S^{2n-3})^d \to F(\mathbb{C},d)_+ \wedge_{\Sigma_d} (\wedge^d S^{2n-3}) = D(n,d)$$

has a stable section

$$e_d : D(n,d) = F(\mathbb{C},d)_+ \wedge_{\Sigma_d} (\wedge^d S^{2n-3}) \to F(\mathbb{C},d) \times_{\Sigma_d} (S^{2n-3})^d.$$

Theorem 2.9. *Let $j_d : \mathrm{SP}_n^{(d-1)n} \to \mathrm{SP}_n^{dn}$ denote the stabilization map and let $h_d : \Sigma^{2(n-2)d} D_d \to \mathrm{SP}_n^{nd} / \mathrm{SP}_n^{n(d-1)}$ be the stable map given by the composite*

$$\Sigma^{2(n-2)d} D_d \simeq_s D(n,d) \xrightarrow{e_d} F(\mathbb{C},d) \times_{\Sigma_d} (S^{2n-3})^d \simeq$$
$$F(\mathbb{C},d) \times_{\Sigma_d} (\mathrm{SP}_n^n)^d \xrightarrow{\mathcal{I}_d} \mathrm{SP}_n^{nd} \xrightarrow{\mathrm{proj}} \mathrm{SP}_n^{nd} / \mathrm{SP}_n^{n(d-1)}$$

where $\mathcal{I}_d$ is the C_2-structure map. Then $h_d : \Sigma^{2(n-2)d} D_d \xrightarrow{\simeq_s} \mathrm{SP}_n^{nd} / \mathrm{SP}_n^{n(d-1)}$ is a stable homotopy equivalence.

The proof of theorem 2.9 will be given in the next section. Assuming theorem 2.9 we now complete the proofs of theorems 1, 2 and corollary 3.

Proof of theorem 1. Let $f_d : \vee_{1\leq j\leq d} \Sigma^{2(n-2)j} D_j \to \mathrm{SP}_n^{nd}$ be the stable map given by the composite of maps

$$f_d : \vee_{j=1}^d \Sigma^{2(n-2)j} D_j \xrightarrow{\vee e_j} \vee_{j=1}^d (F(\mathbb{C},j) \times_{\Sigma_j} (\mathrm{SP}_n^n)^j) \xrightarrow{\vee \mathcal{I}_j} \vee_{j=1}^d \mathrm{SP}_n^{jn} \xrightarrow{\vee \iota_j} \mathrm{SP}_n^{dn}$$

We want to show that f_d is a stable homotopy equivalence. We proceed by induction on d. Since $D_1 \simeq S^1$, the case $d=1$ follows from lemma 2.7.

Assume that the result holds for $d-1$, i.e. the map

$$f_{d-1} : \vee_{1\leq j\leq d-1} \Sigma^{2(n-2)j} D_j \xrightarrow{\simeq_s} \mathrm{SP}_n^{(d-1)n}$$

is a stable homotopy equivalence.

Note that the stable map $f_d : \vee_{1\le j\le d}\Sigma^{2(n-2)j}D_j \to \mathrm{SP}^{nd}_n$ is equal to the stable map

$$\begin{aligned}&\vee_{1\le j\le d}\,\Sigma^{2(n-2)j}D_j\\ &= (\vee_{1\le j\le d-1}\Sigma^{2(n-2)j}D_j)\vee\Sigma^{2(n-2)d}D_d \xrightarrow{f_{d-1}\vee\mathcal{I}_d\circ e_d} \mathrm{SP}^{(d-1)n}_n\vee\mathrm{SP}^{dn}_n\\ &\xrightarrow{j_d\vee\mathrm{id}} \mathrm{SP}^{dn}_n\vee\mathrm{SP}^{dn}_n \xrightarrow{\text{folding map}} \mathrm{SP}^{dn}_n\end{aligned}$$

where $j_d : \mathrm{SP}^{(d-1)n}_n \to \mathrm{SP}^{dn}_n$ is the stabilization map and the map $\mathcal{I}_d\circ e_d$ is the composite

$$\begin{aligned}\Sigma^{2(n-2)d}D_d \simeq_s D(n,d) &\xrightarrow{e_d} F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d\\ &\simeq F(\mathbb{C},d)\times_{\Sigma_d}(\mathrm{SP}^n_n)^d \xrightarrow{\mathcal{I}_d} \mathrm{SP}^{nd}_n.\end{aligned}$$

