SHARP WEAK-TYPE INEQUALITIES FOR HILBERT TRANSFORM AND RIESZ PROJECTION

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Abstract. The paper is devoted to the study of the weak norms of the classical operators in the vector-valued setting.

(i) Let $\mathcal{S}$, $\mathcal{H}$ denote the involution operator and the Hilbert transform on $L^p(\mathbb{T},\ell^2_{\mathbb{C}})$, respectively. Then for $1\leq p\leq 2$ and any f,

$$||\mathcal{S}f||_{p,\infty}\leq\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{-1/p}||f||_p,$$

$$||\mathcal{H}f||_{p,\infty}\leq\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{-1/p}||f||_p.$$

Both inequalities are sharp.

(ii) Let P_+ and P_- stand for the Riesz projection and the co-analytic projection on $L^p(\mathbb{T},\ell^2_{\mathbb{C}})$, respectively. Then for $1\leq p\leq 2$ and any f,

$$||P_+f||_{p,\infty}\leq||f||_p,$$
$$||P_-f||_{p,\infty}\leq||f||_p,$$

Both inequalities are sharp.

(iii) We establish the sharp versions of the estimates above in the nonperiodic case.

The results are new even if the operators act on complex-valued functions. The proof rests on the construction of an appropriate plurisubharmonic function and probabilistic techniques.

1. Introduction

Let $f(\zeta)=\sum_{n\in\mathbb{Z}}\hat{f}(n)\zeta^n$ be a complex-valued integrable function on the unit circle $\mathbb{T}=\{\zeta\in\mathbb{C}:|\zeta|=1\}$. Here and in what follows, $\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})e^{-in\theta}d\theta$ denotes the n-th Fourier coefficient of f. For $p\geq 1$, the space $H^p(\mathbb{T},\mathbb{C})$, a closed subspace of $L^p(\mathbb{T},\mathbb{C})$, consists of those f, which satisfy $\hat{f}(n)=0$ for $n<0$. As usual, $H^p(\mathbb{T},\mathbb{C})$ may be identified with the space of analytic functions on the unit disc $\mathbb{D}$, and

$$||f||_{H^p(\mathbb{T},\mathbb{C})}=\sup_{0<r<1}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\mathrm{d}\theta\right)^{1/p}.$$

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The Riesz projection (or analytic projection) $P_+ : L^p(\mathbb{T},\mathbb{C}) \to H^p(\mathbb{T},\mathbb{C})$, is the operator given by

$$P_+f(\zeta) = f_+(\zeta) = \sum_{n\geq 0} \hat{f}(n)\zeta^n.$$

We will also use a co-analytic projection $P_- = I - P_+$ and the involution operator $\mathcal{S} = P_+ - P_-$, which can be written in the form

$$P_-f(\zeta) = \overline{f_-(\zeta)} = \sum_{n<0} \hat{f}(n)\zeta^n, \qquad \mathcal{S}f(\zeta) = \sum_{n\in\mathbb{Z}} \sigma(n)\hat{f}(n)\zeta^n,$$

with $\sigma(n) = 1$ for $n \geq 0$ and $\sigma(n) = -1$ otherwise. These are closely related to another classical operator, the Hilbert transform (conjugate function) on $\mathbb{T}$, which is defined by

$$\mathcal{H}f(\zeta) = -i\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n)\hat{f}(n)\zeta^n, \qquad \zeta \in \mathbb{T}.$$

Here $\operatorname{sgn}(n) = \sigma(n)$ for $n \neq 0$ and $\operatorname{sgn}(0) = 0$, so that $Sf = i\mathcal{H}f + \hat{f}(0)$. We have the following representation of $\mathcal{H}$ via singular integrals:

$$\mathcal{H}f(e^{i\theta}) = \frac{1}{2\pi}\text{p.v.}\int_{-\pi}^{\pi} f(e^{it})\cot\frac{\theta - t}{2}\mathrm{d}t. \tag{1.1}$$

Now, for $1 \leq p < \infty$ and any operator T on $L^p(\mathbb{T},\mathbb{C})$, define its strong and weak p-th norm by

$$||T||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \sup\{||Tf||_p : ||f||_p \leq 1\}$$

and

$$||T||_{L^p(\mathbb{T},\mathbb{C})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = \sup\big\{||Tf||_{p,\infty} : ||f||_p \leq 1\big\},$$

respectively. Here

$$||Tf||_{p,\infty} = \sup\big\{\lambda\big(|\{\theta \in [-\pi,\pi] : |Tf(e^{i\theta})| \geq \lambda\}|/(2\pi)\big)^{1/p} : \lambda > 0\big\}.$$

A classical theorem of M. Riesz states that the operator P_+ (equivalently, $\mathcal{H}$ or $\mathcal{S}$) is bounded on $L^p(\mathbb{T},\mathbb{C})$ for $1 < p < \infty$. The question about the precise value of the norms of these operators has gathered some interest in the literature. For $p = 2^k$, $k = 1, 2, \ldots$, the exact values of the norms of $\mathcal{S}$ and $\mathcal{H}$ were determined by Gohberg and Krupnik [5]. Using the remarkable identity $f^2 + (\mathcal{S}f)^2 = 2\mathcal{S}(f\mathcal{S}f)$, they showed that

$$||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = ||\mathcal{S}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \cot(\pi/(2p)).$$

For the remaining values of $1 < p < \infty$, the norms of the operator $\mathcal{S}$ and $\mathcal{H}$ acting on real L^p spaces were found by Pichorides [10] and, independently, by Cole (unpublished work, see Gamelin [4]):

$$||\mathcal{S}||_{L^p(\mathbb{T},\mathbb{R})\to L^p(\mathbb{T},\mathbb{C})} = ||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{R})\to L^p(\mathbb{T},\mathbb{C})} = \cot(\pi/(2p^*)),$$

where $p^* = \max\{p, p/(p-1)\}$. These norms do not change while passing to the complex L^p spaces (see e.g. Pełczyński [9]):

$$||\mathcal{S}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = ||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \cot(\pi/(2p^*)), \qquad 1 < p < \infty.$$

