SHARP WEAK TYPE INEQUALITIES FOR THE HAAR SYSTEM AND RELATED ESTIMATES FOR NON-SYMMETRIC MARTINGALE TRANSFORMS

ADAM OSĘKOWSKI

ABSTRACT. For any $1 \leq p < \infty$, we determine the optimal constant C_p such that the following holds. If $(h_k)_{k\geq 0}$ is the Haar system, then for any vectors a_k from a separable Hilbert space $\mathcal{H}$ and $\theta_k \in \{0,1\}$, $k = 0, 1, 2, \ldots$, we have

$$\left|\left|\sum_{k=0}^{n} \theta_k a_k h_k\right|\right|_{p,\infty} \leq C_p \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p.$$

This is generalized to the weak-type inequality

$$||g||_{p,\infty} \leq C_p ||f||_p$$

where f is an $\mathcal{H}$-valued martingale and g is its transform by a predictable sequence taking values in $[0,1]$. We extend this further to the estimate

$$||Y||_{p,\infty} \leq C_p ||X||_p,$$

valid for any two $\mathcal{H}$-valued continuous-time martingales X, Y, such that $([Y, X - Y]_t)$ is nondecreasing and nonnegative as a function of t.

1. Introduction

Let $1 < p < \infty$ and let $(h_k)_{k\geq 0}$ be the Haar system in L^p. In his classical result, Marcinkiewicz [9] proved that there is a universal finite constant c_p such that

$$c_p^{-1} \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p \leq \left|\left|\sum_{k=0}^{n} \varepsilon_k a_k h_k\right|\right|_p \leq c_p \left|\left|\sum_{k=0}^{n} a_k h_k\right|\right|_p \tag{1.1}$$

for any n and any $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1, 1\}$, $k = 0, 1, 2, \ldots, n$. This result was extended by Burkholder [1] to the martingale setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_k)_{k\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Let $f = (f_k)_{k\geq 0}$ be a real-valued martingale with the difference sequence $(df_k)_{k\geq 0}$ given by $df_0 = f_0$ and $df_k = f_k - f_{k-1}$ for $k \geq 1$. Let g be a transform of f by a real predictable sequence $v = (v_k)_{k\geq 0}$ bounded in absolute value by 1: that is, $dg_k = v_k df_k$ for all $k \geq 0$ and by predictability we mean that each term v_k is measurable with respect to $\mathcal{F}_{(k-1)\vee 0}$. Then (cf. [1]) for $1 < p < \infty$ there is an absolute constant c_p' for which

$$||g||_p \leq c_p' ||f||_p. \tag{1.2}$$

Here we have used the notation $||f||_p = \sup_n ||f_n||_p$. Let $c_p(1.1)$, $c_p'(1.2)$ denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space

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being the Lebesgue's unit interval) and hence so is $(a_k h_k)_{k\geq 0}$, for given fixed real numbers $a_0,\ a_1,\ a_2,\ \ldots$. Therefore, $c_p(1.1) \leq c'_p(1.2)$ for all $1 < p < \infty$. It follows from the results of Burkholder [2] and Maurey [10] that in fact the constants coincide: $c_p(1.1) = c'_p(1.2)$ for all $1 < p < \infty$. The question about the precise value of $c_p(1.1)$ was answered by Burkholder in [3]: $c_p(1.1) = p^* - 1$ (where $p^* = \max\{p, p/(p-1)\}$) for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the terms a_k to take values in a separable Hilbert space $\mathcal{H}$. This determines the complex unconditional basis constant of the Haar system:

$$\sup\left\{\left|\left|\sum_{k=0}^n \varepsilon_k a_k h_k\right|\right|_p\right\} = p^* - 1, \qquad 1 < p < \infty,$$

where the supremum is taken over all n, all sequences $\varepsilon_0,\ \varepsilon_1,\ \varepsilon_2,\ \ldots$ of signs and all complex numbers $a_0,\ a_1,\ a_2,\ \ldots$ satisfying $||\sum_{k=0}^n a_k h_k||_p = 1$ (cf. [5]).

For $p = 1$ the inequalities (1.1) and (1.2) do not hold with any finite constant, but one can establish a corresponding weak type estimate. Burkholder [3] proved the following sharp bound, for a wider range of parameters: if $1 \leq p \leq 2$, then

$$||g||_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} ||f||_p, \tag{1.3}$$

where $||g||_{p,\infty} = \sup_{\lambda>0} \lambda(\mathbb{P}(\sup_n |g_n| \geq \lambda))^{1/p}$. For $p > 2$, Suh [12] showed that

$$||g||_{p,\infty} \leq \left(p^{p-1}/2\right)^{1/p} ||f||_p. \tag{1.4}$$

Both (1.3), (1.4) remain sharp for the Haar system, even for $\mathcal{H}$-valued coefficients. In fact, all the martingale inequalities above are valid under less restrictive assumption of differential subordination, and can be extended to the continuous-time setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and equip it with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Let X, Y be two adapted cadlag martingales taking values in $\mathcal{H}$; with no loss of generality we assume, from now on, that $\mathcal{H} = \ell^2$. Following [13], we say that Y is *differentially subordinate* to X, if the process $([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. Here $[X,Y] = \sum_{j=0}^\infty [X^j, Y^j]$, where X^j, Y^j stand for the j-th coordinates of X and Y, respectively, and $[X^j, Y^j]$ is the quadratic covariance process of X^j and Y^j (see e.g. Dellacherie and Meyer [7]). If we treat the discrete-time martingales $f = (f_k)_{k=0}^\infty$, $g = (g_k)_{k=0}^\infty$ as continuous-time processes (by $X_t = f_{\lfloor t\rfloor}$ and $Y_t = g_{\lfloor t\rfloor}$ for $t \geq 0$), then the above condition reads

