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FUNCTIONAL EQUATIONS AND SHARP WEAK–TYPE INEQUALITIES FOR THE MARTINGALE SQUARE FUNCTION

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Abstract. The paper aims at the identification of the best constants in the weak-type (p,p) inequalities for the martingale square function, $1 \leq p < \infty$. To accomplish this, a related optimal stopping problem for the space-time Brownian motion is investigated. Interestingly, the analysis of the cases $1 \leq p \leq 2$ and $2 < p < \infty$ requires completely different methods. Namely, in the first case the corresponding value function can be written down explicitly; in the second case the approach rests on the careful analysis of an interesting, integral functional equation.

1. Introduction

Square function inequalities play an important role in both classical and noncommutative probability theory, harmonic analysis, potential theory and many other areas of mathematics. The purpose of this paper is to establish a class of sharp weak-type bounds for the square function of a continuous-path martingale. These results are motivated by closely related works of Burkholder [3], Davis [7], Novikov [13], Pedersen and Peskir [17], Shepp [22], Wang [23] and many others.

We begin by introducing the necessary background and notation. Suppose that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, filtered by a nondecreasing family $(\mathscr{F}_n)_{n=0}^{\infty}$ of sub-σ-fields of $\mathscr{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted real-valued martingale and let $df = (df_n)$ stand for its difference sequence:

$$df_0 = f_0, \qquad df_n = f_n - f_{n-1}, \quad n = 1, 2, \ldots.$$

Then $S(f)$, the square function of f, is given by

$$S(f) = \left(\sum_{n=0}^{\infty} |df_n|^2 \right)^{1/2}.$$

There is an interesting general question about the comparison of the sizes of f and $S(f)$, which is, for instance, of fundamental importance to the theory of stochastic integration. The literature on the subject is very large, the results are connected with

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other areas of mathematics and it is impossible to give even a brief review here. For some of the aspects of this subject, we refer the interested reader to the survey [5] by Burkholder, the exposition [15] by the author or the monograph [21] by Revuz and Yor. To present our motivation, let us mention the moment inequalities

$$c_p||S(f)||_p \leq ||f||_p \leq C_p||S(f)||_p, \qquad 1 \leq p < \infty, \tag{1.1}$$

where $||f||_p = \sup_{n\geq 0}||f_n||_p$ and c_p, C_p are absolute constants depending only on p. These estimates go back to the classical works of Khintchine [10], Littlewood [11], Marcinkiewicz [12] and Paley [16] (of course, the concept of a martingale did not appear in those papers; the results were formulated in terms of partial sums of the Rademacher functions and the Haar system). Burkholder [4] proved that if $1 < p < \infty$, then (1.1) holds with $c_p^{-1} = C_p = p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. It turns out that this choice of c_p is optimal for $1 < p \leq 2$, and C_p is the best for $p \geq 2$. Furthermore, if $p = 1$, then the left inequality in (1.1) does not hold with any finite c_1, while the best choice for C_1 is 2 (cf. [14]).

Using straightforward approximation arguments, one can transfer (1.1) from the discrete- to the continuous-time setting. Assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathscr{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathscr{F}$, such that $\mathscr{F}_0$ contains all the events of probability 0. Let X be an adapted, real-valued cadlag martingale. Then the role of the square function is played by $[X,X] = ([X,X]_t)_{t\geq 0}$, the quadratic covariance process (or square bracket) of X. See e.g. Dellacherie and Meyer [9] for detailed definition of this object and examples. Then we have

$$c_p||[X,X]_\infty^{1/2}||_p \leq ||X||_p \leq C_p||[X,X]_\infty^{1/2}||_p, \qquad 1 \leq p < \infty, \tag{1.2}$$

where c_p, C_p are the same as in (1.1) and $||X||_p = \sup_{t\geq 0}||X_t||_p$. From the point of view of the applications, it is often interesting to study the above bound for a special class of martingales. We will be particularly interested in the case when X has continuous paths and starts from 0. Then the corresponding sharp versions of (1.2) are due to Davis [7]. Let ν_p be the smallest positive zero of M_p, the confluent hypergeometric function, and μ_p be the largest positive zero of D_p, the parabolic cylinder function of parameter p (see Abramovitz and Stegun [1] for the necessary definitions). Then the best possible constants for C_p are ν_p when $0 < p < 2$ and μ_p for $2 \leq p < \infty$. On the other hand, the best possible constants for c_p are μ_p when $1 < p < 2$ and ν_p when $2 \leq p < \infty$.

When $p = 1$, then the left inequality in (1.1) does not hold with any finite constant, even for continuous-path martingales. However, as usual, one can establish the weak-type bound

$$||[X,X]_\infty^{1/2}||_{1,\infty} \leq C||X||_1$$

for some universal C. Here

$$||[X,X]_\infty^{1/2}||_{p,\infty} = \sup_\lambda \lambda \left(\mathbb{P}\big([X,X]_\infty^{1/2} \geq \lambda\big)\right)^{1/p}, \qquad 1 \leq p < \infty,$$

denotes the weak p-th norm of $[X,X]_\infty^{1/2}$. This estimate, with various constants, can be found in many papers. For instance, Burkholder [3] showed that one can take $C = 3$, and it follows from the work of Bollobás [2] that $C = 1.4623$ suffices. What about the optimal choice for C? The primary goal of this paper is to study this question for a wider class of parameters. Namely, for any $1 \leq p < \infty$, we will identify the optimal constant β_p in the inequality

$$||[X,X]_\infty^{1/2}||_{p,\infty} \leq \beta_p ||X||_p \tag{1.3}$$

under the assumption that X is a real-valued continuous-path martingale satisfying $X_0 = 0$.