Concerning Riesz and co-analytic projections, Hollenbeck and Verbitsky [6] proved that

$$||P_\pm||_{L^p(\mathbb{T},\mathbb{C})\to L^p(\mathbb{T},\mathbb{C})} = \csc(\pi/p), \qquad 1 < p < \infty.$$

Now let us turn to the weak-type estimates. As shown by Davis [1],

$$||\mathcal{S}||_{L^1(\mathbb{T},\mathbb{R})\to L^{1,\infty}(\mathbb{T},\mathbb{C})} = ||\mathcal{H}||_{L^1(\mathbb{T},\mathbb{R})\to L^{1,\infty}(\mathbb{T},\mathbb{C})} = K_1 = \frac{1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots}{1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\dots}.$$

For $1 < p \leq 2$, it follows from the results of Janakiraman [7] and the author [8] (also see Section 3 below) that

$$||\mathcal{S}||_{L^p(\mathbb{T},\mathbb{R})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = ||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{R})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = K_p, \tag{1.2}$$

where

$$K_p = \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{-1/p}. \tag{1.3}$$

For $p > 2$, the question about the precise value of the weak p-th norm of $\mathcal{S}$ or $\mathcal{H}$ (on the real or complex L^p space) is open, to the best of the author's knowledge.

For the Riesz and co-analytic projections, the following can be extracted from the results of Tomaszewski [12]: for $1 \leq p \leq 2$,

$$||P_+||_{L^p(\mathbb{T},\mathbb{R})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} \geq ||P_-||_{L^p(\mathbb{T},\mathbb{R})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = \left(\frac{\sqrt{\pi}}{2^{p+1}}\frac{p\Gamma(p/2)}{\Gamma((p+1)/2)}\right)^{1/p}.$$

For $p > 2$, the weak p-th norm of $P_\pm$, on the real or complex L^p space, is not known.

A related problem, to be solved in the present paper, is the question about the precise values of the weak p-th norms of $\mathcal{S}$, $\mathcal{H}$ and $P_\pm$ on complex L^p spaces, that is, $||\mathcal{S}||_{L^p(\mathbb{T},\mathbb{C})\to L^{p,\infty}(\mathbb{T},\mathbb{C})}$, $||\mathcal{H}||_{L^p(\mathbb{T},\mathbb{C})\to L^{p,\infty}(\mathbb{T},\mathbb{C})}$ and $||P_\pm||_{L^p(\mathbb{T},\mathbb{C})\to L^{p,\infty}(\mathbb{T},\mathbb{C})}$, for $1 \leq p \leq 2$. In fact, we will study these operators in a more general, Hilbert space-valued setting. Let

$$L^p(\mathbb{T},\ell^2_{\mathbb{C}}) = \left\{ f:\mathbb{T}\to\ell^2_{\mathbb{C}} : ||f||_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^p\mathrm{d}\theta\right)^{1/p} < \infty \right\}$$

denote the corresponding L^p space for $\ell^2_{\mathbb{C}}$-valued functions on the unit circle. It is easy to see that $P_\pm$, $\mathcal{S}$, $\mathcal{H}$ can be extended to the operators on $L^p(\mathbb{T},\ell^2_{\mathbb{C}})$, either by defining them coordinatewise, or simply by noting that the previous definitions make sense in this new setting.

Now we turn to the results of the present paper. The first of them can be stated as follows.

Theorem 1.1. *Let $1 \leq p \leq 2$. Then*

$$||\mathcal{S}||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{T},\ell^2_{\mathbb{C}})} = ||\mathcal{H}||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{T},\ell^2_{\mathbb{C}})} = K_p, \tag{1.4}$$

where K_p is given by (1.3).

Comparing this to (1.2), we see that the weak norms of $\mathcal{S}$ and $\mathcal{H}$ do not change while passing from the real to the complex L^p spaces.

Our second result concerns Riesz and co-analytic projection.

Theorem 1.2. *Let $1 \leq p \leq 2$. Then*

$$||P_\pm||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{T},\ell^2_{\mathbb{C}})} = ||P_\pm||_{L^p(\mathbb{T},\mathbb{C})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = ||P_+||_{L^p(\mathbb{T},\mathbb{R})\to L^{p,\infty}(\mathbb{T},\mathbb{C})} = 1.$$

Therefore the passage from real to complex L^p spaces does not affect the weak norm of P_+; on the other hand, the norm of P_- does change.

Using standard arguments of Zygmund [13] and Davis [1], the results above can be transferred to the nonperiodic case, that is, to the case when the operators act on $L^p(\mathbb{R},\ell^2_{\mathbb{C}})$. Let us briefly introduce the necessary notation. The analytic projection on the real line, $P_+=P_+^{\mathbb{R}}$, is given by

$$P_+^{\mathbb{R}}f(x)=\mathcal{F}^{-1}(1_{[0,\infty)}\mathcal{F}f)(x),\qquad x\in\mathbb{R},$$

where $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ are the direct and inverse Fourier transforms of f. Furthermore, $P_-=P_-^{\mathbb{R}}$, the co-analytic projection on $\mathbb{R}$, is given by $P_-^{\mathbb{R}}=I-P_+^{\mathbb{R}}$. The Hilbert transform $\mathcal{H}=\mathcal{H}^{\mathbb{R}}$ and the involution operator $\mathcal{S}=\mathcal{S}^{\mathbb{R}}$ are defined by $\mathcal{H}^{\mathbb{R}}=-i\mathcal{S}^{\mathbb{R}}=-i(P_+^{\mathbb{R}}-P_-^{\mathbb{R}})$. The operator $\mathcal{H}^{\mathbb{R}}$ admits the following representation by singular integrals:

$$\mathcal{H}^{\mathbb{R}}f(x)=\frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}}\frac{f(t)}{x-t}\mathrm{d}x. \tag{1.5}$$

The weak p-th norms are defined analogously to the periodic setting. We will establish the following fact.