Now we can see that the diagram

$$\begin{array}{ccccc}\vee_{1\le j\le d-1}\Sigma^{2(n-2)j}D_j & \xrightarrow{\subset} & \vee_{1\le j\le d}\Sigma^{2(n-2)j}D_j & \longrightarrow & \Sigma^{2(n-2)d}D_d\\ f_{d-1}\big\downarrow\simeq_s & & f_d\big\downarrow & & h_d\big\downarrow\simeq_s\\ \mathrm{SP}^{(d-1)n}_n & \xrightarrow{j_d} & \mathrm{SP}^{nd}_n & \xrightarrow{\text{proj}} & \mathrm{SP}^{nd}_n/\mathrm{SP}^{(d-1)n}_n\end{array}$$

where the horizontal sequences are cofibrations, is homotopy commutative.

Since f_{d-1} and h_d are stable homotopy equivalences, f_d is also a stable homotopy equivalence. □

Let $J_2(X)$ denote the May-Milgram model for $\Omega^2\Sigma^2X$ ([11])

$$J_2(X) = (\coprod_{j\ge1}F(\mathbb{C},j)\times_{\Sigma_j}X^j)/\sim$$

and let $J_2(X)_d\subset J_2(X)$ be the subspace

$$\begin{aligned}J_2(X)_d = (\coprod_{1\le j\le d}F(\mathbb{C},j)\times_{\Sigma_j}X^j)/\sim\\ \subset J_2(X)\simeq\Omega^2\Sigma^2X.\end{aligned}$$

where $\sim$ denotes the well known equivalence relation.

Proof of theorem 2. It follows from theorem 5 that it suffices to prove that there is a homotopy equivalence

$$\mathrm{SP}^{dn}_n \simeq \mathrm{Hol}^*_d(S^2,\mathbb{C}P^{n-1}).$$

Since the C_2 structure of the SP^d_n's is compatible with that induced from the double loop sums, the maps $\mathcal{I}_j$ induce a map $\epsilon_d : J_2(S^{2n-3})_d \to \mathrm{SP}^{dn}_n$ such that the diagram

$$\begin{array}{ccc} \vee_{j=1}^d (F(\mathbb{C},j)\times_{\Sigma_j}(\mathrm{SP}^n_n)^j) & \xrightarrow{\vee \mathcal{I}_j} & \vee_{j=1}^d \mathrm{SP}^{dn}_n \\ \vee q_j \big\downarrow & & \vee \iota_j \big\downarrow \\ J_2(S^{2n-3})_d & \xrightarrow{\epsilon_d} & \mathrm{SP}^{dn}_n \end{array}$$

is homotopy commutative. Since the stable maps e_j are stable sections of the Snaith splitting, the stable map

$$J = (\vee q_j)\circ(\vee e_j) : \vee_{j=1}^d \Sigma^{2(n-2)j} D_j \xrightarrow{\simeq_s} J_2(S^{2n-3})_d$$

is a stable homotopy equivalence.

Consider the (stable homotopy commutative) diagram

$$\begin{array}{ccccc} \vee_{j=1}^d \Sigma^{2(n-2)j} D_j & \xrightarrow{\vee e_j} & \vee_{j=1}^d (F(\mathbb{C},j)\times_{\Sigma_j}(\mathrm{SP}^n_n)^j) & \xrightarrow{\vee \mathcal{I}_j} & \vee_{j=1}^d \mathrm{SP}^{dn}_n \\ = \big\downarrow & & \vee q_j \big\downarrow & & \vee \iota_j \big\downarrow \\ \vee_{j=1}^d \Sigma^{2(n-2)j} D_j & \underset{\simeq_s}{\xrightarrow{J}} & J_2(S^{2n-3})_d & \xrightarrow{\epsilon_d} & \mathrm{SP}^{dn}_n \end{array}$$