A few words about our approach. By Dambis-Dubins-Schwarz theorem (cf. [6] and [8]; a convenient reference is Chapter V of [21]), it suffices to study the above weak-type inequalities in the case when X is a stopped Brownian motion. Let $1 \leq p < \infty$ be a fixed number. To put the problem in the right framework, let us introduce the gain function $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by the formula

$$G(x,t) = 1_{\{t \geq 1\}} - |x|^p.$$

For a fixed $t \in \mathbb{R}$, consider the optimal stopping problem

$$\sup_\tau \mathbb{E} G(B_\tau, t+\tau), \tag{1.4}$$

where B is a standard Brownian motion and the supremum is taken over all adapted bounded stopping times. As we will see, the (partial) solution to (1.4) will immediately yield the optimal constants β_p in the weak-type bound (1.3).

It may be a little surprising that the analysis in the cases $1 \leq p \leq 2$ and $p > 2$ is completely different. The first case is very easy: the optimal stopping rule and the value of the supremum in (1.4) can be written down explicitly (in particular, β_p can be expressed by a compact formula). On the other hand, the case $p > 2$ requires the solution of an underlying free-boundary problem. This is a much more difficult task and will involve the analysis of a certain integral functional equation. The constant β_p will be expressed in terms of the solution to this equation; no compact explicit formula for β_p seems to exist in this case. The approach is of independent interest and, as we hope, can be applied in a number of related important results.

We have organized this paper as follows. In the next section we analyze the optimal stopping problem (1.4) in the case $1 \leq p \leq 2$. Section 3 is the main part of the paper and contains the study of (1.4) for $2 < p < \infty$.

2. Easy case

We start with some general observations which will also be useful in the case $p > 2$. A successful treatment of (1.4) requires the extension of the problem so that the space-time process (B_t, t) can start at arbitrary points in the state space $\mathbb{R} \times \mathbb{R}$. This is standard: consider the function $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by the formula

$$V(x,t) = \sup \mathbb{E} G(x + B_\tau, t+\tau), \tag{2.1}$$

where the supremum is taken over all adapted bounded stopping times τ. The process $(|x+B_t|^p)_{t\geq 0}$ is a submartingale, so for $t\geq 1$ and any bounded τ we have $\mathbb{E}G(x+B_\tau,t+\tau)\leq 1-|x|^p$. On the other hand, $\tau\equiv 0$ gives equality here, so we conclude that

$$V(x,t)=1-|x|^p \qquad \text{for } t\geq 1. \tag{2.2}$$

In addition, if $t<1$, then in the computation of $V(x,t)$, we may restrict ourselves to stopping times bounded by $1-t$; indeed, this follows from the inequality

$$\mathbb{P}(t+\tau\geq 1)-\mathbb{E}|x+B_\tau|^p\leq \mathbb{P}(t+(\tau\wedge(1-t))\geq 1)-\mathbb{E}|x+B_{\tau\wedge(1-t)}|^p.$$

Now we assume that $1\leq p\leq 2$. Actually, it will be enough for us to find the solution to (2.1) (i.e., the formula for V) on a part of the domain $\mathbb{R}\times\mathbb{R}$. Let

$$t_0=t_0(p)=1-\frac{\pi^{1/p}}{2\Gamma(\frac{p}{2}+\frac{1}{2})^{2/p}}\leq 0.$$

For the motivation for t_0, look at the last line in (2.5) below. To see the inequality $t_0\leq 0$, observe that

$$t_0=1-(\mathbb{E}|B_1|^p)^{-2/p}\leq 1-(\mathbb{E}|B_1|^2)^{-1}=0.$$

We will show that if $x\in\mathbb{R}$ and $t\geq t_0$, then the optimal stopping time for $V(x,t)$ equals $\tau\equiv(1-t)_+$; thus the value function V can be expressed explicitly on $[t_0,\infty)\times\mathbb{R}$ in the following manner.

LEMMA 2.1. *If $t\geq t_0$ and $x\in\mathbb{R}$, then*

$$V(x,t)=\begin{cases}1-|x|^p & \text{if } t\geq 1,\\ 1-\mathbb{E}|x+B_{1-t}|^p & \text{if } t_0\leq t<1.\end{cases} \tag{2.3}$$

Proof. Let U denote the right-hand side of (2.3). By (2.2), the formula (2.3) holds true for $t\geq 1$. If $t_0\leq t<1$, observe that

$$U(x,t)=\mathbb{E}G(x+B_{1-t},t+(1-t)),$$

which implies $U\leq V$, by the very definition of V. To prove the reverse bound, note that by the strong Markov property, U satisfies the heat equation

$$U_t+\frac{1}{2}U_{xx}=0 \qquad \text{on } \mathbb{R}\times(t_0,1). \tag{2.4}$$

Furthermore, for any $a,b\in\mathbb{R}$ we have

$$|a+b|^p+|a-b|^p\leq 2|a|^p+2|b|^p.$$

Hence, by the symmetry of Brownian motion,

$$\begin{aligned}2\mathbb{E}|x+B_{1-t}|^p&=\mathbb{E}|x+B_{1-t}|^p+\mathbb{E}|x-B_{1-t}|^p\\&\leq 2|x|^p+2\mathbb{E}|B_{1-t}|^p\\&\leq 2|x|^p+2\mathbb{E}|B_{1-t_0}|^p\\&=2|x|^p+2(1-t_0)^{p/2}\mathbb{E}|B_1|^p=2|x|^p+2,\end{aligned} \tag{2.5}$$

which is equivalent to $G(x,t) \leq U(x,t)$. Therefore, if τ is a stopping time bounded by $1-t$, then, by Itô's formula and (2.4),

$$\mathbb{E}G(x+B_\tau,t+\tau) \leq \mathbb{E}U(x+B_\tau,t+\tau) = U(x,t).$$

Taking supremum over τ gives $V(x,t) \leq U(x,t)$. This completes the proof.