Theorem 1.3. *Let $1\leq p\leq 2$ and $f\in L^p(\mathbb{R},\ell^2_{\mathbb{C}})$. Then*

$$||\mathcal{H}^{\mathbb{R}}||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})}=||\mathcal{S}^{\mathbb{R}}||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})}=K_p \tag{1.6}$$

and

$$||P_\pm||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})}=1. \tag{1.7}$$

A few words about the proof. To describe our approach, let us first recall the method used by Hollenbeck and Verbitsky in [6]. In order to establish the inequality

$$||P_\pm f||_{L^p(\mathbb{T},\mathbb{C})}\leq \csc(\pi/p)||f||_{L^p(\mathbb{T},\mathbb{C})},\qquad 1<p\leq 2, \tag{1.8}$$

they invented a plurisubharmonic function U on $\mathbb{C}^2$ satisfying $U(0,0)=0$ and

$$U(w,z)\leq \csc^p(\pi/p)|w+\overline{z}|^p-\max(|w|^p,|z|^p)\qquad \text{for all } (w,z)\in\mathbb{C}^2.$$

The existence of such a function yields (1.8): it suffices to take $w=P_+f$, $z=\overline{P_-f}$ in the preceding inequality, integrate both sides over $\mathbb{T}$ and use the sub-mean-value property for the subharmonic function $U(P_+f,\overline{P_-f})$ (in fact, one obtains then the stronger estimate $||\max\{|P_+f|,|P_-f|\}||_{L^p(\mathbb{T},\mathbb{C})}\leq\csc(\pi/p)||f||_{L^p(\mathbb{T},\mathbb{C})}$).

In the proof of Theorems 1.1 and 1.2 we proceed similarly and try to construct an appropriate plurisubharmonic function. We slightly change the language and establish the announced results using probabilistic tools. This approach has the following additional advantage: by a stopping time argument, we do not have to construct the special functions U on the whole $\ell^2_{\mathbb{C}}\times\ell^2_{\mathbb{C}}$, but only on certain subdomains of this product. For example, in the proof of the weak-type inequality for Hilbert transform, it suffices to invent the function U on

$$E=\{(w,z)\in\ell^2_{\mathbb{C}}\times\ell^2_{\mathbb{C}}:|w-\overline{z}|\leq 1\}. \tag{1.9}$$

The remainder of the paper is organized as follows. We introduce the necessary probabilistic background in the next section. Then, in Section 3, we study the norm for the involution operator and Hilbert transform, while Section 4 is devoted to the result for the Riesz projection. The final section, Section 5, contains the proof of Theorem 1.3.

2. Main Lemma

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. In addition, assume that $\mathcal{F}_0$ contains all the events of probability 0. Let Z be a continuous-path adapted martingale, taking values in $\ell^2_{\mathbb{C}}$; the standard norm and scalar product in this Hilbert space is denoted by $|\cdot|$ and $\cdot$, respectively. The conjugation operation is given by $\overline{z} = (\overline{z}_1, \overline{z}_2, \ldots)$ for $z = (z_1, z_2, \ldots) \in \ell^2_{\mathbb{C}}$. We say that a martingale Z is *conformal* (or *analytic*), if for any j, ℓ we have $[Z^j, Z^\ell] = 0$. Here Z^j, Z^ℓ denote the j-th and the ℓ-th coordinate of the process Z, respectively, and

$$[Z^j, Z^\ell] = [\mathrm{Re}Z^j, \mathrm{Re}Z^\ell] - [\mathrm{Im}Z^j, \mathrm{Im}Z^\ell] + i([\mathrm{Re}Z^j, \mathrm{Im}Z^\ell] + [\mathrm{Im}Z^j, \mathrm{Re}Z^\ell]),$$

where, for real $(\mathcal{F}_t)$-martingales M and N, the symbol $[M, N]$ stands for their quadratic covariance process (see e.g. Dellacherie and Meyer [2]). We will also use the notation $[M, N]^t_s = [M, N]_t - [M, N]_s$ for any $s \leq t$. We define the *maximal function* of Z by $Z^* = \sup_{s\geq 0} |Z_s|$; we will also use the truncated maximal functions, given by $Z^*_t = \sup_{0\leq s\leq t} |Z_s|$ for $t \geq 0$. The p-th norm of a martingale Z is defined by $||Z||_p = \sup_{t\geq 0} ||Z_t||_p$, $1 \leq p < \infty$.

Our main tool is described in the following lemma.

Theorem 2.1. *Let D be a fixed subdomain of $\ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}}$ and suppose that $U : \overline{D} \to \mathbb{R}$ is a continuous and plurisubharmonic function. Let (W, Z) be a bounded analytic martingale taking values in $\ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}}$, such that $(W_0, Z_0) \in D$, and let $\eta = \eta_D = \inf\{t > 0 : (W_t, Z_t) \notin D\}$ be the exit time of (W, Z) from D. Then for any $t \geq 0$ we have*

$$\mathbb{E}U(W_{\eta\wedge t}, Z_{\eta\wedge t}) \geq \mathbb{E}U(W_0, Z_0).$$

Proof. By the boundedness of (W, Z), continuity of U and Lebesgue's dominated convergence theorem, it suffices to prove the theorem for W and Z taking values in a finite-dimensional subspace of $\ell^2_{\mathbb{C}}$; say, W, $Z \in \mathbb{C}^m$ for some positive integer m. Let g be a C^∞ nonnegative function on $\mathbb{C}^m \times \mathbb{C}^m$, supported on the ball of center 0 and radius 1, satisfying $\int_{\mathbb{C}^m\times\mathbb{C}^m} g = 1$. For a fixed $\varepsilon > 0$, let $D_\varepsilon = \{z \in D : \mathrm{dist}(z, \partial D) > \varepsilon\}$ and consider the stopping time