Since the stable maps

$$f_d = (\vee \iota_j)\circ(\vee \mathcal{I}_j)\circ(\vee e_j) : \vee_{j=1}^d \Sigma^{2(n-2)j} D_j \xrightarrow{\simeq_s} \mathrm{SP}^{dn}_n$$

and

$$J = (\vee q_j)\circ(\vee e_j) : \vee_{j=1}^d \Sigma^{2(n-2)j} D_j \xrightarrow{\simeq_s} J_2(S^{2n-3})_d$$

are both stable homotopy equivalences, the map ϵ_d is also a stable homotopy equivalence. Hence the induced homomorphism

$$(\epsilon_d)_* : H_*(J_2(S^{2n-3})_d, \mathbb{Z}) \xrightarrow{\cong} H_*(\mathrm{SP}^{dn}_n, \mathbb{Z})$$

is an isomorphism. Since both spaces $J_2(S^{2n-3})_d$ and SP^{dn}_n are simply connected, the map

$$\epsilon_d : J_2(S^{2n-3})_d \xrightarrow{\simeq} \mathrm{SP}^{dn}_n$$

is a homotopy equivalence.

On the other hand, it follows from [7] that there is a homotopy equivalence

$$J_2(S^{2n-3})_d \simeq \mathrm{Hol}^*_d(S^2, \mathbb{C}P^{n-1}).$$

Hence there is a homotopy equivalence $\mathrm{SP}^{dn}_n \simeq \mathrm{Hol}^*_d(S^2, \mathbb{C}P^{n-1})$. □

Proof of corollary 3. Since the homotopy equivalence given in theorem 2 is natural, there is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathrm{SP}^d_n & \xrightarrow{\iota_d} & \lim_{d'\to\infty} \mathrm{SP}^{d'}_n & \xrightarrow{\simeq} & \Omega^2 S^{2n-1} \\ \simeq \big\downarrow & & & & = \big\downarrow \\ \mathrm{Hol}^*_{[d/n]}(S^2, \mathbb{C}P^{n-1}) & \longrightarrow & \lim_{d'\to\infty} \mathrm{Hol}^*_{d'}(S^2, \mathbb{C}P^{n-1}) & \xrightarrow{\simeq} & \Omega^2 S^{2n-1} \end{array}$$

Since the bottom horizontal map $\mathrm{Hol}^*_{[d/n]}(S^2, \mathbb{C}P^{n-1}) \to \Omega^2 S^{2n-1}$ is a homotopy equivalence up to dimension $(2n-3)[d/n]$ from the main result of Segal ([13]), the result follows. □

§3. Proof of theorem 2.9.

In this section we shall prove theorem 2.9.

Lemma 3.1. *If* $i_d : \mathrm{Hol}_d^*(S^2, \mathbb{C}P^{n-1}) \to \Omega_d^2\mathbb{C}P^{n-1}$ *is the inclusion map, the map*

$$\coprod_d (i_{nd}\circ T_{nd}) : \coprod_{d\geq 0} \mathrm{SP}_n^{nd} \to \coprod_{d\in\mathbb{Z}} \Omega_d^2\mathbb{C}P^{n-1} = \Omega^2\mathbb{C}P^{n-1}$$

is a C_2*-map up to homotopy.*

Proof. Analogous to (4.16) of [2]. □

Let

$$\mathrm{SP}_n^\infty = \lim_{d\to\infty} \mathrm{SP}_n^{dn}(\mathbb{C}) = \lim_{d\to\infty} \mathrm{SP}_n^{dn}$$

be the (homotopy) limit induced by the stabilization maps

$$\mathrm{SP}_n^n \to \mathrm{SP}_n^{2n} \to \mathrm{SP}_n^{3n} \to \mathrm{SP}_n^{4n} \to \cdots\cdots$$

and let $\iota_d : \mathrm{SP}_n^{dn} \to \mathrm{SP}_n^\infty$ be the natural inclusion map.