THEOREM 2.1. *Let $1 \leq p \leq 2$ be fixed. For any continuous-path martingale X starting from 0 we have the sharp inequality*

$$||[X,X]_\infty^{1/2}||_{p,\infty} \leq \frac{\pi^{1/(2p)}}{2^{1/2}\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)^{1/p}}||X||_p. \tag{2.6}$$

Proof. We will use the Dambis-Dubins-Schwarz theorem and represent X as a time-changed Brownian motion. Clearly, we may and do assume that $||X||_p < \infty$, since otherwise there is nothing to prove. Then the pointwise limit X_∞ of $(X_t)_{t\geq 0}$ exists and is finite almost surely. For any $t \geq 0$, introduce the stopping time

$$T_t = \inf\{s : [X,X]_s > t\}$$

and define the process W by

$$W_t = \begin{cases} X_{T_t} & \text{for } t < [X,X]_\infty, \\ X_\infty & \text{for } t \geq [X,X]_\infty. \end{cases}$$

Then there is an enlargement $(\tilde{\mathscr{F}}_t)_{t\geq 0}$ of the filtration $(\mathscr{F}_t)_{t\geq 0}$ such that W is an $(\tilde{\mathscr{F}}_t)_{t\geq 0}$-Brownian motion stopped at $[X,X]_\infty$ (see Theorem 1.7 of Chapter V in [21]). Consequently, by Lemma 2.1, we may write

$$\mathbb{P}([X,X]_\infty \geq 1-t_0) = \mathbb{P}(t_0+[X,X]_\infty \geq 1) \leq \mathbb{E}\left|W_{[X,X]_\infty}\right|^p + V(0,t_0) = ||X||_p^p,$$

since $V(0,t_0) = 1-\mathbb{E}|B_{1-t_0}|^p = 1-(1-t_0)^{p/2}\mathbb{E}|B_1|^p = 0$. By homogeneity, the above bound implies that for any $\lambda > 0$,

$$\lambda^p\mathbb{P}([X,X]_\infty^{1/2} \geq \lambda) \leq (1-t_0)^{p/2}||X||_p^p = \frac{\pi^{1/2}}{2^{p/2}\Gamma\left(\frac{p}{2}+\frac{1}{2}\right)}||X||_p^p,$$

and taking the supremum over λ gives (2.6). To show that this estimate is sharp, let B be a Brownian motion and pick $\tau = 1-t_0$. Then $||\tau^{1/2}||_{p,\infty} = (1-t_0)^{1/2}$ and $||B_\tau||_p = 1$, so both sides of (2.6) are equal.

3. Difficult case

Now we turn to the much more elaborate case of large p. As previously, we extend the optimal stopping problem to the form (2.1). Then $V(x,t) = 1-|x|^p$ for $t \geq 1$, and if $t < 1$, we may restrict ourselves to stopping times τ which are bounded by $1-t$. For the sake of clarity and convenience, we have decided to split the remaining reasoning into eleven intermediate steps.

Step 1. First we will show that the function V is continuous. Pick $t<1$, $x,y\in\mathbb{R}$ and let τ be an arbitrary stopping time bounded by $1-t$. The process $(|x+B_s|^{p-1})_{s\geq 0}$ is a submartingale, so by mean value property,

$$\mathbb{E}|y+B_\tau|^p-\mathbb{E}|x+B_\tau|^p\leq p\mathbb{E}|y+B_\tau|^{p-1}|y-x|\leq p|y-x|\cdot\mathbb{E}|y+B_{1-t}|^{p-1}$$

and thus

$$V(x,t)\leq V(y,t)+p|y-x|\cdot\mathbb{E}|y+B_{1-t}|^{p-1},$$

by the definition of V. Therefore, by symmetry,

$$|V(x,t)-V(y,t)|\leq p|y-x|\mathbb{E}(|x|+|y|+|B_{1-t}|)^{p-1},\tag{3.1}$$

so for each fixed t, the function $x\mapsto V(x,t)$ is locally Lipschitz. Next, fix $s<t\leq 1$. For a given x, the function $u\mapsto G(x,u)$ is nondecreasing and hence V also has this property: thus, $V(x,s)\leq V(x,t)$. Furthermore, if τ is an arbitrary stopping time bounded by $1-t$ and we put $\sigma=\tau+t-s$, then

$$\begin{aligned}
&\mathbb{P}(\tau\geq 1-t)-\mathbb{E}|x+B_\tau|^p\\
&\quad=\mathbb{P}(\sigma\geq 1-s)-\mathbb{E}|x+B_\sigma|^p+\mathbb{E}\Big[|x+B_{\tau+t-s}|^p-|x+B_\tau|^p\Big]\\
&\quad\leq V(x,s)+p\mathbb{E}|x+B_\tau|^{p-1}|B_{\tau+t-s}-B_\tau|\\
&\quad\leq V(x,s)+p\mathbb{E}|x+B_\tau|^{p-1}\mathbb{E}|B_{\tau+t-s}-B_\tau|\\
&\quad\leq V(x,s)+\sqrt{\frac{2(t-s)}{\pi}}p\mathbb{E}|x+B_{1-t}|^{p-1}.
\end{aligned}$$

Therefore, we have

$$0\leq V(x,t)-V(x,s)\leq\sqrt{\frac{2(t-s)}{\pi}}p\mathbb{E}|x+B_{1-t}|^{p-1},$$

which combined with (3.1) yields the continuity of V.