$$|dg_k| \leq |df_k| \qquad \text{for } k \geq 0,$$

which is the original definition of the differential subordination due to Burkholder [3]. Clearly, this condition is satisfied by the martingale transforms studied above. Thus the following theorem (cf. [12], [13]) generalizes the previous inequalities (1.2), (1.3) and (1.4). We use the notation $||X||_p = \sup_t ||X_t||_p$ and $||X||_{p,\infty} = \sup_{\lambda>0} \lambda(\mathbb{P}(\sup_t |X_t| \geq \lambda))^{1/p}$, analogous to that of the discrete-time setting.

Theorem 1.1. *If Y is differentially subordinate to X, then*

$$||Y||_p \leq c_p ||X||_p, \qquad 1 < p < \infty, \tag{1.5}$$

$$||Y||_{p,\infty} \leq \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} ||X||_p, \quad 1 \leq p \leq 2,$$

$$||Y||_{p,\infty} \leq \left(\frac{p^{p-1}}{2}\right)^{1/p} ||X||_p, \quad 2 \leq p < \infty$$

and the inequalities are sharp.

Let us now turn to the non-symmetric case, a setting we will be particularly interested in. An alternative way of defining the unconditional basis constant is

$$\sup\left\{\left|\left|\sum_{k\in J} a_k h_k\right|\right|_p\right\}$$

where the supremum is taken over all n, all subsets $J \subseteq \{0, 1, 2, \dots, n\}$ and all complex numbers $a_0, a_1, a_2, \dots, a_n$ satisfying $||\sum_{k=0}^n a_k h_k||_p = 1$ (see e.g. [8]). This leads to another natural transformation of Haar series: to throw out some of its terms, instead of changing their signs. In other words, it suggests to consider in (1.1) the case when each ε_k takes values in $\{0,1\}$. Under this non-symmetric assumption, Choi [6] found the best constant c_p'' in (1.1) for real coefficients $a_0, a_1, a_2, \dots$. He also showed if a martingale f is real and each v_k takes values in $[0,1]$, then (1.2) holds with the same constant c_p''. Since the description of the constant is quite complicated, we do not present it here and refer the interested reader to [6].

There is a natural question about the best constants in the corresponding weak type estimates for the Haar system and the extension of these bounds to continuous-time martingales. We will study this problem in the general case when the coefficients $a_0, a_1, a_2, \dots$ as well as the processes take values in a Hilbert space $\mathcal{H}$. The role of "non-symmetric differential subordination" is played by the condition

$$([X,Y]_t - [Y,Y]_t) \text{ is nonnegative and nondecreasing as a function of } t. \tag{1.6}$$

This generalizes non-symmetric martingale transforms: assume that f is a martingale and g is its transform by a predictable sequence v. If we treat these as continuous-time processes, we see that the condition (1.6) reads $(v_k - v_k^2)df_k^2 \geq 0$ for all k, and hence it is satisfied if the variables v_k take values in $[0,1]$.

We turn to the formulation of our main result. Let

$$C_p = \begin{cases} 1 & \text{if } 1 \leq p \leq 2, \\ \frac{1}{2}\left[\frac{(2c+p-1)^{p-1}}{c+1}\right]^{1/p} & \text{if } p > 2, \end{cases}$$

where $c = c(p) > 1$ is the unique positive number satisfying

$$c^{p-1} = 2c + 1. \tag{1.7}$$

Theorem 1.2. *Let X, Y be two Hilbert-space valued martingales satisfying* (1.6). *Then for any $1 \leq p < \infty$ we have*

$$||Y||_{p,\infty} \leq C_p ||X||_p \tag{1.8}$$

and the constant C_p is the best possible. It is already the best possible in the following one-sided bound for the Haar system:

$$\left|\left\{r \in [0,1] : \sum_{k=0}^n \theta_k a_k h_k(r) \geq 1\right\}\right| \leq C_p^p \left|\left|\sum_{k=0}^n a_k h_k\right|\right|_p^p \tag{1.9}$$

for all n, $a_k \in \mathbb{R}$ and $\theta_k \in \{0,1\}$, $k = 0, 1, 2, \dots, n$.

A few words about the proof and the organization of the paper. Our approach is based on Burkholder's technique, which exploits special functions which have certain convex-type properties. To be more precise, the inequality (1.8) reduces to $\mathbb{E}V_p(X_t, Y_t) \leq 0$ for some appropriate function $V_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and all $t \geq 0$. The key to study this inequality is to find a majorant U_p of V_p such that $(U_p(X_t, Y_t))_{t\geq 0}$ is an $(\mathcal{F}_t)$-supermartingale satisfying $U_p(X_0, Y_0) \leq 0$. This is the way we prove Theorem 1.2 for $1 \leq p \leq 2$; see Section 2 below. For $p > 2$ our argument turns out to be substantially different and uses an "integration trick" developed by the author in [11]. First we show that $\mathbb{E}U_\infty(X_t, Y_t) \leq 0$ for some simple $U_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and all $t \geq 0$, and then complicate the function by integrating it against certain positive kernel, thus obtaining the appropriate majorant; see Section 3. The final part of the paper contains the proof of a technical fact needed in the earlier considerations.

