HOMOLOGICAL STABILITY OF ORIENTED CONFIGURATION SPACES

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§1. Introduction.

For a connected space M, let $F(M,d)$ be the space of *ordered configurations* of d distinct points in M, which is defined by

$$F(M,d) = \{(x_1, \cdots, x_d) \in M^d : x_i \neq x_j \text{ if } i \neq j\}.$$

Let Σ_d be the symmetric group of d letters $\{1, 2, \cdots, d\}$. Σ_d acts on $F(M,d)$ freely in the usual manner. The orbit space

$$C_d(M) = F(M,d)/\Sigma_d$$

is called the space of *configurations* of d distinct points in M. In this paper we shall assume that M is an open manifold, i.e. each component is non-compact and without boundary. Adding a point near one of the ends of M gives (up to homotopy) a stabilization map

$$j_d : C_d(M) \to C_{d+1}(M).$$

The following is well-known:

Theorem 0 ([Se]).
If M is an open manifold, then the stabilization map $j_d : C_d(M) \to C_{d+1}(M)$ is a homology equivalence up to dimension $[d/2]$. $\square$

(We shall call a map $f : X \to Y$ a homology equivalence up to dimension m if the induced homomorphism

$$f_* : H_i(X, \mathbf{Z}) \to H_i(Y, \mathbf{Z})$$

is bijective when $i < m$ and surjective when $i = m$.)

Remarks. Various special cases of this result were known earlier. For example:
(1) Let $M = \mathbf{R}^q$ $(q > 2)$. Then $\lim_{q\to\infty} C_d(\mathbf{R}^q) = K(\Sigma_d, 1)$. The homology stabilization of the maps $K(\Sigma_d, 1) \to K(\Sigma_{d+1}, 1)$ follows from work of Nakaoka ([Na]). This also follows from Theorem 0.
(2) Let $M = \mathbf{R}^2$. Then $C_d(M) = K(Br_d, 1)$. The statement of Theorem 0 in this case was proved by Arnold ([A]).

Let $\tilde{C}_d(M) = F(M,d)/A_d$, where $A_d \subset \Sigma_d$ is the alternating group of d letters $\{1, \cdots, d\}$. We shall call $\tilde{C}_d(M)$ the space of *oriented configurations* of d distinct points in M. There is a non-trivial double covering $\tilde{C}_d(M) \to C_d(M)$. Adding a point near an end of M gives a stabilization map

$$\tilde{j}_d : \tilde{C}_d(M) \to \tilde{C}_{d+1}(M).$$

In this note we shall determine the homological stability dimension for the spaces $\tilde{C}_d(M)$, when M is obtained from a compact Riemann surface by removing a finite number of points.

More precisely, we shall prove:

Theorem 1. *Let M be a compact Riemann surface, and let*

$$M' = M \setminus \{n \text{ points}\}$$

where $n \geq 1$. Then the stabilization map

$$\tilde{j}_d : \tilde{C}_d(M') \to \tilde{C}_{d+1}(M')$$

is a homology equivalence up to dimension $[(d-1)/3]$. Moreover, this bound is the best possible.

We shall give a proof in the next section, based on the calculations of [BCT] and [BCM]. First we make some remarks and pose a question:

Remarks. (1) It seems somewhat surprising that the answer is (about) $d/3$, not $d/2$ as in the un-oriented case.
(2) An analogous argument proves a similar result for McDuff's configuration space $C_n^{\pm}(M)$ of "positive and negative particles "([McD]). An application of this will be given in [GKY].

Question. *Is Theorem 1 true for any open manifold?*

§2. Proof of Theorem 1.

Without loss of generality we shall from now on assume that

$$M' = \mathbf{C} - \{l \text{ points}\}$$

and write C_n for $C_n(M')$ and $\tilde{C}_n$ for $\tilde{C}_n(M')$. We shall only consider the case $l \geq 1$. The case $l = 0$ can be dealt with in a similar way.

We shall show that

$$(*) \qquad H_q(\tilde{C}_d, \mathbf{F}) \to H_q(\tilde{C}_{d+1}, \mathbf{F})$$

is bijective for $q < n(d)$ and surjective for $q = n(d)$ if $\mathbf{F} = \mathbf{Z}/p$ (p is any prime) or $\mathbf{F} = \mathbf{Q}$, where

$$n(d) = \begin{cases} [d/2] & \text{if } \mathbf{F} \neq \mathbf{Z}/3 \\ [(d-1)/3] & \text{if } \mathbf{F} = \mathbf{Z}/3 \end{cases}$$

Theorem 1 follows from this and the universal coefficient theorem. (The case $\mathbf{F} = \mathbf{Z}/2$ is trivial. Indeed, since $\tilde{C}_d \to C_d$ is a double covering and the stabilization map $C_d \to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, the result follows from the Gysin exact sequence.)