Step 2. Let us provide an abstract formula for the optimal stopping time in (2.1). Introduce the continuation set C and the stopping region D by

$$C=\{(x,t)\in\mathbb{R}\times\mathbb{R}:V(x,t)>G(x,t)\}$$

and

$$D=\{(x,t)\in\mathbb{R}\times\mathbb{R}:V(x,t)=G(x,t)\}.$$

The gain function G is upper semicontinuous and the function V is lower semicontinuous (since it is continuous, in view of Step 1). Therefore, by the general theory of optimal stopping (cf. Corollary 2.9 in Peskir and Shiryaev [20]), for a given state $(x,t)\in\mathbb{R}\times\mathbb{R}$, the stopping time

$$\tau_D=\inf\{s\geq 0:(t+s,x+B_s)\in D\}$$

is optimal in (2.1). Now, standard arguments based on the strong Markov property and classic results from PDEs (see e.g. Chapter III in [20]) show that V is of class $C^{2,1}$ on C and satisfies the heat equation

$$V_t + \frac{1}{2}V_{xx} = 0 \tag{3.2}$$

on this set. These facts will be freely used in the considerations below.

Step 3. Let us provide some insight into the shape of C and D. By the symmetry of Brownian motion, we immediately get that C and D are symmetric with respect to y-axis. By the upper semicontinuity of G and the continuity of V, we conclude that C is open and D is closed. As we have already observed, $\mathbb{R}\times[1,\infty)\subset D$. Now, for any $x\in\mathbb{R}$ we have $\lim_{t\uparrow 1}(1-|x+B_{1-t}|^p) = 1-|x|^p > -|x|^p$; thus, for a given x, we have $V(x,t)>G(x,t)$ for t sufficiently close to 1, and hence $(x,t)\in C$. Next, take $0\leq x<y$ and a bounded stopping time τ. Put $2\delta = y-x$ and set $\sigma = \inf\{s : B_s = -x-\delta\}$. Consider the Brownian motion reflected at time σ, given by

$$W_s = \begin{cases} B_s & \text{if } s<\sigma,\\ 2B_\sigma - B_s & \text{if } s\geq \sigma. \end{cases} \tag{3.3}$$

It is easy to check the majorization

$$|x+B_s| \leq |y+W_s| \qquad \text{for all } s. \tag{3.4}$$

Indeed, if $s\leq\sigma$, then $B_s\geq -x-\delta$, so

$$|y+W_s| = |x+2\delta+B_s| = x+2\delta+B_s \geq |x+B_s|.$$

On the other hand, if $s>\sigma$, then $y+W_s = x+2\delta-2x-2\delta-B_s = -x-B_s$ and $|y+W_s| = |x+B_s|$. Thus, (3.4) follows. Now, if τ is a stopping time bounded by $1-t$, then Itô's formula gives

$$\begin{aligned} V(x,t)-G(x,t) &\geq \mathbb{P}(t+\tau\geq 1) - \mathbb{E}|x+B_\tau|^p - G(x,t)\\ &= \mathbb{P}(t+\tau\geq 1) - \mathbb{E}|x+B_\tau|^p + |x|^p\\ &= \mathbb{P}(t+\tau\geq 1) - \frac{p(p-1)}{2}\mathbb{E}\int_0^\tau |x+B_s|^{p-2}\mathrm{d}s\\ &\geq \mathbb{P}(t+\tau\geq 1) - \frac{p(p-1)}{2}\mathbb{E}\int_0^\tau |y+W_s|^{p-2}\mathrm{d}s\\ &= \mathbb{P}(t+\tau\geq 1) - \mathbb{E}|y+W_\tau|^p + |y|^p. \end{aligned}$$

Hence, taking supremum over τ yields

$$V(x,t)-G(x,t) \geq V(y,t)-G(y,t).$$

Therefore, C enjoys the following: if $(y,t)\in C$, then $[-y,y]\times\{t\}\subset C$.

To prove another geometrical property of the continuation set, note that for a fixed x, the function $t\mapsto V(x,t)$ is nondecreasing (we have already pointed this out above).

Hence the function $t \mapsto V(x,t) - G(x,t)$, $t \in (-\infty, 1)$, also satisfies this condition; thus we may conclude that if $(x,t) \in C$, then the whole line segment $\{x\} \times [t,1)$ is contained in C.

Now we will show that if $t < 1$ is fixed, then $(x,t) \in D$ for sufficiently large $|x|$. Otherwise, by the previous facts, we would have $\mathbb{R} \times [t,1) \subset C$ and hence $\tau \equiv 1-t$ would be optimal for (x,t). But, by Itô's formula,

$$\mathbb{E}|x+B_{1-t}|^p - |x|^p = \frac{p(p-1)}{2} \int_0^{1-t} \mathbb{E}|x+B_s|^{p-2} \mathrm{d}s \to \infty$$

as $|x| \to \infty$. So, if $|x|$ is sufficiently large, then $V(x,t) = 1 - \mathbb{E}|x+B_{1-t}|^p < -|x|^p = G(x,t)$, a contradiction.