$$\tau_\varepsilon = \inf\{t : (W_t, Z_t) \notin D_\varepsilon\}.$$

For $\delta < \varepsilon$, define $\hat{U} = \hat{U}^\delta : (D_\delta \cap \mathbb{C}^m) \times (D_\delta \cap \mathbb{C}^m) \to \mathbb{R}$ by the convolution

$$\hat{U}(w, z) = \int_{\mathbb{C}^m\times\mathbb{C}^m} U(w - u\delta, z - v\delta)g(u, v)\mathrm{d}u\mathrm{d}v,$$

with the convention $U(w, z) = U((w, 0, 0, \ldots), (z, 0, 0, \ldots))$ for w, $z \in \mathbb{C}^m$. It follows directly from the definition that $\hat{U}$ is plurisubharmonic (since U has this property). Furthermore, $\hat{U}$ is of class C^∞. Applying Itô's formula and using the fact that (W, Z) is analytic, we get

$$\hat{U}(W_{\tau_\varepsilon\wedge t}, Z_{\tau_\varepsilon\wedge t}) = I_0 + I_1 + I_2/2, \tag{2.1}$$

where

$$I_0 = \hat{U}(W_0, Z_0),$$

$$I_1 = \sum_{j=1}^{m} \Bigg[\int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{w_j}(W_s, Z_s)\mathrm{d}W_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{z_j}(W_s, Z_s)\mathrm{d}Z_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{\overline{w_j}}(W_s, Z_s)\mathrm{d}\overline{W}_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{\overline{z_j}}(W_s, Z_s)\mathrm{d}\overline{Z}_s \Bigg],$$

$$I_2 = \sum_{j,\ell\leq m} \Bigg[\int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{w_j\overline{w_\ell}}(W_s, Z_s)\mathrm{d}[W^j, \overline{W^\ell}]_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{w_j\overline{z_\ell}}(W_s, Z_s)\mathrm{d}[W^j, \overline{Z^\ell}]_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{z_j\overline{w_\ell}}(W_s, Z_s)\mathrm{d}[Z^j, \overline{W^\ell}]_s + \int_{0+}^{\tau_\varepsilon\wedge t} \hat{U}_{z_j\overline{z_\ell}}(W_s, Z_s)\mathrm{d}[Z^j, \overline{Z^\ell}]_s \Bigg].$$

Note that $\mathbb{E}I_1 = 0$ by the properties of stochastic integrals. To deal with the term I_2, observe that for any h, $k \in \mathbb{C}^m$ we have, by plurisubharmonicity of $\hat{U}$,

$$\sum_{j,\ell\leq m} \Big(\hat{U}_{w_j\overline{w_\ell}}(w,z)h_j\overline{h}_\ell + \hat{U}_{w_j\overline{z_\ell}}(w,z)h_j\overline{k}_\ell + \hat{U}_{z_j\overline{w_\ell}}(w,z)k_j\overline{h}_\ell + \hat{U}_{z_j\overline{z_\ell}}(w,z)k_j\overline{k}_\ell \Big) \geq 0. \tag{2.2}$$

Fix $s < s_1 \leq t$ and for each n, let $(T_r^n)_{1\leq r\leq r_n}$ be a nondecreasing sequence of finite stopping times with $T_1^n = s$ and $T_{r_n}^n = s_1$, satisfying the condition $\lim_{n\to\infty} \max_{1\leq r\leq r_n} |T_{r+1}^n - T_r^n| = 0$. Apply (2.2) to $w = W_{\tau_\varepsilon\wedge s}$, $z = Z_{\tau_\varepsilon\wedge s}$, $h = W_{\tau_\varepsilon\wedge T_{r+1}^n} - W_{\tau_\varepsilon\wedge T_r^n}$ and $k = Z_{\tau_\varepsilon\wedge T_{r+1}^n} - Z_{\tau_\varepsilon\wedge T_r^n}$ for each $r = 1, 2, \ldots, r_n$. Summing over r and letting $n \to \infty$ gives

$$\sum_{j,\ell\leq m} \Big[\hat{U}_{w_j\overline{w_\ell}}(W_{\tau_\varepsilon\wedge s}, Z_{\tau_\varepsilon\wedge s})[W^j, \overline{W^\ell}]_{\tau_\varepsilon\wedge s}^{\tau_\varepsilon\wedge s_1} + \hat{U}_{w_j\overline{z_\ell}}(W_{\tau_\varepsilon\wedge s}, Z_{\tau_\varepsilon\wedge s})[W^j, \overline{Z^\ell}]_{\tau_\varepsilon\wedge s}^{\tau_\varepsilon\wedge s_1} + \hat{U}_{z_j\overline{w_\ell}}(W_{\tau_\varepsilon\wedge s}, Z_{\tau_\varepsilon\wedge s})[Z^j, \overline{W^\ell}]_{\tau_\varepsilon\wedge s}^{\tau_\varepsilon\wedge s_1} + \hat{U}_{z_j\overline{z_\ell}}(W_{\tau_\varepsilon\wedge s}, Z_{\tau_\varepsilon\wedge s})[Z^j, \overline{Z^\ell}]_{\tau_\varepsilon\wedge s}^{\tau_\varepsilon\wedge s_1} \Big] \geq 0.$$

This yields $I_2 \geq 0$: simply approximate the integrals by discrete sums. Thus, combining the above facts with (2.1) gives $\mathbb{E}\hat{U}(W_{\tau_\varepsilon\wedge t}, Z_{\tau_\varepsilon\wedge t}) \geq \mathbb{E}\hat{U}(W_0, Z_0)$. Take $\delta \to 0$, and then $\varepsilon \to 0$ to obtain

$$\mathbb{E}U(W_{\eta\wedge t}, Z_{\eta\wedge t}) \geq \mathbb{E}U(W_0, Z_0),$$

by the continuity of U, continuity of the paths of W and Z, and Lebesgue's dominated convergence theorem. □

We conclude this section by recalling a well-known fact from complex analysis (see e.g. Theorem 4.13 in [11]). Note that for w, $z \in \ell^2_{\mathbb{C}}$ we have $w \cdot \overline{z} = \sum_{j=1}^\infty w_j z_j$.