Lemma 3.2. *There is a homotopy commutative diagram*

$$\begin{array}{ccccc}
\vee_{d=1}^\infty F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d & \xrightarrow{\simeq} & \vee_{d=1}^\infty F(\mathbb{C},d)\times_{\Sigma_d}(\mathrm{SP}_n^n)^d & \xrightarrow{\vee\mathcal{I}_d} & \vee_{d=1}^\infty \mathrm{SP}_n^{nd} \\
\vee q_d \downarrow & & & & \vee\iota_d \downarrow \\
J_2(S^{2n-3}) \simeq \Omega^2 S^{2n-1} & \xrightarrow{\simeq} & \Omega_0^2\mathbb{C}P^{n-1} & \xleftarrow{\simeq} & \mathrm{SP}_n^\infty
\end{array}$$

where $q_d : F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d \to J_2(S^{2n-3})$ *denotes the natural projection map.*

Proof. It follows from lemma 3.1 and the group completion theorem that there is an induced C_2-map

$$\tilde{j} : \mathrm{SP}_n^\infty \xrightarrow{\simeq} \Omega_0^2\mathbb{C}P^{n-1} \simeq \Omega^2 S^{2n-1}$$

such that the diagram

(a)
$$\begin{array}{ccc}
J_2(\mathrm{SP}_n^\infty) & \xrightarrow[\simeq]{C(\tilde{j})} & J_2(\Omega^2 S^{2n-1}) \\
r_1 \downarrow & & r_2 \downarrow \\
\mathrm{SP}_n^\infty & \xrightarrow[\simeq]{\tilde{j}} & \Omega^2 S^{2n-1}
\end{array}$$

is homotopy commutative, where r_1 and r_2 are natural retractions. Note that, by theorem 3, $\tilde{j}$ is also a homotopy equivalence.

Similarly, since $\mathrm{SP}^n_n \simeq S^{2n-3}$, it follows from lemma 3.1 that the diagram

$$\text{(b)}\qquad \begin{array}{ccccc} \vee_{d=1}^{\infty} F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d & \xrightarrow{\vee q_d} & J_2(S^{2n-3}) & \xrightarrow{J_2(\iota)} & J_2(\mathrm{SP}^{\infty}_n) \\ \downarrow{\scriptstyle \vee\mathcal{I}_d} & & & & \downarrow{\scriptstyle r_1} \\ \vee_{d=1}^{\infty}\mathrm{SP}^{dn}_n & & & \xrightarrow{\vee\iota_d} & \mathrm{SP}^{\infty}_n \end{array}$$

is homotopy commutative, where $\iota : S^{2n-3} \simeq \mathrm{SP}^n_n \xrightarrow{\iota_1} \mathrm{SP}^{\infty}_n$ denotes the natural inclusion map.

It follows from (a) and (b) that the following diagram is also homotopy commutative:

$$\begin{array}{ccccccc} \vee_{d=1}^{\infty} F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d & \xrightarrow{\vee q_d} & J_2(S^{2n-3}) & \xrightarrow{J_2(\iota)} & J_2(\mathrm{SP}^{\infty}_n) & \xrightarrow{J_2(\tilde{j})} & J_2(\Omega^2 S^{2n-1}) \\ \downarrow{\scriptstyle \vee\mathcal{I}_d} & & & & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle r_2} \\ \vee_{d=1}^{\infty}\mathrm{SP}^{nd}_n & & & \xrightarrow{\vee\iota_d} & \mathrm{SP}^{\infty}_n & \underset{\simeq}{\xrightarrow{\tilde{j}}} & \Omega^2 S^{2n-1} \end{array}$$

Since the homotopy class of the map $S^{2n-3} \xrightarrow{\iota} \mathrm{SP}^{\infty}_n \underset{\simeq}{\xrightarrow{\tilde{j}}} \Omega^2 S^{2n-1}$ is the generator of $\pi_{2n-3}(\Omega^2 S^{2n-1}) \cong \mathbb{Z}$, this map is homotopic to the natural inclusion of the bottom cell $E^2 : S^{2n-3} \to \Omega^2 S^{2n-1}$. Hence there is a homotopy commutative diagram

$$\begin{array}{ccc} J_2(S^{2n-3}) & \xrightarrow{J_2(\tilde{j}\circ\iota)} & J_2(\Omega^2 S^{2n-1}) \\ \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle r_2} \\ J_2(S^{2n-3}) & \xrightarrow{\simeq} & \Omega^2 S^{2n-1} \end{array}$$