2. The case $1 \leq p \leq 2$

The main object in this section is the function $U_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ given by

$$U_p(x, y) = \begin{cases} py \cdot (y - x) & \text{if } |x| + |2y - x| < 2, \\ p - p|x| & \text{if } |x| + |2y - x| \geq 2. \end{cases}$$

Here and below, the dot $\cdot$ denotes the scalar product in $\mathcal{H}$ and $|x|$ stands for the norm of $x \in \mathcal{H}$. Let $V_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be defined by

$$V_p(x, y) = 1_{\{|y|\geq 1\}} - |x|^p.$$

We have the following majorization.

Lemma 2.1. *For all x, $y \in \mathcal{H}$ we have*

$$U_p(x, y) \geq V_p(x, y). \tag{2.1}$$

Proof. If $|x| + |2y - x| < 2$, then $|y| \leq |x/2| + |y - x/2| < 1$ and, consequently,

$$1_{\{|y|\geq 1\}} - |x|^p = -|x|^p \leq -\frac{p|x|^2}{4} \leq p|y|(|y| - |x|) \leq py \cdot (y - x).$$

On the other hand, if $|x| + |2y - x| \geq 2$, then (2.1) follows immediately from the estimate $p - ps \geq 1 - s^p$, valid for all $s \geq 0$, by virtue of the mean-value theorem. □

Lemma 2.2. *Suppose that martingales X, Y satisfy the condition* (1.6). *Then for any $t \geq 0$ we have*

$$\mathbb{E}U_p(X_t, Y_t) \leq 0. \tag{2.2}$$

Proof. Let $\mathcal{U} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be given by

$$\mathcal{U}(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } |x| + |y| < 1, \\ 1 - 2|x| & \text{if } |x| + |y| \geq 1. \end{cases}$$

This is Burkholder's special function corresponding to the weak-type inequality (1.3) for $p = 1$ (cf. [3]). As shown by Wang (see the proof of Theorem 3 in [13]), if $\zeta = (\zeta_t)$ is differentially subordinate to $\xi = (\xi_t)$, then for any $t \geq 0$ we have

$$\mathbb{E}\mathcal{U}(\zeta_t, \xi_t) \leq 0.$$

We apply this to the martingales $\xi = X/2$ and $\zeta = -X/2 + Y$; the differential subordination follows from the identity

$$[X/2, X/2]_t - [-X/2 + Y, -X/2 + Y]_t = [X, Y]_t - [Y, Y]_t.$$

The proof is completed by noting that $U_p(x,y) = p\mathcal{U}(x/2, -x/2+y)$ for all x, y. □

Now we turn to the proof of Theorem 1.2 in the case $1 \leq p \leq 2$.

Proof of Theorem 1.2. Obviously, the constant $C_p = 1$ is the best possible in (1.9): take $a_0 = \theta_0 = 1$ and $a_k = \theta_k = 0$ for $k \geq 1$. Therefore all we need is to establish the estimate (1.8). Note that we may assume that X is bounded in L^p, otherwise there is nothing to prove. By homogeneity, we will be done if we show that

$$\mathbb{P}(Y^* \geq 1) \leq ||X||_p^p, \tag{2.3}$$

where $Y^* = \sup_{t\geq 0} |Y_t|$ is the maximal function of Y. Observe that by virtue of (2.1) and (2.2) we have

$$\mathbb{P}(|Y_t| \geq 1) \leq \mathbb{E}|X_t|^p \quad \text{for } t \geq 0. \tag{2.4}$$

Now take $\varepsilon \in (0,1)$ and introduce the stopping time $\tau = \inf\{s \geq 0 : |Y_s| \geq 1-\varepsilon\}$. We have that

$$\{Y^* \geq 1\} \subset \{|Y_t| \geq 1-\varepsilon \text{ for some } t\} = \{|Y_{\tau\wedge t}| \geq 1-\varepsilon \text{ for some } t\}.$$

Since the family $(\{|Y_{\tau\wedge s}| \geq 1-\varepsilon\})_s$ is nondecreasing and

$$\{|Y_{\tau\wedge t}| \geq 1-\varepsilon \text{ for some } t\} = \bigcup_{t\geq 0}\{|Y_{\tau\wedge t}| \geq 1-\varepsilon\},$$

we get $\mathbb{P}(Y^* \geq 1) \leq \lim_{t\to\infty} \mathbb{P}(Y_{\tau\wedge t} \geq 1-\varepsilon)$. Now it is easy to see that the pair $(X_{\tau\wedge t}/(1-\varepsilon))$, $(Y_{\tau\wedge t}/(1-\varepsilon))$ satisfies (1.6). Applying (2.4) to this pair gives

$$\lim_{t\to\infty} \mathbb{P}(Y_{\tau\wedge t} \geq 1-\varepsilon) \leq (1-\varepsilon)^{-p}\mathbb{E}|X_{\tau\wedge t}|^p \leq (1-\varepsilon)^{-p}||X||_p^p.$$

Thus (2.3) follows, since ε was arbitrary. □

3. The case $p > 2$

This is more involved. Define an auxiliary function $U_\infty : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$U_\infty(x,y) = \begin{cases} 0 & \text{if } |x| + |2y-x| < 1, \\ (|2y-x|-1)^2 - |x|^2 & \text{if } |x| + |2y-x| \geq 1. \end{cases}$$

Later on, we will need the following properties of this function.