Theorem 2.2. *Let $m \geq 1$ be a fixed integer. Suppose D is a given subdomain of $\mathbb{C}$ and let $D' = \{(w, z) \in \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} : w \cdot \overline{z} \in D\}$. Then if $\phi : D \to \mathbb{R}$ is subharmonic, then $U : D' \to \mathbb{R}$ given by $U(w, z) = \phi(w \cdot \overline{z})$ is plurisubharmonic.*

3. Weak-type estimates for Hilbert transform and involution operator

Throughout this section, p is a fixed number lying in the interval $[1, 2]$. We start by introducing a certain special function V_p, invented by Janakiraman in [7]. Let

$H=\{(x,y):y\geq 0\}$ be the upper half-space in $\mathbb{R}^2$, $S=\{(x,y)\in\mathbb{R}^2:|y|\leq 1\}$ denote the horizontal strip in $\mathbb{R}^2$ and set $S_+=\{(x,y):x\geq 0,\ y\in[0,1)\}$. Define, for $\alpha\in\mathbb{R}$ and $\beta>0$,

$$\mathcal{V}_p(\alpha,\beta)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\beta\left|\frac{2}{\pi}\log|t|\right|^p}{(\alpha-t)^2+\beta^2}\mathrm{d}t. \tag{3.1}$$

Then $\mathcal{V}_p$ is a harmonic function on H which vanishes as $\beta\to\infty$ and satisfies

$$\lim_{(\alpha,\beta)\to(t,0)}\mathcal{V}_p(\alpha,\beta)=\left|\frac{2}{\pi}\log|t|\right|^p. \tag{3.2}$$

Consider a conformal map $\phi(z)=i\exp(\pi z/2)$, which maps S onto H, and introduce $V_p:S\to\mathbb{R}$ by

$$V_p(x,y)=\begin{cases}|x|^p & \text{if } |y|=1,\\ \mathcal{V}_p(\phi(x,y)) & \text{if } |y|<1.\end{cases}$$

We see that V_p is harmonic in the interior of S, since it is a real part of a certain holomorphic function:

$$V_p=\mathrm{Re}G_p. \tag{3.3}$$

In addition, in view of (3.2), V_p is a continuous function on the whole strip S. It is not difficult to see that V_p satisfies the condition

$$V_p(x,y)=V_p(x,-y)=V_p(-x,y)\qquad\text{for all } (x,y)\in S. \tag{3.4}$$

Indeed, this is equivalent to

$$\mathcal{V}_p(\alpha,\beta)=\mathcal{V}_p(-\alpha,\beta)=\mathcal{V}_p\left(\frac{\alpha}{\alpha^2+\beta^2},\frac{\beta}{\alpha^2+\beta^2}\right),$$

which can be verified by substitution $t:=-t$ and $t:=1/t$ in (3.1).

We will need the following further properties of V_p. Recall that

$$K_p=\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{1/p}.$$

Lemma 3.1. *(i) We have $V_p(x,0)\geq V_p(0,0)=K_p^{-p}$ for all $x\in\mathbb{R}$.*

(ii) For any $(x,y)\in S^+$, we have $V_{pxxx}(x,y)\leq 0$.

(iii) For any x, $y\in\mathbb{R}$ we have $|x|^p\leq V_p(x,y)\leq K_p^{-p}1_{\{|y|<1\}}+|x|^p$.

(iv) For any $(x,y)\in S^+$, we have $yV_{px}(x,y)+xV_{py}(x,y)\geq 0$.

(v) There are a_0, a_1, a_2, $\ldots\in\mathbb{C}$ such that the holomorphic function G_p given by (3.3) satisfies $G_p(z)=\sum_{n=0}^{\infty}a_nz^{2n}$ for all $z\in S$.

Proof. (i) After a change of variables,

$$V_p(x,0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{2}{\pi}\log|u|+x\right|^p}{u^2+1}\mathrm{du},$$

which implies, by Fubini's theorem, that

$$V_{pxx}(x,0)=\frac{p(p-1)}{\pi}\int_{-\infty}^{\infty}\frac{\left|\frac{2}{\pi}\log|u|+x\right|^{p-2}}{u^2+1}\mathrm{du}\geq 0,$$

so, by (3.4), $V_p(x,0)\geq V_p(0,0)=\mathcal{V}_p(0,1)=K_p^{-p}$.

(ii) See Lemma 7.4 in [8].

(iii) See Lemma 2.1 in [7].

(iv) We start from the observation that

$$V_{pxy} \geq 0 \quad \text{on } S^+. \tag{3.5}$$

Indeed, by (3.4), we have $V_{py}(x, 0) = 0$ for any $x \in \mathbb{R}$; this implies $V_{pxy}(x, 0) = 0$ for all x. Furthermore, note that by (ii) and the fact that V_p is harmonic, we have $V_{pxyy} = -V_{pxxx} \geq 0$ on S; hence (3.5) follows.

Fix $x \geq 0$ and let $F(y) = yV_{px}(x, y) + xV_{py}(x, y)$, $y \in [0, 1)$. We see that $F(0) = 0$, and hence it suffices to prove that F is nondecreasing. Using the harmonicity of V_p, we get

$$\begin{aligned} F'(y) &= yV_{pxy}(x, y) + V_{px}(x, y) + xV_{pyy}(x, y) \\ &= yV_{pxy}(x, y) + (V_{px}(x, y) - xV_{pxx}(x, y)) \geq 0, \end{aligned}$$

in virtue of (3.5) and (ii).