The final insight into the structure of C is gained with the use Davis' inequality (1.2). Namely, for any bounded stopping time τ we have

$$\mathbb{P}(t+\tau \geq 1) \leq \mathbb{P}(\tau \geq 1-t) \leq \frac{||\tau^{1/2}||_p^p}{(1-t)^{p/2}} \leq \frac{C_p^p \mathbb{E}|B_\tau|^p}{(1-t)^{p/2}}.$$

Therefore, if $1-t > C_p^2$, then $\mathbb{P}(t+\tau \geq 1) - \mathbb{E}|B_\tau|^p \leq 0 = G(0,t)$ and hence $(0,t) \in D$ (which implies that the whole horizontal line $\mathbb{R} \times \{t\}$ is contained in D, in view of the above properties). The combination of the facts proved above leads to the following statement: there is a nondecreasing function $b : (-\infty, 1) \to [0, \infty)$ which vanishes on some interval $(-\infty, t_0]$ and tends to ∞ as $t \uparrow 1$, such that

$$C = \{(x,t) \in \mathbb{R} \times (-\infty, 1) : |x| < b(t)\}.$$

See Figure 1 below.

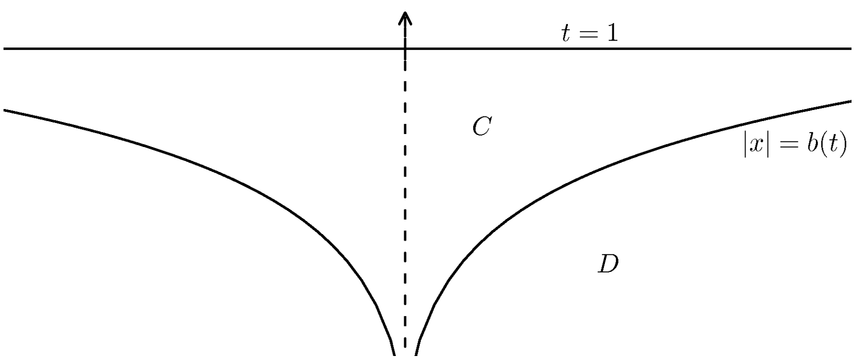

Figure 1: The continuation set lies between the curves $|x| = b(t)$ and $t = 1$.

Step 4. Our next task is to prove that the function b is continuous. Let us first focus on the left-continuity of b. For this, fix $t < 1$ and consider a sequence $(t_n)_{n\geq 0}$ which increases to t as $n \to \infty$. Since b is nondecreasing, the limit $b(t-) = \lim_{n\to\infty} b(t_n)$

exists. Because $(b(t_n),t_n)$ belongs to D for all n and D is closed, it follows that $(b(t-),t)$ belongs to D. This implies $b(t-) \geq b(t)$, and we get the reverse bound by the monotonicity of b. Consequently, b is left-continuous, as claimed.

We turn to the right-continuity of b. Assume on contrary that there is $t > 0$ such that $b(t) < b(t+)$ and pick $x, y \in (b(t), b(t+))$ such that $x < y$. Define the stopping time

$$\tau' = \inf\{s > 0 : (y + B_s, t + s) \in D\}$$

(the difference between τ' and τ_D lies in the fact that in the above infimum we consider *positive* s). The process $(y + B_s, t + s)_{0<s<\tau'}$ takes values in C, so by Itô's formula and (3.2), we have

$$\begin{aligned} 0 = V(y,t) + |y|^p &= \mathbb{E}V(y + B_{\tau'}, t + \tau') - G(y,t) \\ &= \mathbb{E}G(y + B_{\tau'}, t + \tau') - G(y,t) \\ &= \mathbb{P}(t + \tau' \geq 1) - \frac{p(p-1)}{2}\mathbb{E}\int_0^{\tau'} |y + B_s|^{p-2} \mathrm{d}s. \end{aligned} \tag{3.5}$$

Now we repeat the coupling argument from Step 3: let W be the reflected Brownian motion, corresponding to $x < y$, given by (3.3). Directly from its construction and the monotonicity of b, we infer that the process $(x + W_s, t + s)_{0<s<\tau'}$ takes values in C. Indeed, if $(x + W_s, t + s) \in D$ for some $s \in (0, \tau')$, then $\sigma > s$, since otherwise we would have $|x + W_s| = |y + B_s|$, a contradiction with the definition of τ'. But if $\sigma > s$, then $-b(t + s) < -\delta < x + W_s < |y + B_s| < b(t + s)$, which again makes the condition $(x + W_s, t + s) \in D$ impossible. Thus, $(x + W_s, t + s) \in C$ for $s \in (0, \tau)$; furthermore, we have $x + B_{\tau'} > 0$ with positive probability, which implies that $\mathbb{P}((x + W_{\tau'}, t + \tau') \in C) > 0$. Therefore, by Itô's formula,

$$\begin{aligned} 0 = V(x,t) + |x|^p &= \mathbb{E}V(x + W_{\tau'}, t + \tau') - G(y,t) \\ &> \mathbb{E}G(x + W_{\tau'}, t + \tau') - G(y,t) \\ &= \mathbb{P}(t + \tau' \geq 1) - \frac{p(p-1)}{2}\mathbb{E}\int_0^{\tau'} |x + W_s|^{p-2} \mathrm{d}s. \end{aligned}$$

Combining this with (3.5) yields

$$\mathbb{E}\int_0^{\tau'} |x + W_s|^{p-2} \mathrm{d}s > \mathbb{E}\int_0^{\tau'} |y + B_s|^{p-2} \mathrm{d}s,$$

which contradicts the inequality $|x + W_s| \leq |y + B_s|$ which can be proved as in Step 3. This gives the desired continuity of b.