Lemma 3.1. *(i) There is an absolute constant $A > 0$ such that for all x, $y \in \mathcal{H}$,*

$$U_\infty(x,y) \leq A(|x|^2 + |y|^2 + 1). \tag{3.1}$$

(ii) For all x, $y \in \mathcal{H}$ we have

$$U_\infty(x,y) \leq (|2y-x|-1)^2 - |x|^2. \tag{3.2}$$

(iii) If x, y, h, $k \in \mathcal{H}$ satisfy

$$|x| + |2y-x| \leq 1, \quad |x+h| + |2(y+k) - (x+h)| \geq 1 \tag{3.3}$$

and

$$|2k-h| \leq |h|, \tag{3.4}$$

then $U_\infty(x+h, y+k) \leq 0$.

(iv) If x, $y \in \mathcal{H}$ satisfy $x \cdot y - |y|^2 \geq 0$, then $U_\infty(x,y) \leq 0$.

Proof. (i), (ii) Evident from the very definition of U_∞.

(iii) The desired inequality can be written in the form

$$-|x+h| \leq |2(y+k)-(x+h)|-1 \leq |x+h|.$$

The left inequality is precisely the second condition in (3.3). To get the right one, note that by a triangle inequality, (3.3) and (3.4),

$$|2(y+k)-(x+h)|-1 \leq |2y-x|+|2k-h|-1 \leq |2y-x|-1+|h| \leq -|x|+|h| \leq |x+h|.$$

(iv) The estimate is trivial if $|x|+|2y-x| \leq 1$. If the reverse holds, note that $x \cdot y - |y|^2 \geq 0$ is equivalent to $|x|^2 \geq |2y-x|^2$ and hence

$$U_\infty(x,y) = (|2y-x|+|x|-1)(|2y-x|-|x|-1) \leq 0. \quad \square$$

The next result is a dual version of Lemma 2.2.

Lemma 3.2. *Suppose that martingales X, Y are bounded in L^2 and satisfy the condition* (1.6). *Then for any $t \geq 0$,*

$$\mathbb{E}U_\infty(X_t,Y_t) \leq 0. \tag{3.5}$$

Proof. First note that by (3.1), the random variable $U_\infty(X_t,Y_t)$ is integrable. Let $\tau = \inf\{s \geq 0 : |X_s|+|2Y_s-X_s| > 1\}$. We will show the following three statements:

$$\mathbb{E}U_\infty(X_t,Y_t)1_{\{|X_0|+|2Y_0-X_0|>1\}} \leq \mathbb{E}U_\infty(X_0,Y_0)1_{\{|X_0|+|2Y_0-X_0|>1\}}, \tag{3.6}$$

$$U_\infty(X_t,Y_t) = U_\infty(X_0,Y_0) = 0 \quad \text{ on } \{|X_0|+|2Y_0-X_0| \leq 1,\, \tau > t\} \tag{3.7}$$

and

$$\mathbb{E}U_\infty(X_t,Y_t)1_{\{|X_0|+|2Y_0-X_0|\leq 1,\,\tau\leq t\}} \leq \mathbb{E}U_\infty(X_0,Y_0)1_{\{|X_0|+|2Y_0-X_0|\leq 1,\,\tau\leq t\}}. \tag{3.8}$$

These three facts yield the claim: indeed, they give $\mathbb{E}U(X_t,Y_t) \leq \mathbb{E}U(X_0,Y_0)$ and it suffices to note that $U(X_0,Y_0) \leq 0$, in view of (1.6) and part (iv) of Lemma 3.1.

To prove (3.6), use (3.2) to get

$$\mathbb{E}\big[U_\infty(X_t,Y_t)|\mathcal{F}_0\big] \leq 4\mathbb{E}\big[Y_t \cdot (Y_t-X_t)|\mathcal{F}_0\big] - 2\mathbb{E}(|2Y_t-X_t||\mathcal{F}_0)+1.$$

Clearly, $\mathbb{E}(|2Y_t-X_t||\mathcal{F}_0) \geq |2Y_0-X_0|$. Moreover, by (1.6), we have that

$$\mathbb{E}\big[Y_t(Y_t-X_t)-Y_0(Y_0-X_0)|\mathcal{F}_0\big] = -\mathbb{E}\big[([X,Y]_t-[Y,Y]_t)-([X,Y]_0-[Y,Y]_0)|\mathcal{F}_0\big]$$

is nonpositive. Consequently, on the set $\{|X_0|+|2Y_0-X_0|>1\}$,

$$\mathbb{E}\big[U_\infty(X_t,Y_t)|\mathcal{F}_0\big] \leq 4Y_0 \cdot (Y_0-X_0) - 2|2Y_0-X_0|+1 = U_\infty(X_0,Y_0)$$

and (3.6) follows. The condition (3.7) is obvious, by the definition of U_∞ and τ. To get (3.8), we proceed as previously: by (3.2) and (1.6) we have, on the set $\{|X_0|+|2Y_0-X_0| \leq 1,\, \tau \leq t\}$,

$$\begin{aligned}\mathbb{E}\big[U_\infty(X_t,Y_t)|\mathcal{F}_\tau\big] &= 4\mathbb{E}\big[Y_t \cdot (Y_t-X_t)|\mathcal{F}_\tau\big] - 2\mathbb{E}(|2Y_t-X_t||\mathcal{F}_\tau)+1 \\ &\leq 4Y_\tau \cdot (Y_\tau-X_\tau) - 2|2Y_\tau-X_\tau|+1 \\ &= U_\infty(X_\tau,Y_\tau).\end{aligned}$$