(v) By (3.4), the odd order partial derivatives of V_p vanish and hence so do those of $\mathrm{Im}G_p$, by Cauchy-Riemann equations. This implies $G_p^{(2n+1)}(0) = 0$, as desired. □

Consider the parabolic region $D = \{z \in \mathbb{C} : |2\mathrm{Im}z^{1/2}| \leq 1\}$.

Lemma 3.2. *The function $z \mapsto V_p(2z^{1/2})$, $z \in D$, is harmonic.*

Proof. First notice that the function is well defined: in view of (3.4) it does not matter which square root of z we take. The assertion is an immediate consequence of Lemma 3.1 (v). Indeed, the function $z \mapsto G_p(2z^{1/2})$ is holomorphic, and hence its real part is harmonic. □

Now we are ready to introduce the main special function. Let E be given by (1.9) and let $V_p^{\mathbb{C}} : E \to \mathbb{R}$ be defined by the formula $V_p^{\mathbb{C}}(w, z) = V_p(2(w \cdot \overline{z})^{1/2})$. Note that the definition makes sense: we have the following fact.

Lemma 3.3. *For any w, $z \in \ell^2_{\mathbb{C}}$ we have*

$$2|Re(w \cdot \overline{z})^{1/2}| \leq |w + \overline{z}| \quad \text{and} \quad 2|Im(w \cdot \overline{z})^{1/2}| \leq |w - \overline{z}|. \tag{3.6}$$

Proof. It suffices to establish the first estimate; the second one follows after substitution $-z$ in the place of z. By continuity, we may and do assume that there is a positive integer m such that $w_j \neq 0$ for all $j = 1, 2, \ldots, m$, and $w_j = z_j = 0$ for $j > m$. We have

$$\begin{aligned} |w + \overline{z}|^2 &= \sum_{j=1}^m \left[|w_j|^2 + \frac{|w_j z_j|^2}{|w_j|^2} + 2\mathrm{Re}(w_j z_j) \right] \\ &\geq \sum_{j=1}^m [2|w_j z_j| + 2\mathrm{Re}(w_j z_j)] = \sum_{j=1}^m (2|\mathrm{Re}(w_j z_j)^{1/2}|)^2 \end{aligned} \tag{3.7}$$

and it remains to prove that

$$\sum_{j=1}^m |\mathrm{Re}(w_j z_j)^{1/2}|^2 \geq |\mathrm{Re}(w \cdot \overline{z})^{1/2}|^2 = \left| \mathrm{Re}\left(\sum_{j=1}^m w_j z_j \right)^{1/2} \right|^2.$$

By induction, it suffices to establish this bound for $m = 2$. The substitution $(w_1 z_1)^{1/2} = x_1 + iy_1$ and $(w_2 z_2)^{1/2} = x_2 + iy_2$, for x_1, y_1, x_2, $y_2 \in \mathbb{R}$, transforms the inequality into

$$x_1^2+y_1^2 \geq \frac{x_1^2-y_1^2+x_2^2-y_2^2+\sqrt{(x_1^2-y_1^2+x_2^2-y_2^2)^2+4(x_1y_1+x_2y_2)^2}}{2}.$$

This can be simplified to $(x_1y_2-x_2y_1)^2 \geq 0$. The proof is complete. □

Lemma 3.4. *For any w, $z \in \ell^2_{\mathbb{C}}$ such that $|w-\overline{z}| \leq 1$, we have*

$$1-K_p^pV_p^{\mathbb{C}}(w,z) \geq 1_{\{|w-\overline{z}|\geq 1\}} - K_p^p|w+\overline{z}|^p. \tag{3.8}$$

Proof. The inequality is equivalent to

$$V_p(2(w\cdot\overline{z})^{1/2}) \leq K_p^{-p}1_{\{|w-\overline{z}|<1\}} + |w+\overline{z}|^p. \tag{3.9}$$

By (3.6), if $|w-\overline{z}|<1$, then $2|\mathrm{Im}(w\cdot\overline{z})^{1/2}|$ is also smaller than 1; in consequence, by the part (iii) of Lemma 3.1 and (3.6),

$$V_p(2(w\cdot\overline{z})^{1/2}) \leq K_p^{-p} + (2|\mathrm{Re}(w\cdot\overline{z})^{1/2}|)^p \leq K_p^{-p}+|w+\overline{z}|^p.$$

Suppose then, that w, z satisfy $|w-\overline{z}|=1$ and fix $s\geq 0$. By the part (iv) of Lemma 3.1, the function $F_s:[\sqrt{(s^2-1)_+},s]\to\mathbb{R}$, given by $F_s(x)=V_p(x,\sqrt{x^2+1-s^2})$, is nondecreasing: indeed,

$$F_s'(x)=V_{px}(x,\sqrt{x^2+1-s^2})+\frac{x}{\sqrt{x^2+1-s^2}}V_{py}(x,\sqrt{x^2+1-s^2})\geq 0.$$

Therefore, by (3.6),

$$\begin{aligned}|w+\overline{z}|^p &= V_p(|w+\overline{z}|,|w-\overline{z}|)=F_{|w+\overline{z}|}(|w+\overline{z}|)\\ &\geq F_{|w+\overline{z}|}(2|\mathrm{Re}(w\cdot\overline{z})^{1/2}|)=V_p(2(w\cdot\overline{z})^{1/2}),\end{aligned}$$

where the latter equation is due to the definition of $F_{|w+\overline{z}|}$ and the identity

$$(2|\mathrm{Re}(w\cdot\overline{z})^{1/2}|)^2+|w-\overline{z}|^2-|w+\overline{z}|^2=(2|\mathrm{Im}(w\cdot\overline{z})^{1/2}|)^2. \quad \square$$

Now we are ready to establish the probabilistic version of Theorem 1.1.