We shall make use of the following well known fact (cf. [**B**]):

Lemma 2. *Let G be a group and $H \subset G$ a subgroup of G of index 2. Let $\mathbf{F}$ be any field of characteristic not equal to 2. Then there is a natural additive isomorphism*

$$H_q(H, \mathbf{F}) \cong H_q(G, \mathbf{F}) \oplus H_q(G, \mathbf{F}(-1))$$

for any $q \geq 1$, where $\mathbf{F}(-1)$ denotes the field $\mathbf{F}$ with the G-module structure given by

$$g \cdot f = \begin{cases} -f & g \notin H \\ f & g \in H \end{cases}$$

for $f \in \mathbf{F}$ and $g \in G$. $\square$

Let us take $G = \pi_1(C_d)$ and $H = \pi_1(\tilde{C}_d)$. Since $\tilde{C}_d \to C_d$ is a double covering, H can be identified with a subgroup of G of index 2. We have $\tilde{C}_d \simeq K(H,1)$, $C_d \simeq K(G,1)$ and we can identify the covering map with the map $K(H,1) \to K(G,1)$ induced by the inclusion $H \subset G$. We can thus apply Lemma 2 to obtain:

Lemma 3. *If $\mathbf{F} = \mathbf{Z}/p$ (p any odd prime) or $\mathbf{F} = \mathbf{Q}$, then there is a natural additive isomorphism*

$$H_q(\tilde{C}_d, \mathbf{F}) \cong H_q(C_d, \mathbf{F}) \oplus H_q(C_d, \mathbf{F}(-1))$$

for any $q \geq 1$ $\square$

Now, since $C_d \to C_{d+1}$ is a homology equivalence up to dimension $[d/2]$, Theorem 1 follows directly from the following result:

Lemma 4. *Let q and d be positive integers such that $1 \leq q \leq [d/2]$ and $(q,d) \neq (1,2)$.*

(1) If $\mathbf{F} = \mathbf{Z}/p$ (p prime, $p \geq 7$) or $\mathbf{F} = \mathbf{Q}$, then

$$H_q(C_d, \mathbf{F}(-1)) = 0$$

(2) If $\mathbf{F} = \mathbf{Z}/5$ and $(q,d) \neq (3,6)$, then

$$H_q(C_d, \mathbf{Z}/5(-1)) = 0$$

(3) If $\mathbf{F} = \mathbf{Z}/3$ and $d \geq 3q + 2$, then

$$H_q(C_d, \mathbf{Z}/3(-1)) = 0$$

Proof. Let $1 \leq q \leq [d/2]$.

By (8.4) of [BCM], if n is sufficiently large, then

$$H_q(C_d, \mathbf{F}(-1)) \cong H_{q+(2n+1)d}(\Omega^2 S^{2n+3} \times (\Omega S^{2n+3})^l, \mathbf{F})$$

Note that

$$H_j((\Omega S^{2n+3})^l, \mathbf{F}) \cong \begin{cases} \mathbf{F}^{m(\beta)} & \text{if } j = (2n+2)\beta, \quad \beta \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and there is a stable splitting ([CMM], [S])

$$\Omega^2 S^{2n+3} \simeq_s \vee_{\alpha \geq 1} \Sigma^{2n} D_\alpha$$

where we take

$$m(\beta) = \binom{\beta + l - 1}{l - 1} \quad \text{and} \quad D_\alpha = F(\mathbf{C}, \alpha)_+ \wedge_{\Sigma_\alpha} (\wedge^\alpha S^1).$$

Since D_α has the homotopy type of a CW complex of dimension $2\alpha - 1$, $H_j(D_\alpha, \mathbf{Z}/p) = 0$ for any $j \geq 2\alpha$.

Applying the Künneth formula one can show that

$$(**) \qquad H_q(C_d, \mathbf{F}(-1)) \cong \oplus_{\alpha=1}^d \tilde{H}_{q+2\alpha-d}(D_\alpha, \mathbf{F})^{m(d-\alpha)}$$

From now on we shall only consider the case $\mathbf{F} = \mathbf{Z}/p$ (p an odd prime). The case $\mathbf{F} = \mathbf{Q}$ can be dealt with analogously.

The following is well known:

Lemma 5. *Let $p \geq 3$ be any odd prime.*

(1) There is a multiplicative isomorphism

$$\text{(a)} \qquad H_*(\Omega^2 S^3, \mathbf{Z}/p) = \mathbf{Z}/p[x_1, x_2, \cdots] \otimes E[y_0, y_1, y_2, \cdots]$$

where deg$(x_i) = 2p^i - 2$ and deg$(y_i) = 2p^i - 1$.