Now use part (iii) of Lemma 3.1 with $x = X_{\tau-}$, $y = Y_{\tau-}$, $h = \Delta X_\tau$ and $k = \Delta Y_\tau$: the condition (3.3) follows from the definition of τ, while (3.4) is a consequence of (1.6). Thus, $U_\infty(X_\tau,Y_\tau) \leq 0 = U_\infty(X_0,Y_0)$ and the proof is complete. $\square$

We are ready to introduce the special function $U_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, corresponding to the weak type estimate for $p > 2$. Recall $c = c(p)$ given by (1.7) and let

$$b = b(p) = \frac{2(p-1)}{2c+p-1}. \tag{3.9}$$

Set

$$U_p(x,y) = \int_0^b t^{p-1} U_\infty(x/t, y/t)\mathrm{d}t. \tag{3.10}$$

Some lengthy, but straightforward calculations give that

$$U_p(x,y) = \frac{2}{p(p-1)(p-2)}(|x|+|2y-x|)^{p-1}\big(|2y-x|-(p-1)|x|\big)$$

if $|x|+|2y-x| \leq b$, and

$$U_p(x,y) = b^{p-2}\left[\frac{|2y-x|^2-|x|^2}{p-2} - \frac{2b|2y-x|}{p-1} + \frac{b^2}{p}\right]$$

for $|x|+|2y-x| > b$. We will also need the function $V_p : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$, given by

$$V_p(x,y) = \alpha_p(C_p^{-p} 1_{\{|y|\geq 1\}} - |x|^p),$$

where

$$\alpha_p = \frac{2(p-1)^{p-2}}{p(p-2)}. \tag{3.11}$$

We have the following majorization.

Lemma 3.3. *For all x, $y \in \mathcal{H}$, we have*

$$U_p(x,y) \geq V_p(x,y). \tag{3.12}$$

The justification of this estimate is quite involved, so for the sake of clarity we postpone it to Section 4 and proceed with the proof of Theorem 1.2.

Proof of (1.8). We may assume that X is bounded in L^p. Then so is Y: since $-X/2+Y$ is differentially subordinate to $X/2$ (see the proof of Lemma 2.2 above), the inequality (1.5) implies

$$||Y||_p \leq ||-X/2+Y||_p + ||X/2||_p \leq p^*||X/2||_p < \infty.$$

As in the case $1 \leq p \leq 2$, we reduce the desired estimate to

$$\mathbb{P}(|Y_t| \geq 1) \leq C_p^p |X_t|^p, \qquad t \geq 0.$$

By (3.12), this will be done if we show that $\mathbb{E}U_p(X_t, Y_t) \leq 0$. This follows immediately from (3.5), the definition of U_p and Fubini's theorem. To see that the latter is applicable, note that by (3.1),

$$\mathbb{E}|U_p(X_t,Y_t)| \leq A\mathbb{E}\left[\frac{b^{p-2}(|X_t|^2+|Y_t|^2)}{p-2} + \frac{b^p}{p}\right] < \infty,$$

since X, Y are bounded in L^p. □

We turn to the sharpness of (1.9), which is the most technical element of the paper. We will need the following fact, which relates the validity of a given inequality for the Haar system to a certain boundary value problem (for similar results, see e.g. Section 11 in [3] or Section 7 in [4]).

Lemma 3.4. *Let $V : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a given Borel function, locally bounded from below. Assume that*

$$\int_0^1 V\left(\sum_{k=0}^n a_k h_k(r), \sum_{k=0}^n a_k\theta_k h_k(r)\right) dr \leq 0 \tag{3.13}$$

for all n and all $a_k\in\mathbb{R}$, $\theta_k\in\{0,1\}$, $k=0, 1, 2, \ldots, n$. Then there is a function $W:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying the following properties.

(a) We have $W(x,x)\leq 0$ for any $x\in\mathbb{R}$.

(b) For all x, $y\in\mathbb{R}$ we have $W(x,y)\geq V(x,y)$.

(c) The function W is concave along any line of slope 0 or 1.

Proof. Define $W:\mathbb{R}\times\mathbb{R}\to(-\infty,\infty]$ by

$$W(x,y)=\sup\left\{\int_0^1 V\left(x+\sum_{k=1}^n a_k h_k(r), y+\sum_{k=1}^n a_k\theta_k h_k(r)\right)\mathrm{d}r\right\} \tag{3.14}$$

where the supremum is taken over all n and all $a_k\in\mathbb{R}$, $\theta_k\in\{0,1\}$, $k=1, 2, \ldots, n$. Then the property (a) is a consequence of (3.13), while (b) follows from the definition of W by considering the sequence $a_1=a_2=\ldots=0$. To get (c), we use Burkholder's "splicing" argument: take any line L of slope 1 and any point (x,y) lying on it. Take $d>0$, an integer N, sequences a_1^+, a_2^+, $\ldots$, a_N^+, a_1^-, a_2^-, $\ldots$, a_N^- of real numbers and θ_1^+, θ_2^+, $\ldots$, θ_N^+, θ_1^-, θ_2^-, $\ldots$, θ_N^- with $\theta_k^\pm\in\{0,1\}$. Let

$$Z_{x,y}^\pm(r)=\left(x\pm d+\sum_{k=1}^N a_k^\pm h_k(r), y\pm d+\sum_{k=1}^N a_k^\pm\theta_k^\pm h_k(r)\right).$$

and splice the functions Z^+ and Z^- are together in the following way:

$$Z_{x,y}(r)=\begin{cases}Z_{x,y}^-(2r) & \text{if } 0\leq r\leq 1/2,\\ Z_{x,y}^+(2r-1) & \text{if } 1/2<r\leq 1.\end{cases}$$