Hence the map

$$J_2(S^{2n-3}) \xrightarrow{J_2(\tilde{j})\circ J_2(\iota)} J_2(\Omega^2 S^{2n-1}) \xrightarrow{r_2} \Omega^2 S^{2n-1}$$

is homotopic to the natural homotopy equivalence $J_2(S^{2n-3}) \xrightarrow{\simeq} \Omega^2 S^{2n-1}$. Thus the above diagram reduces to the diagram in the statement of the lemma. □

Lemma 3.3. *The stable map*

$$(\vee\iota_d)\circ(\vee\mathcal{I}_d\circ e_d):\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d\to\vee_{d=1}^{\infty}\mathrm{SP}_n^{nd}\to\mathrm{SP}_n^{\infty}$$

is a stable homotopy equivalence.

Proof. Consider the homotopy commutative diagram of lemma 3.2:

$$\begin{array}{ccccc}
\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d & & & & \\
\downarrow{\scriptstyle \vee e_d} & & & & \\
\vee_{d=1}^{\infty}F(\mathbb{C},d)\times_{\Sigma_d}(S^{2n-3})^d & \xrightarrow{\simeq} & \vee_{d=1}^{\infty}F(\mathbb{C},d)\times_{\Sigma_d}(\mathrm{SP}_n^n)^d & \xrightarrow{\vee\mathcal{I}_d} & \vee_{d=1}^{\infty}\mathrm{SP}_n^{nd} \\
\downarrow{\scriptstyle \vee q_d} & & & & \downarrow{\scriptstyle \vee\iota_d} \\
J_2(S^{2n-3})\simeq\Omega^2S^{2n-1} & \xrightarrow{\simeq} & \Omega_0^2\mathbb{C}P^{n-1} & \xleftarrow{\simeq} & \mathrm{SP}_n^{\infty}
\end{array}$$

Since the e_d's are stable sections of the Snaith splitting $\Omega^2S^{2n-1}\simeq_s\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d$, the map $(\vee q_d)\circ(\vee e_d):\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d\xrightarrow{\simeq_s}\Omega^2S^{2n-1}$ is a stable homotopy equivalence. Hence the map

$$(\vee\iota_d)\circ(\vee(\mathcal{I}_d\circ e_d)):\vee_{d=1}^{\infty}\Sigma^{(2(n-2)d}D_d\to\vee_{d=1}^{\infty}\mathrm{SP}_n^{nd}\to\mathrm{SP}_n^{\infty}$$

is also a stable homotopy equivalence. $\Box$

The following lemma is the key to the proof of theorem 2.9.

Lemma 3.4. *(1) The induced homomorphism $(j_d)_*:H_*(\mathrm{SP}_n^{(d-1)n},\mathbb{Z})\to H_*(\mathrm{SP}_n^{dn},\mathbb{Z})$ is injective.*

(2) The induced homomorphism

$$(h_d)_*:H_*(\Sigma^{2(n-2)d}D_d,F)\to H_*(\mathrm{SP}_n^{dn}/\mathrm{SP}_n^{(d-1)n},F)$$

is injective for $F=\mathbb{Q}$ or $\mathbb{Z}/p$ (p: any prime).

We shall prove theorem 2.9 using lemma 3.4, whose proof will be postponed to the next section.

Proof of theorem 2.9.