(2) There is an additive isomorphism

$$\text{(b)} \qquad \tilde{H}_*(D_\alpha, \mathbf{Z}/p) = \oplus_{J=(\epsilon_0, m_1, \epsilon_1, \cdots) \in \mathcal{J}} \mathbf{Z}/p\{\prod_{j \geq 1} x_j^{m_j} \cdot \prod_{j \geq 0} y_j^{\epsilon_j}\}$$

where we take:

$$\mathcal{J} = \{J = (\epsilon_0, m_1, \epsilon_1, \cdots) : \epsilon_j \in \{0, 1\}, m_j \geq 0, w(J) = \alpha\}$$

and

$$w(J) = \epsilon_0 + \sum_{j \geq 1} p^j (m_j + \epsilon_j). \quad \square$$

¿From Lemma 5

(c) $$\dim_{\mathbf{Z}/p} \tilde{H}_{q+2\alpha-d}(D_\alpha, \mathbf{Z}/p) = \mathrm{card}(\mathcal{F})$$

where

$$\mathcal{F} = \{J = (\epsilon_0, m_1, \epsilon_1, \cdots) \neq (0, 0, \cdots) : \epsilon_j \in \{0,1\}, m_j \geq 0, D(J) = q + 2\alpha - d, w(J) = \alpha\}$$

and

$$D(J) = \epsilon_0 + \sum_{j\geq 1} \{2(p^j - 1)m_j + (2p^j - 1)\epsilon_j\}.$$

Here $\mathrm{card}(S)$ denotes the cardinality of a finite set S.

Note that for $J = (\epsilon_0, m_1, \epsilon_1, \cdots)$, if $w(J) = \alpha$, then

$$D(J) = q + 2\alpha - d \Leftrightarrow H(J) = \epsilon_0 + \sum_{j\geq 1}(2m_j + \epsilon_j) = d - q$$

Hence

(d) $\mathcal{F} = \{J = (\epsilon_0, m_1, \epsilon_1, \cdots) \neq (0, 0, \cdots) : \epsilon_j \in \{0,1\}, m_j \geq 0, w(J) = \alpha, H(J) = d - q\}.$

By (c) and (d) it suffices to show:

Claim. *Let $1 \leq q \leq [d/2]$, $1 \leq \alpha \leq d$ and $(q, d) \neq (1, 2)$.*
(1) If $p \geq 7$ is an odd prime or $p = 5$ and $(q, d) \neq (3, 6)$, then $\mathcal{F} = \emptyset$
(2) If $p = 3$ and $d \geq 3q + 2$, $\mathcal{F} = \emptyset$.

Proof of Claim. (1) Assume that $p \geq 5$ is a prime and $J = (\epsilon_0, m_1, \epsilon_1, \cdots) \in \mathcal{F}$.

Since $1 \leq q \leq [d/2] \leq d/2$,

$$\epsilon_0 + \sum_{j\geq 1}(2m_j + \epsilon_j) = H(J) = d - q \geq d/2 \geq \alpha/2 = \{\epsilon_0 + \sum_{j\geq 1} p^j(m_j + \epsilon_j)\}/2.$$

Hence

(e) $$\epsilon_0 + \sum_{j\geq 1}\{(4 - p^j)m_j + (2 - p^j)\epsilon_j\} \geq 0$$

Since $J \neq (0, 0, \cdots)$, one can deduce from (e) that

$$J = (\epsilon_0, m_1, \epsilon_1, m_2, \epsilon_2, \cdots) = \begin{cases} (1, 0, 0, 0, 0, \cdots) & \text{if } p \geq 7 \\ (1, 0, 0, 0, 0, \cdots) \text{ or } (1, 1, 0, 0, 0, \cdots) & \text{if } p = 5 \end{cases}$$

Hence

$$(q, d) = \begin{cases} (1, 2) & p \geq 7 \\ (1, 2), (3, 6) & p = 5 \end{cases}$$

This is a contradiction.

(2) Assume $d \geq 3q+2$ and $p=3$. Then

$$\begin{aligned}
\alpha - d &= w(J) - (q + H(J)) \\
&= \{\epsilon_0 + \sum_{j\geq 1} 3^j(m_j + \epsilon_j)\} - \{\epsilon_0 + \sum_{j\geq 1}(2m_j + \epsilon_j)\} - q \\
&= \sum_{j\geq 1}\{(3^j - 2)m_j + (3^j - 1)\epsilon_j\} - q \\
&\geq \frac{1}{2}\sum_{j\geq 1}(2m_j + \epsilon_j) - q \\
&= \frac{1}{2}(d - q - \epsilon_0) - q \qquad (\text{by } H(J) = d - q) \\
&= \frac{1}{2}(d - 3q - \epsilon_0) \\
&\geq \frac{1}{2}\{(3q+2) - 3q - 1\} = \frac{1}{2} > 0
\end{aligned}$$

Hence $\alpha = w(J) > d$, which is a contradiction. □

This completes the proof of Theorem 2. □

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