Then it is easily seen that

$$Z_{x,y}(r)=\left(x+\sum_{k=1}^{2N} a_k h_k(r), y+\sum_{k=1}^{2N} a_k\theta_k h_k(r)\right),$$

where for any $1\leq k\leq 2N$, there is $\ell\leq N$ such that $(a_k,\theta_k)=(a_\ell',\theta_\ell')$ or $(a_k,\theta_k)=(a_\ell'',\theta_\ell'')$. Thus

$$W(x,y)\geq\int_0^1 V(Z_{x,y}(r))\mathrm{d}r=\int_0^{1/2} V(Z_{x,y}^-(2r))\mathrm{d}r+\int_{1/2}^1 V(Z_{x,y}^+(2r-1))\mathrm{d}r$$

and taking supremum over all N, $a_k^\pm$ and $\theta_k^\pm$ as above yields

$$W(x,y)\geq(W(x-d,y-d)+W(x+d,y+d))/2.$$

Since x, y, and d were arbitrary, W is midpoint concave along L. Analogous arguments lead to the midpoint concavity along the lines of slope 0. This yields the finiteness of W: indeed, for any x, $y\in\mathbb{R}$ we have, by (a) and (b),

$$0\geq W(y,y)\geq\frac{1}{2}W(x,y)+\frac{1}{2}W(2y-x,y)\geq\frac{1}{2}W(x,y)+\frac{1}{2}V(2y-x,y),$$

so $W(x,y)\leq -V(2y-x,y)$. Finally, W is locally bounded from below, in virtue of (b) and the fact that V also has this property. This, combined with the midpoint concavity of W along the lines of slope 0 or 1, yields (c). $\square$

We are ready to study (1.9). Let $p>2$, $0<\gamma_p<C_p$ and assume that

$$\left|\left\{r\in[0,1]:\sum_{k=0}^n a_k\theta_k h_k(r)\geq 1\right\}\right| \leq \gamma_p^p\int_0^1\left|\sum_{k=0}^n a_kh_k(r)\right|^p \mathrm{d}r, \tag{3.15}$$

for all n and all $a_k\in\mathbb{R}$, $\theta_k\in\{0,1\}$, $k=0,1,2,\ldots,n$. Take $\beta_p\in(\gamma_p,C_p)$ and

$$V_p^{\beta_p}(x,y)=1_{\{y\geq 1\}}-\beta_p^p|x|^p.$$

Let W_p be given by (3.14), with $V=V_p^{\beta_p}$. For clarity, we split the remaining part of the proof into a few steps. Recall b and c, given by (3.9) and (1.7), respectively.

Step 1. The starting point is the equation

$$W_p(0,y)=0 \qquad \text{for sufficiently small } y>0. \tag{3.16}$$

To see this, note first that if $y<1$, then, by (b), $W_p(0,y)\geq 0$. On the other hand, let y be a positive number satisfying $\beta_p(1-y)\geq\gamma_p$. Take an integer n, numbers $a_1,a_2,\ldots,a_n$ belonging to $\mathbb{R}$ and $\theta_1,\theta_2,\ldots,\theta_n$ belonging to $\{0,1\}$. We have

$$\begin{aligned}
&\left|\left\{r\in[0,1]:y+\sum_{k=1}^n a_k\theta_kh_k(r)\geq 1\right\}\right|-\beta_p^p\int_0^1\left|\sum_{k=1}^n a_kh_k(r)\right|^p\mathrm{d}r\\
&=\left|\left\{r\in[0,1]:\sum_{k=1}^n \frac{a_k}{1-y}\theta_kh_k(r)\geq 1\right\}\right|-\beta_p^p(1-y)^p\int_0^1\left|\sum_{k=1}^n \frac{a_k}{1-y}h_k(r)\right|^p\mathrm{d}r\\
&\leq\left|\left\{r\in[0,1]:\sum_{k=1}^n \frac{a_k}{1-y}\theta_kh_k(r)\geq 1\right\}\right|-\gamma_p^p\int_0^1\left|\sum_{k=1}^n \frac{a_k}{1-y}h_k(r)\right|^p\mathrm{d}r\leq 0,
\end{aligned}$$

where the latter estimate follows from (3.15). Since n and the numbers a_k and θ_k were arbitrary, this gives $W_p(0,y)\leq 0$ and we are done.