Step 5. Now we will show the following smooth-fit property: for each $t < 1$ the function $x \mapsto V(x,t)$ is differentiable at the point $b(t)$ and satisfies $V_x(b(t),t) = G_x(b(t),t)$. Clearly, it suffices to compare the left derivatives of V and G. Since $V(b(t),t) = G(b(t),t)$, we may write

$$\frac{V(b(t),t) - V(b(t) - \varepsilon, t)}{\varepsilon} \leq \frac{G(b(t),t) - G(b(t) - \varepsilon, t)}{\varepsilon}$$

for all $\varepsilon > 0$ and hence

$$\limsup_{\varepsilon \downarrow 0} \frac{V(b(t),t) - V(b(t)-\varepsilon,t)}{\varepsilon} \le G_x(b(t),t).$$

Let $\tau_\varepsilon = \tau_D(b(t)-\varepsilon,t)$ be optimal for $V(b(t)-\varepsilon,t)$. Then by the mean value theorem we have

$$\begin{aligned}
&\frac{V(b(t),t) - V(b(t)-\varepsilon,t)}{\varepsilon} \\
&\qquad \ge \frac{1}{\varepsilon}\Big(\mathbb{E}G(b(t)+B_{\tau_\varepsilon},t+\tau_\varepsilon) - \mathbb{E}G(b(t)-\varepsilon+B_{\tau_\varepsilon},t+\tau_\varepsilon)\Big) \\
&\qquad = \mathbb{E}G_x(\xi_\varepsilon,t+\tau_\varepsilon),
\end{aligned}$$

where ξ_ε lies between $b(t)-\varepsilon+B_{\tau_\varepsilon}$ and $b(t)+B_{\tau_\varepsilon}$. Since $t \mapsto b(t)$ is nondecreasing and $t \mapsto \lambda t$ is a lower function for B at $0+$ for every $\lambda \in \mathbb{R}$, one easily proves that $\tau_\varepsilon \to 0$ as $\varepsilon \downarrow 0$. Consequently, $\xi_\varepsilon \to b(t)$ and $G_x(\xi_\varepsilon,t+\tau_\varepsilon) \to G_x(b(t),t)$ as $\varepsilon \downarrow 0$. In addition,

$$|G_x(\xi_\varepsilon,t+\tau_\varepsilon)| \le p\big(|b(t)+B_{\tau_\varepsilon}|+\varepsilon\big)^{p-1} \le p\big(\sup_{0\le s\le 1-t}|b(t)+B_s|+\varepsilon\big)^{p-1}$$

and the latter variable is integrable. Therefore, we may conclude that

$$\liminf_{\varepsilon \downarrow 0} \frac{V(b(t),t) - V(b(t)-\varepsilon,t)}{\varepsilon} \ge G_x(b(t),t)$$

by Lebesgue's dominated convergence theorem. This proves the desired smoothness of V.

Step 6. Now we will apply a local time-space formula of Peskir [19]. Let us first list the necessary properties of V, which will allow us to use this result. We denote by A^o the interior of a set A.

$$V \text{ is of class } C^{2,1} \text{ on } C \cup D^o, \tag{3.6}$$

$$V_t + V_{xx}/2 \text{ is locally bounded on } C \cup D^o, \tag{3.7}$$

$$t \mapsto V_x(b(t),t) \text{ is continuous on } (-\infty,1), \tag{3.8}$$

$$x \mapsto V(x,t) \text{ is concave.} \tag{3.9}$$

Indeed: (3.6) is obvious; we have $V_t + V_{xx}/2 = 0$ on C and $V_t + V_{xx}/2 = -p(p-1)|x|^{p-2}/2$ on D^o, which gives (3.7); by Step 5, we have $V_x(b(t),t) = -pb(t)^{p-1}$, which is a continuous function of t: see Step 4. Finally, we have $V_t \ge 0$ on C and hence, by (3.2), $V_{xx} \le 0$ on this set; furthermore, $V_{xx}(x,t) = -p(p-1)|x|^{p-2} \le 0$ on D^o. Combining these facts with Step 5 gives that $x \mapsto V(x,t)$ is of class C^1 for all t and the concavity follows. Furthermore, the function b obviously satisfies the condition

$$\mathbb{P}(x+B_s = b(t+s)) = 0 \qquad \text{for all } x \in \mathbb{R} \text{ and } t,\, s > 0.$$

Therefore, by the result of Peskir [19], the following change-of-variable formula holds: for $t < 1$ and $s \leq 1-t$,

$$V(x+B_s, t+s) = V(x,t) + I + II + III, \tag{3.10}$$

where

$$\begin{aligned}
I &= \int_0^s V_x(x+B_u, t+u)\mathrm{d}B_u, \\
II &= \int_0^s \left(V_t + \frac{1}{2}V_{xx}\right)(x+B_u, t+u)1_{\{|x+B_u| \neq b(t+u)\}}\mathrm{d}u \\
III &= \int_0^s \big(V_x(x+B_u+, t+u) - V_x(x+B_u-, t+u)\big)1_{\{x+B_u=-b(t+u)\}}\mathrm{d}\ell_{t+u}^{-b} \\
&\quad + \int_0^s \big(V_x(x+B_u+, t+u) - V_x(x+B_u-, t+u)\big)1_{\{x+B_u=b(t+u)\}}\mathrm{d}\ell_{t+u}^{b}.
\end{aligned}$$

Here ℓ_s^c is the local time of Brownian motion B at curve c given by the limit in probability

$$\ell_s^c = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^s 1_{\{c(r)-\varepsilon < X_r < c(r)+\varepsilon\}}\mathrm{d}r.$$

Let us analyze the terms I through III separately. The process in I is a local martingale; in fact, by (3.9), the partial derivative V_x is locally bounded, so the process is actually a true martingale and hence the stochastic integral has mean zero. The term II can be computed directly:

$$II = -\frac{p(p-1)}{2}\int_0^s |x+B_u|^{p-2}1_{\{|x+B_u|>b(t+u)\}}\mathrm{d}u.$$