Theorem 3.5. *Let (W,Z) be an analytic martingale taking values in $\ell^2_{\mathbb{C}}\times\ell^2_{\mathbb{C}}$, satisfying $Z_0=a\overline{W}_0$ for some $a\geq 0$. Then*

$$\mathbb{P}((W-\overline{Z})^*\geq 1)\leq K_p^p||W+\overline{Z}||_p^p, \qquad 1\leq p\leq 2, \tag{3.10}$$

and the constant K_p^p is the best possible.

Proof. It suffices to prove that

$$\mathbb{P}(|W_t-\overline{Z}_t|\geq 1 \text{ for some } t)\leq K_p^p||W+\overline{Z}||_p^p. \tag{3.11}$$

Indeed, fix $\varepsilon\in(0,1)$ and consider the pair $(W_t/(1-\varepsilon),Z_t/(1-\varepsilon))$. Since

$$\{|W-\overline{Z}|^*\geq 1\}\subseteq\{|W_t-\overline{Z}_t|\geq 1-\varepsilon \text{ for some } t\},$$

the inequality (3.11), applied to this new pair, yields

$$\mathbb{P}(|W-\overline{Z}|^*\geq 1)\leq\frac{K_p||W+\overline{Z}||_p^p}{(1-\varepsilon)^p}$$

and the claim follows, since ε was arbitrary.

For $L>0$, let

$$D_L=\{(w,z)\in\ell^2_{\mathbb{C}}\times\ell^2_{\mathbb{C}}:|w-\overline{z}|<1 \text{ and } |w+\overline{z}|\leq L\}$$

and consider a stopping time $\tau_L=\inf\{t\geq 0:(W_t,Z_t)\notin D_L\}$. Clearly, $\overline{D}_L\subset E$ (with E given by (1.9)). By Theorem 2.2 and Lemma 3.2, we have that the function

$U = -1 + K_p^p V_p^{\mathbb{C}}$, restricted to D_L, is plurisubharmonic. Apply Theorem 2.1 to this function and the bounded analytic martingale $(W_{\tau_L \wedge t} 1_{\{\tau_L>0\}}, Z_{\tau_L\wedge t}1_{\{\tau_L>0\}})_{t\geq 0}$. We get that for any $t \geq 0$,

$$\mathbb{E}[1 - K_p^p V_p^{\mathbb{C}}(W_{\tau_L\wedge t}, Z_{\tau_L\wedge t})]1_{\{\tau_L>0\}} \leq \mathbb{E}[1 - K_p^p V_p^{\mathbb{C}}(W_0, Z_0)]1_{\{\tau_L>0\}} \leq 0,$$

since $Z_0 = a\overline{W}_0$ and $V_p^{\mathbb{C}}(w, a\overline{w}) \geq K_p^{-p}$ for any $w \in \ell^2_{\mathbb{C}}$ (by part (i) of Lemma 3.1). Using (3.8), we obtain

$$\mathbb{P}(|W_{\tau_L\wedge t} - \overline{Z}_{\tau_L\wedge t}| \geq 1,\, \tau_L > 0) \leq K_p^p \mathbb{E}|W_{\tau\wedge t} + \overline{Z}_{\tau\wedge t}|^p 1_{\{\tau_L>0\}}. \tag{3.12}$$

Furthermore, since $Z_0 = a\overline{W}_0$ for $a \geq 0$, we have, by Czebyshev's inequality,

$$\begin{aligned}\mathbb{P}(|W_{\tau_L\wedge t} - \overline{Z}_{\tau_L\wedge t}| \geq 1,\, \tau_L = 0) &\leq |1-a|^p \mathbb{E}|W_0|^p 1_{\{\tau_L=0\}}\\ &\leq |1+a|^p \mathbb{E}|W_0|^p 1_{\{\tau_L=0\}}\\ &\leq K_p^p \mathbb{E}|W_{\tau_L\wedge t} + \overline{Z}_{\tau_L\wedge t}|^p 1_{\{\tau_L=0\}}.\end{aligned}$$

Adding this to (3.12), we get

$$\mathbb{P}(|W_{\tau_L\wedge t} - \overline{Z}_{\tau_L\wedge t}| \geq 1) \leq K_p^p \mathbb{E}|W_{\tau\wedge t} + \overline{Z}_{\tau\wedge t}|^p \leq K_p^p ||W + \overline{Z}||_p^p,$$

where the latter estimate follows from Doob's optional sampling theorem. Letting $L \to \infty$ and then $t \to \infty$, we get (3.11).

The sharpness of the estimate will be clear from the proof of Theorem 1.1, to be presented below. $\square$

Proof of Theorem 1.1. First we will prove that

$$||\mathcal{H}||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{T},\ell^2_{\mathbb{C}})} \leq K_p. \tag{3.13}$$

To do this, it suffices to show that for any trigonometric polynomial $f(\zeta) = \sum_{n=-k}^{\ell} \hat{f}(n)\zeta^n$, $\zeta = e^{i\theta} \in \mathbb{T}$, we have

$$\frac{1}{2\pi}|\{\theta \in [-\pi,\pi] : |\mathcal{H}f(e^{i\theta})| \geq 1\}| \leq K_p^p ||f||_p^p. \tag{3.14}$$

Note that

$$f_+(\zeta) = \sum_{n=0}^{\ell} \hat{f}(n)\zeta^n, \qquad f_-(\zeta) = \sum_{n=-k}^{-1} \overline{\hat{f}(n)}\zeta^{-n} \tag{3.15}$$

are analytic trigonometric polynomials on $\mathbb{T}$, and hence they can be extended to polynomials defined on $\mathbb{C}$. Let $B = (B_t)_{t\geq 0}$ be a standard two-dimensional Brownian motion (i.e., starting from $0 \in \mathbb{C}$), $\tau = \inf\{t \geq 0 : |B_t| = 1\}$ and set

$$W_t = f_+(B_{\tau\wedge t}) - \frac{1}{2}\hat{f}(0), \quad Z_t = f_-(B_{\tau\wedge t}) + \frac{1}{2}\overline{\hat{f}(0)} \qquad \text{for } t \geq 0.$$