Step 2. Note that the points

$$P_1=\big(1-b/2,1\big),\ P_2=\big(0,b/2\big),\ P_3=\big(-b/(p-1),b(p-3)/(2(p-1))\big)$$

lie, in this order, on a certain line of slope 1. Moreover, by (b),

$$W_p(P_1)\geq 1-\beta_p^p\left(1-\frac{b}{2}\right)^p=1-\beta_p^pc^p\left(\frac{b}{p-1}\right)^p$$

and

$$W_p(P_3)\geq-\beta_p^p\left(\frac{b}{p-1}\right)^p.$$

Thus, combining this with (c),

$$\begin{aligned}
W_p(P_2)&\geq\frac{|P_3-P_2|}{|P_3-P_1|}W_p(P_1)+\frac{|P_2-P_1|}{|P_3-P_1|}W_p(P_3)\\
&\geq\frac{1}{c+1}-\frac{c}{c+1}\beta_p^p\left(\frac{b}{p-1}\right)^p(1+c^{p-1}).
\end{aligned}\tag{3.17}$$

Step 3. Fix positive numbers y and δ. Using (c) and then (b), we get

$$\begin{aligned}
W_p(0,y)&\geq\frac{\delta(p-1)}{\delta(p-1)+2}W_p\left(\frac{2y}{p-1},y\right)+\frac{2}{\delta(p-1)+2}W_p(-\delta y,y)\\
&\geq-\frac{\delta(p-1)}{\delta(p-1)+2}\beta_p^p\left(\frac{2y}{p-1}\right)^p+\frac{2}{\delta(p-1)+2}W_p(-\delta y,y).
\end{aligned}$$

Similarly, we get

$$W_p(-\delta y, y) \geq -\frac{\delta(p-1)}{2(1+\delta)}\beta_p^p \left(\frac{2y(1+\delta)}{p-1}\right)^p + \frac{2(1+\delta)-\delta(p-1)}{2(1+\delta)} W_p(0, y(1+\delta))$$

and plugging this into the preceding estimate gives

$$W_p(0,y) \geq A(\delta)\beta_p^p y^p + B(\delta) W_p(0, y(1+\delta)), \tag{3.18}$$

where

$$A(\delta) = -\left(\frac{2}{p-1}\right)^p \frac{\delta(p-1)}{\delta(p-1)+2}(1+(1+\delta)^{p-1})$$

and

$$B(\delta) = \frac{2(1+\delta)-\delta(p-1)}{(1+\delta)(\delta(p-1)+2)}.$$

One easily verifies that we have the following asymptotics:

$$\lim_{\delta\to 0} \frac{A(\delta)}{\delta} = -2\left(\frac{2}{p-1}\right)^{p-1}, \tag{3.19}$$

$$\lim_{\delta\to 0} \frac{B(\delta)(1+\delta)^{p-1}-1}{\delta^2} = 0. \tag{3.20}$$

By induction, (3.18) leads to

$$W_p(0,y) \geq \beta_p^p y^p A(\delta) \frac{[B(\delta)(1+\delta)^p]^N - 1}{B(\delta)(1+\delta)^p - 1} + B(\delta)^N W_p(0, y(1+\delta)^N) \tag{3.21}$$

for any positive integer N.

Step 4. This is the final part. Take $\delta = N^{-1/2}$, $y = b(1+\delta)^{-N}/2$, divide both sides of (3.21) by y^{p-1} and let $N \to \infty$. We obtain the estimate

$$0 \geq -b\beta_p^p \left(\frac{2}{p-1}\right)^{p-1} + \left(\frac{2}{b}\right)^{p-1} W_p\left(0, \frac{b}{2}\right). \tag{3.22}$$

To see this, note that $\delta \to 0$, $y \to 0$ as $N \to \infty$ and therefore, by (3.16), the left-hand side of (3.21), divided by y^{p-1}, converges to 0. To deal with the right-hand side, observe that, by (3.19) and (3.20),

$$\lim_{\delta\to 0} \frac{A(\delta)}{B(\delta)(1+\delta)^p - 1} = \lim_{\delta\to 0} \frac{A(\delta)}{\delta B(\delta)(1+\delta)^{p-1}} = -2\left(\frac{2}{p-1}\right)^{p-1}.$$

Furthermore, again by (3.20), we have

$$y\big([B(\delta)(1+\delta)^p]^N - 1\big) = \frac{b}{2}[B(\delta)(1+\delta)^{p-1}]^{1/\delta^2} \to \frac{b}{2}$$

and, similarly,

$$\frac{B(\delta)^N}{y^{p-1}} = \left(\frac{2}{b}\right)^{p-1} (B(\delta)(1+\delta)^{p-1})^{1/\delta^2} \to \left(\frac{2}{b}\right)^{p-1}.$$

The above three limits yield (3.22). Combining this estimate with (3.17) gives

$$\beta_p^p \left[b\left(\frac{2}{p-1}\right)^{p-1} + \left(\frac{2}{b}\right)^{p-1} \frac{c}{c+1}\left(\frac{b}{p-1}\right)^p (1+c^{p-1})\right] \geq \left(\frac{2}{b}\right)^{p-1} \frac{1}{c+1}$$

After some cancellations and manipulations, it can be written in the form

$$\beta_p^p \geq \frac{(p-1)^p}{b^p(c+1)}\left[p-1+\frac{c}{c+1}(1+c^{p-1})\right]^{-1} = C_p^p,$$

where the equality follows from (1.7). This contradicts the initial assumption $\beta_p \in (\gamma_p, C_p)$ and completes the proof.