Let $F = \mathbb{Q}$ or $F = \mathbb{Z}/p$ (p: any prime). It follows from the Snaith splitting, (1) of lemma 3.4 and theorems 3, 4 that there is an isomorphism of F-vector spaces

$$H_*(\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d, F) \cong H_*(\vee_{d=1}^{\infty}\,\mathrm{SP}_n^{dn}\,/\,\mathrm{SP}_n^{(d-1)n}, F).$$

Hence for each j

$$\dim_F H_j(\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d, F) = \dim_F H_j(\vee_{d=1}^{\infty}\,\mathrm{SP}_n^{dn}\,/\,\mathrm{SP}_n^{(d-1)n}, F) < \infty.$$

However, from (2) of lemma 3.4

$$(\vee h_d)_* : H_*(\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d, F) \to H_*(\vee_{d=1}^{\infty}\,\mathrm{SP}_n^{dn}\,/\,\mathrm{SP}_n^{(d-1)n}, F)$$

is injective and so that

$$(\vee h_d)_* : H_*(\vee_{d=1}^{\infty}\Sigma^{2(n-2)d}D_d, F) \xrightarrow{\cong} H_*(\vee_{d=1}^{\infty}\,\mathrm{SP}_n^{dn}\,/\,\mathrm{SP}_n^{(d-1)n}, F)$$

is an isomorphism. Hence

$$(h_d)_* : H_*(\Sigma^{2(n-2)d}D_d, F) \xrightarrow{\cong} H_*(\mathrm{SP}_n^{dn}\,/\,\mathrm{SP}_n^{(d-1)n}, F)$$

is also an isomorphism. Thus from the universal coefficient theorem, h_d induces an isomorphism on integral homology. Hence h_d is a stable homotopy equivalence. $\square$

§4. Transfer homomorphisms.

In this section we shall prove lemma 3.4. For this purpose, we use Dold-type transfer homomorphisms ([8]).

For a based space (X, x_0), let $\mathrm{Sp}^{\infty}(X)$ denote the infinite symmetric product

$$\mathrm{Sp}^{\infty}(X) = \lim_{d\to\infty} X^d/\Sigma_d.$$

An element of $\mathrm{Sp}^{\infty}(X)$ may be thought of as a formal finite sum $\alpha = \sum_j x_j$, where $x_j \in X$.

Assume that $n \geq 3$. Then by theorem 5, $\mathrm{SP}_n^{(d-1)n} \simeq \mathrm{SP}_n^{dn-1}$.

Define the transfer map

$$\tau : \mathrm{SP}_n^{dn} \to \mathrm{Sp}^{\infty}(\mathrm{SP}_n^{dn-1}) \simeq \mathrm{Sp}^{\infty}(\mathrm{SP}_n^{(d-1)n})$$

by

$$f(z) = \prod_{j=1}^{dn}(z-\alpha_j) \mapsto \sum_{i=1}^{dn}\prod_{j=1, j\neq i}^{dn}(z-\alpha_j)$$

The map τ naturally extends to a homomorphism of abelian monoids

$$\tau_{d-1} : \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \to \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n)$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{SP}^{dn}_n & \xrightarrow{=} & \mathrm{SP}^{dn}_n \\ \cap\downarrow & & \tau\downarrow \\ \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) & \xrightarrow{\tau_{d-1}} & \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n) \end{array}$$

The next result follows easily from the definition.

Lemma 4.1. *The diagram*

$$\begin{array}{ccc} \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n) & \xrightarrow{j_d} & \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \\ \mathrm{proj}\downarrow & & \tau_{d-1}\downarrow \\ \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n\,/\,\mathrm{SP}^{(d-1)n}_n) & \xleftarrow{\mathrm{proj}} & \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n) \end{array}$$

is homotopy commutative.

For $0 \leq j \leq d$, define the transfer map $\tau_{d,j} : \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \to \mathrm{Sp}^\infty(\mathrm{SP}^{jn}_n)$ as the composite

$$\mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \xrightarrow{\tau_{d-1}} \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n) \xrightarrow{\tau_{d-2}} \cdots \to \mathrm{Sp}^\infty(\mathrm{SP}^{(j+1)n}_n) \xrightarrow{\tau_j} \mathrm{Sp}^\infty(\mathrm{SP}^{jn}_n),$$

where we take $\tau_{d,d} = \mathrm{id}$.

Lemma 4.2. *(1) The induced homomorphism*

$$(j_d)_* : H_*(\mathrm{SP}^{(d-1)n}_n, \mathbb{Z}) \to H_*(\mathrm{SP}^{dn}_n, \mathbb{Z})$$

is injective.