Finally, by the smooth-fit principle studied in Step 5, the term III vanishes. Consequently, taking $s = 1-t$ and integrating both sides of (3.10), we get

$$1 - \mathbb{E}|x+B_{1-t}|^p = V(x,t) - \frac{p(p-1)}{2}\int_0^{1-t} \mathbb{E}|x+B_u|^{p-2}1_{\{|x+B_u|>b(t+u)\}}\mathrm{d}u.$$

Now take $x = b(t)$. Then $V(x,t) = -b(t)^p$ and by Itô's formula, we get

$$\int_0^{1-t} \mathbb{E}|x+B_u|^{p-2}1_{\{|x+B_u|\leq b(t+u)\}}\mathrm{d}u = \frac{2}{p(p-1)}.$$

Since $x+B_u \sim \mathcal{N}(x,u)$, we obtain the following functional equation for b:

$$\int_t^1 \int_{|r|\leq b(u)} \frac{|r|^{p-2}}{\sqrt{2\pi(u-t)}}\exp\left(-\frac{(r-b(t))^2}{2(u-t)}\right)\mathrm{d}r\mathrm{d}u = \frac{2}{p(p-1)}. \tag{3.11}$$

Step 7. The purpose of the four steps below is to show that the solution to (3.11) is unique. The reasoning we are going to present is a modification of the technique introduced by Peskir [18] to study an open problem concerning American options. So, suppose that $c:(-\infty,1)\to[0,\infty)$ is a nondecreasing continuous function satisfying the above functional equation. Note that $\lim_{t\uparrow 1} c(t)=\infty$; otherwise, the left-hand side of (3.11) would be bounded from above by an expression of the form

$$\int_t^1 \frac{\alpha_1}{\sqrt{u-t}}\exp\left(-\frac{\alpha_2}{2(u-t)}\right)\mathrm{d}u$$

(for some positive constants α_1, α_2), which converges to 0 as $t\uparrow 1$.

Motivated by the above considerations leading to (3.11), we introduce the auxiliary function $U^c:\mathbb{R}\times(-\infty,1]\to\mathbb{R}$ given by

$$U^c(x,t)=\mathbb{E}G(x+B_{1-t},1)-\frac{p(p-1)}{2}\int_0^{1-t}\mathbb{E}|x+B_u|^{p-2}1_{\{|x+B_u|>c(t+u)\}}\mathrm{d}u.$$

The assumption that c solves (3.11) is equivalent to saying that

$$\int_0^{1-t}\mathbb{E}|x+B_u|^{p-2}1_{\{|x+B_u|\le c(t+u)\}}\mathrm{d}u=\frac{2}{p(p-1)},$$

which, in turn, implies that $U^c(c(t),t)=G(c(t),t)$ for all t. In the remainder of this step we shall prove that $U^c(x,t)=G(x,t)$ provided $|x|\ge c(t)$; this will be accomplished with the use of the following martingale methods.

Observe that if X is a Markov process and we set $F(x,t)=\mathbb{E}_xG(X_{1-t})$ for an integrable function G (where $\mathbb{P}_x$ is a probability measure on the sample space such that $\mathbb{P}_x(X_0=x)=1$), the the Markov property of X implies that $\big(F(X_t,t)\big)_{t\in[0,1]}$ is a martingale under $\mathbb{P}_x$. Similarly, if we set $F(x,t)=\mathbb{E}_x\left(\int_0^{1-t}H(X_s)\mathrm{d}s\right)$ for a sufficiently regular function H, then $\big(F(X_t,t)+\int_0^t H(X_s)\mathrm{d}s\big)_{t\in[0,1]}$ is a martingale under $\mathbb{P}_x$. Applying these facts to the space-time Markov process $((x+B_s,t+s))_{s\ge 0}$, we get that for a fixed number t,

$$\left(U^c(x+B_s,t+s)+\frac{p(p-1)}{2}\int_0^s|x+B_u|^{p-2}1_{\{|x+B_u|>c(t+u)\}}\mathrm{d}u\right)_{s\le 1-t}\tag{3.12}$$

is a martingale. On the other hand, we have

$$G(x+B_s,t+s)=G(x,t)-\frac{p(p-1)}{2}\int_0^s|x+B_u|^{p-2}\mathrm{d}u+M_s,$$

where

$$M_s=p\int_0^s|x+B_u|^{p-2}(x+B_u)\mathrm{d}B_u,\qquad 0<s<1-t,$$

is a martingale. Suppose that $|x|>c(t)$ and consider the stopping time

$$\sigma_c=\inf\{0<s<1-t:|x+B_s|=c(t+s)\}.$$

The stopping time σ_c is bounded by $1-t$, since c is continuous and explodes at the right endpoint of its domain. As we have already pointed out above, we have the equality $U^c(x+B_{\sigma_c},t+\sigma_c)=G(x+B_{\sigma_c},t+\sigma_c)$ and thus

$$\begin{aligned}
&U^c(x,t)\\
&=\mathbb{E}\left[U^c(x+B_{\sigma_c},t+\sigma_c)+\frac{p(p-1)}{2}\int_0^{\sigma_c}|x+B_u|^{p-2}1_{\{|x+B_u|>c(t+u)\}}\mathrm{d}u\right]\\
&=\mathbb{E}\left[G(x+B_{\sigma_c},t+\sigma_c)+\frac{p(p-1)}{2}\int_0^{\sigma_c}|x+B_u|^{p-2}\mathrm{d}u\right]\\
&=G(x,t).
\end{aligned}$$