4. Proof of Lemma 3.3

We start with a reduction step: it suffices to establish the majorization in the real case and for x, y satisfying $0 \leq x \leq 2y$. To see this, let us (for a moment) write $U_p^{\mathcal{H}}$, $V_p^{\mathcal{H}}$ instead of U_p, V_p, to indicate the Hilbert space we are working with. For x, $y \in \mathcal{H}$, take $x' = |x|$ and $y' = |x/2| + |x/2 - y|$. Then $0 \leq x' \leq 2y'$, $2y' - x' = |2y - x|$ and $y' \geq |y|$, so

$$U_p^{\mathcal{H}}(x,y) - V_p^{\mathcal{H}}(x,y) \geq U_p^{\mathbb{R}}(x',y') - V_p^{\mathbb{R}}(x',y').$$

This justifies the reduction. We consider the cases $y \leq b/2$, $y \in (b/2, 1)$ and $y \geq 1$ separately in the three lemmas below.

Lemma 4.1. *We have*

$$\frac{2}{p(p-1)(p-2)}(2y)^{p-1}\big(2y - px\big) \geq -\frac{2(p-1)^{p-2}}{p(p-2)}x^p. \tag{4.1}$$

This yields the majorization (3.12) *for* $y \leq b/2$.

Proof. The estimate is clear for $x = 0$. If $x > 0$ and we divide both sides by x^p, the inequality takes the form $F_0(2y/x) \geq 0$, where

$$F_0(s) := \frac{2}{p(p-1)(p-2)}s^{p-1}(s-p) + \frac{2(p-1)^{p-2}}{p(p-2)}$$

for $s > 0$. It suffices to note that F_0 is convex and $F_0(p-1) = F_0'(p-1) = 0$. □

Lemma 4.2. *(i) For all* $s \geq 0$,

$$b^{p-2}\left(-\frac{s}{p-2} + \frac{b^2}{p(p-1)^2}\right) + \frac{2(p-1)^{p-2}}{p(p-2)}s^{p/2} \geq 0. \tag{4.2}$$

(ii) We have

$$b^{p-2}\left[\frac{4y^2 - 4xy}{p-2} - \frac{2b(2y-x)}{p-1} + \frac{b^2}{p}\right] + \frac{2(p-1)^{p-2}}{p(p-2)}x^p \geq 0. \tag{4.3}$$

This yields the majorization (3.12) *for* $y \in (b/2, 1)$.

Proof. (i) Denote the left-hand side of (4.2) by $F_1(s)$. It is evident that the function F_1 is convex on $\mathbb{R}$. In addition, it is straightforward to check that

$$F_1(b^2/(p-1)^2) = F_1'(b^2/(p-1)^2) = 0. \tag{4.4}$$

The claim follows.

(ii) The partial derivative of the left-hand side of (4.3) with respect to y equals

$$\frac{4b^{p-2}}{p-2}\left(2y - x - \frac{b(p-2)}{p-1}\right).$$

Thus it suffices to verify the estimate for $2y = x + b(p-2)/(p-1)$. Plug this into (4.3) to get the inequality $F_1(x^2) \geq 0$, which has been already proved in (i). □

Lemma 4.3. *If $y \geq 1$, then*

$$b^{p-2}\left[\frac{4y^2-4xy}{p-2}-\frac{2b(2y-x)}{p-1}+\frac{b^2}{p}\right]-\frac{2(p-1)^{p-2}}{p(p-2)}\left(\frac{1}{C_p^p}-x^p\right)\geq 0. \tag{4.5}$$

This yields the majorization (3.12) *for $y \geq 1$.*

Proof. We divide the proof into three parts.

Step 1. A reduction. Denoting the left hand side of (4.5) by $F_2(x,y)$, we derive that its partial derivative with respect to y is given by

$$F_{2y}(x,y)=\frac{4b^{p-2}}{p-2}\left(2y-x-\frac{b(p-2)}{p-1}\right).$$

In consequence, it suffices to establish the estimate on the line segment

$$H_1=\left\{(x,y):y=1,\,x\geq 0,\,2y-x-\frac{b(p-2)}{p-1}\geq 0\right\}$$

and the halfline

$$H_2=\left\{(x,y):y>1,\,2y-x-\frac{b(p-2)}{p-1}=0\right\}.$$

Step 2. The segment H_1. It is obvious that $x\mapsto F_2(x,1)$ is convex on the interval $[0,2-b(p-2)/(p-1)]$. After some lengthy, but easy calculations we verify that

$$1-\frac{b}{2}<2-\frac{b(p-2)}{p-1},\quad\text{and}\quad F_2\left(1-\frac{b}{2},1\right)=F_2'\left(1-\frac{b}{2},1\right)=0,$$

which yields the estimate on H_1.

Step 3. The halfline H_2. Plugging $2y=x+b(p-2)/(p-1)$ to the estimate transforms it into

$$b^{p-2}\left(-\frac{x^2}{p-2}+\frac{b^2}{p(p-1)^2}\right)+\frac{2(p-1)^{p-2}}{p(p-2)}x^p\geq\frac{2(p-1)^{p-2}}{p(p-2)C_p^p}. \tag{4.6}$$

The left-hand side is equal to $F_1(x^2)$, where F_1 was defined in the proof of Lemma 4.2. Note that we have

$$x\geq 2-\frac{b(p-2)}{p-1}\geq\frac{b}{p-1},$$

the latter being equivalent to $b\leq 2$, which is obvious. Thus, by the convexity of F_1 and (4.4), we get that the left-hand side of (4.6) attains its minimum at $x=2-b(p-2)/(p-1)$. However, then the estimate reads $F_2(2-b(p-2)/(p-1))\geq 0$, and we have already showed this in the preceding step. The proof is complete. □

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Department of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

E-mail address: `ados@mimuw.edu.pl`