(2) The induced homomorphism, proj $\circ\, (\tau_{d,j})_*$*:*

$$\tilde{H}_*(\mathrm{SP}^{dn}_n, \mathbb{Z}) \xrightarrow{\cong} \oplus_{0\leq k\leq d}\tilde{H}_*(\mathrm{SP}^{kn}_n, \mathbb{Z})/Im\ [(j_k)_* : \tilde{H}_*(\mathrm{SP}^{(k-1)n}_n) \to \tilde{H}_*(\mathrm{SP}^{kn}_n)]$$

is an isomorphism.

Proof. It is well-known that if X is connected $\pi_j(\mathrm{Sp}^\infty(X)) \cong \tilde{H}_j(X, \mathbb{Z})$. It follows from lemma 4.1 that $(\tau_{d,k})_* \circ (j_d)_* \equiv (\tau_{d-1,k})_*$ (mod Im $(j_k)_*$) and $\tau_{d,d} = \mathrm{id}$. Then the assertion follows from lemma 2 of [8]. $\square$

Corollary 4.3. *(1) There is a homotopy equivalence*

$$\mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \xrightarrow[\simeq]{\prod \tilde{\tau}_{d,k}} \prod_{k=1}^{d} \mathrm{Sp}^\infty(\mathrm{SP}^{kn}_n \,/\, \mathrm{SP}^{(k-1)n}_n)$$

where the map $\tilde{\tau}_{d,k}$ is the composite

$$\mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \xrightarrow{\tau_{d,k}} \mathrm{Sp}^\infty(\mathrm{SP}^{kn}_n) \xrightarrow{\mathrm{proj}} \mathrm{Sp}^\infty(\mathrm{SP}^{kn}_n \,/\, \mathrm{SP}^{(k-1)n}_n)$$

(2) In particular, there is a homotopy equivalence

$$\mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \xrightarrow[\simeq]{\mathrm{proj}\times\tau_{d,d-1}} \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n \,/\, \mathrm{SP}^{(d-1)n}_n) \times \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n).$$

Lemma 4.4. *The stable map*

$$\tau_{d,d-1} \circ \mathrm{Sp}^\infty(\mathcal{I}_d) \circ \mathrm{Sp}^\infty(e_d) : \mathrm{Sp}^\infty(\Sigma^{2(n-2)d} D_d) \xrightarrow{\mathrm{Sp}^\infty(e_d)} \mathrm{Sp}^\infty(F(\mathbb{C},d) \times_{\Sigma_d} (S^{2n-3})^d)$$
$$\xrightarrow{\mathrm{Sp}^\infty(\mathcal{I}_d)} \mathrm{Sp}^\infty(\mathrm{SP}^{dn}_n) \xrightarrow{\tau_{d,d-1}} \mathrm{Sp}^\infty(\mathrm{SP}^{(d-1)n}_n)$$

is null-homotopic.

Assuming lemma 4.4, we can prove lemma 3.4.

Proof of lemma 3.4. The assertion (1) was already proved in (1) of lemma 4.2 and it suffices to prove (2).

It follows from lemma 3.3 that the induced homomorphism

$$H_*(\Sigma^{2(n-2)d} D_d) \xrightarrow{(\mathcal{I}_d \circ e_d)_*} H_*(\mathrm{SP}^{dn}_n)$$

is injective. Consider the composite of homomorphisms

$$\begin{array}{ccc} H_*(\Sigma^{2(n-2)d} D_d) & \xrightarrow[\text{injective}]{(\mathcal{I}_d \circ e_d)_*} & H_*(\mathrm{SP}^{dn}_n) \\ & & \cong \big\downarrow (\mathrm{proj}_*, (\tau_{d,d-1})_*) \\ & & H_*(\mathrm{SP}^{dn}_n \,/\, \mathrm{SP}^{(d-1)n}_n) \oplus H_*(\mathrm{SP}^{(d-1)n}_n) \end{array}$$