Step 8. Let us prove that $U^c\leq V$. To do this, fix $x\in\mathbb{R}$, $t<1$ and consider the stopping time

$$\tau_c=\inf\{0\leq s<1-t:|x+B_s|\geq c(t+s)\},$$

with the convention $\inf\emptyset=1-t$. By the previous step and the equality $U^c(x,1)=G(x,1)$ (which is obvious from the formula for U^c) we have $U^c(x+B_{\tau_c},t+\tau_c)=G(x+B_{\tau_c},t+\tau_c)$. Using the martingale property of the process (3.12) and the fact that the integrand appearing in its definition vanishes for $u<\tau_c$, we obtain

$$U^c(x,t)=\mathbb{E}U^c(x+B_{\tau_c},t+\tau_c)=\mathbb{E}G(x+B_{\tau_c},t+\tau_c)\leq V(x,t),$$

as desired.

Step 9. We are ready to show that $c(t)\leq b(t)$ for all $t<1$. Suppose that there is $t<1$ for which the reverse inequality $c(t)>b(t)$ holds. Introduce the stopping time

$$\sigma_b=\inf\{0\leq s<1-t:c(t)+B_s=b(t+s)\}$$

(since $\lim_{u\uparrow 1}b(u)=\infty$, the above definition makes sense). By Itô formula, we have

$$\mathbb{E}V(c(t)+B_{\sigma_b},t+\sigma_b)=V(c(t),t)-\frac{p(p-1)}{2}\mathbb{E}\int_0^{\sigma_b}|c(t)+B_u|^{p-2}\mathrm{d}u$$

and, by the martingale property of the process (3.12),

$$\begin{aligned}
&\mathbb{E}U^c(c(t)+B_{\sigma_b},t+\sigma_b)\\
&=U^c(c(t),t)-\frac{p(p-1)}{2}\mathbb{E}\int_0^{\sigma_b}|c(t)+B_u|^{p-2}1_{\{|c(t)+B_u|>c(t+u)\}}\mathrm{d}u.
\end{aligned}$$

But we have $V(c(t),t)=G(c(t),t)=U^c(c(t),t)$ and, by the previous step, $U^c(c(t)+B_{\sigma_b},t+\sigma_b)\leq V(c(t)+B_{\sigma_b},t+\sigma_b)$. Consequently, the two equalities above imply that

$$\mathbb{E}\int_0^{\sigma_b}|c(t)+B_u|^{p-2}1_{\{|c(t)+B_u|\leq c(t+u)\}}\mathrm{d}u\leq 0,$$

which is impossible in view of the continuity of b and c.

Step 10. Finally, we show that $b \leq c$, which will complete the proof of the uniqueness. Suppose on contrary that there is t for which $c(t) < b(t)$ and pick $x \in (c(t), b(t))$. Let

$$\tau_D = \inf\{s > 0 : (x + B_s, t + s) \in D\}.$$

We have

$$\mathbb{E}G(x + B_{\tau_D}, t + \tau_D) = V(x, t)$$

and, since $G(x + B_{\tau_D}, t + \tau_D) = U^c(x + B_{\tau_D}, t + \tau_D)$, we get

$$\begin{aligned}\mathbb{E}G(x + B_{\tau_D}, t + \tau_D) &= U^c(x, t) \\ &\quad - \frac{p(p-1)}{2}\mathbb{E}\left(\int_0^{\tau_D} |x + B_{t+u}|^{p-2} 1_{\{|x+B_u| > c(t+u)\}} \mathrm{d}u\right).\end{aligned}$$

However, we have $V \geq U^c$ (see Step 8), so the two identities above imply

$$\mathbb{E}\left(\int_0^{\tau_D} |x + B_{t+u}|^{p-2} 1_{\{|x+B_u| > c(t+u)\}} \mathrm{d}u\right) \leq 0,$$

which cannot hold, because of the continuity of b and c. This completes the proof of the uniqueness of the solution to (3.11).

Step 11. Finally, we are ready to establish the main statement of this section. Let $t_0 = \sup\{t : b(t) = 0\}$.

THEOREM 3.1. *Suppose that $2 < p < \infty$. Then for any continuous-path martingale X we have the sharp bound*

$$||[X, X]_\infty^{1/2}||_{p,\infty} \leq (1 - t_0)^{1/2} ||X||_p. \tag{3.13}$$

Proof. We proceed as in the proof of Theorem 2.1. Let W be the stopped Brownian motion introduced there; then

$$\mathbb{P}([X, X]_\infty \geq 1 - t_0) \leq ||X||_p^p + V(0, t_0) = ||X||_p^p,$$

which, by homogeneity, yields

$$\lambda^p \mathbb{P}([X, X]_\infty \geq \lambda) \leq (1 - t_0)^{p/2} ||X||_p^p$$

and (3.13) follows. To see that the constant $(1 - t_0)^{1/2}$ cannot be improved, pick an arbitrary number $t > t_0$ and let τ be the optimal stopping time for $(0, t)$. We have $V(0, t) > 0$ and

$$\mathbb{P}(\tau \geq 1 - t) - \mathbb{E}|B_\tau|^p = V(0, t) > 0,$$

which implies

$$||\tau^{1/2}||_{p,\infty} \geq (1 - t)^{1/2} \big(\mathbb{P}(\tau \geq 1 - t)\big)^{1/p} > (1 - t)^{1/2} ||B_\tau||_p.$$

Therefore, the best constant is not smaller than $(1 - t)^{1/2}$ and letting $t \downarrow t_0$ yields the desired lower bound.

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