Notice that the second homomorphism $(\mathrm{proj}_*, (\tau_{d,d-1})_*)$ is an isomorphism (by corollary 4.3) and that $(\tau_{d,d-1})_* \circ (\mathcal{I}_d \circ e_d)_* = 0$ (by lemma 4.4). Hence the induced homomorphism

$$(h_d)_* : H_*(\Sigma^{2(n-2)d} D_d) \xrightarrow{(\mathcal{I}_d \circ e_d)_*} H_*(\mathrm{SP}^{dn}_n) \xrightarrow{\mathrm{proj}_*} H_*(\mathrm{SP}^{dn}_n \,/\, \mathrm{SP}^{(d-1)n}_n)$$

is injective and this completes the proof. □

Now it remains to prove lemma 4.4. For this purpose, we recall the relation between transfers and covering projections.

Definition 4.5. Assume that $1 \le j < d$.

(1) Let

$$q_{d,j} : F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^d \to F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d$$

denote the natural covering projection corresponding to the subgroup $\Sigma_j \times \Sigma_{d-j} \subset \Sigma_d$. Define the transfer map for $q_{d,j}$,

$$\sigma : F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d \to \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^d)$$

by

$$\sigma(x) = \sum_{\tilde{x} \in q_{d,j}^{-1}(x)} \tilde{x}.$$

(2) Let $\rho_j : F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^d \to F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j$ denote the projection map onto the first j coordinates of $(S^{2n-3})^d$. Define a map

$$\sigma_j : F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d \to \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j)$$

by $\sigma_j = \mathrm{Sp}^\infty(\rho_j) \circ \sigma$.

The map σ_j naturally extends to a map

$$\tilde{\sigma}_j : \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d) \to \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j)$$

by the usual addition: $\tilde{\sigma}_j(\sum_i x_i) = \sum_i \sigma_j(x_i)$.

(3) Define a C_2-structure map

$$\mathcal{I}_{d,j} : F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j \to \mathrm{SP}_n^{jn}$$

similarly to the way $\mathcal{I}_d$ was defined.

The following is easy to verify:

Lemma 4.6. *Let $1 \le j < d$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d) & \xrightarrow{\tilde{\sigma}_j} & \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j) \\
{\scriptstyle \mathrm{Sp}^\infty(\mathcal{I}_d)} \downarrow & & \downarrow {\scriptstyle \mathrm{Sp}^\infty(\mathcal{I}_{d,j})} \\
\mathrm{Sp}^\infty(\mathrm{SP}_n^{dn}) & \xrightarrow{\tau_{d,j}} & \mathrm{Sp}^\infty(\mathrm{SP}_n^{jn})
\end{array}$$

Lemma 4.7. *Let $1 \le j < d$. Then the composite of stable maps*

$$\Sigma^{2(n-2)d} D_d \xrightarrow{e_d} F(\mathbb{C}, d) \times_{\Sigma_d} (S^{2n-3})^d \xrightarrow{\sigma_j} \mathrm{Sp}^\infty(F(\mathbb{C}, d) \times_{\Sigma_j \times \Sigma_{d-j}} (S^{2n-3})^j)$$

is null-homotopic.

Proof. This is well known (cf. [6] p. 44). □

Now we can complete the proof of lemma 4.4.

Proof of lemma 4.4. It follows from (4.6) and (4.7) that

$$\begin{aligned}
\tau_{d,d-1} \circ \mathrm{Sp}^\infty(\mathcal{I}_d) \circ \mathrm{Sp}^\infty(e_d) &\simeq \mathrm{Sp}^\infty(\mathcal{I}_{d,d-1}) \circ \tilde{\sigma}_{d-1} \circ \mathrm{Sp}^\infty(e_d) \\
&= \mathrm{Sp}^\infty(\mathcal{I}_{d,d-1}) \circ \mathrm{Sp}^\infty(\sigma_{d-1} \circ e_d) \\
&\simeq 0. \quad \square
\end{aligned}$$

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Department of Mathematics, University of Rochester, Rochester, New York 14627, USA

Department of Mathematics, Toyama International University, Kaminikawa, Toyama 930-12, Japan

Department of Mathematics, The University of Electro-Communications, Chofu, Tokyo 182, Japan