Since f_+ and f_- are analytic, the process (W, Z) is a conformal martingale. We have $W_t - \overline{Z}_t = i\mathcal{H}f(B_{\tau\wedge t})$, $W_t + \overline{Z}_t = f(B_{\tau\wedge t})$,

$$\frac{1}{2\pi}|\{\theta \in [0, 2\pi) : |\mathcal{H}f(\theta)| \geq 1\}| = \mathbb{P}(|W_\tau - \overline{Z}_\tau| \geq 1) \leq \mathbb{P}((W - \overline{Z})^* \geq 1)$$

and $||W + \overline{Z}||_p = ||f||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})}$. Finally, note that $W_0 = \overline{Z_0}$. Thus it suffices to apply (3.10) to get (3.14). The inequality $||\mathcal{S}||_{L^p(\mathbb{T},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{T},\ell^2_{\mathbb{C}})} \leq K_p$ is proved exactly in the same manner, using (3.10) with the conformal martingale

$$(W_t, Z_t) = (f_+(B_{\tau\wedge t}), f_-(B_{\tau\wedge t})), \quad t \geq 0.$$

To get the reverse inequality for the norms of $\mathcal{S}$ and $\mathcal{H}$, we observe that the constant K_p is the lower bound for these norms even if the operators are restricted to real-valued f. To see this, consider a function $F : \overline{\mathbb{D}} \to S$, given by

$$F(z) = (2/\pi)\log[(iz-1)/(z-i)] - i. \tag{3.16}$$

Clearly, F is analytic on $\mathbb{D}$, which implies $\mathrm{Im}F = \mathcal{H}\mathrm{Re}F$. In addition, it can be easily verified that

$$\mathrm{Re}F(e^{i\theta}) = \frac{2}{\pi}\log\left|\frac{1+\sin\theta}{\cos\theta}\right| \quad \text{and} \quad \mathrm{Im}F(e^{i\theta}) = 1_{\{|\theta|\leq\pi/2\}} - 1_{\{|\theta|>\pi/2\}}$$

for $\theta \in [-\pi,\pi]$. Consequently, we have $||\mathrm{Im}F||_{p,\infty} = 1$ and

$$||\mathrm{Re}F||_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{2}{\pi}\log\left|\frac{1+\sin\theta}{\cos\theta}\right|\right|^p \mathrm{d}\theta\right)^{1/p} = K_p^{-1}, \tag{3.17}$$

where the latter equality follows from the substitution $t = (1+\sin\theta)/\cos\theta$. □

4. Weak-type estimates for Riesz and co-analytic projections

The proof of the weak-type inequality for Riesz and co-analytic projections follows the same pattern as the one presented in the previous section. Let $1 \leq p \leq 2$ be fixed, set

$$E = \{(w,z) \in \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} : |w| \leq 1\}$$

and consider $U_p : E \to \mathbb{R}$ given by

$$U_p(w,z) = p\mathrm{Re}(w\cdot\overline{z}).$$

Lemma 4.1. *For any w, $z \in \ell^2_{\mathbb{C}}$ satisfying $|w| \leq 1$ we have*

$$U_p(w,z) \leq |w+\overline{z}|^p - 1_{\{|w|=1\}}. \tag{4.1}$$

Proof. By continuity, we may and do assume that $w \neq 0$. The claim follows from the chain of inequalities

$$\begin{aligned}
|w+\overline{z}|^p &= |w|^p\left|\frac{w}{|w|} + \frac{\overline{z}}{|w|}\right|^p \geq |w|^2\left(1 + 2\mathrm{Re}\frac{w\cdot\overline{z}}{|w|^2} + \frac{|z|^2}{|w|^2}\right)^{p/2} \\
&\geq |w|^2\left(1 + 2\mathrm{Re}\frac{w\cdot\overline{z}}{|w|^2} + \frac{|w\cdot\overline{z}|^2}{|w|^4}\right)^{p/2} = |w|^2\left|1 + \frac{w\cdot\overline{z}}{|w|^2}\right|^p \\
&\geq |w|^2\left|1 + \mathrm{Re}\frac{w\cdot\overline{z}}{|w|^2}\right|^p \geq |w|^2\left(1 + p\,\mathrm{Re}\frac{w\cdot\overline{z}}{|w|^2}\right) = |w|^2 + p\,\mathrm{Re}(w\cdot\overline{z}).
\end{aligned}$$ □

Theorem 4.2. *Let (W,Z) be an analytic martingale taking values in $\ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}}$, satisfying $W_0 = 0$ or $Z_0 = 0$. Then*

$$\mathbb{P}(W^* \geq 1) \leq ||W + \overline{Z}||_p, \qquad 1 \leq p \leq 2, \tag{4.2}$$

and the constant 1 is the best possible, even if W and Z are assumed to take values in one-dimensional subspace of $\ell^2_{\mathbb{C}}$.

Proof. The proof is similar to that of Theorem 3.5. Given $L > 0$, let

$$E_L = \{(w,z) \in \ell^2_{\mathbb{C}} \times \ell^2_{\mathbb{C}} : |w| \leq 1, |z| \leq L\}$$

and consider a stopping time $\tau_L = \inf\{t : (W,Z) \notin E_L\}$. Clearly, the function U_p is plurisubharmonic, so by Theorem 2.1, applied to the conformal martingale $(W_t 1_{\{\tau_L>0\}}, Z_t 1_{\{\tau_L>0\}})$ and combining this with Czebyshev's inequality yields

$$\mathbb{E}U_p(W_{\tau_L\wedge t}, Z_{\tau_L\wedge t}) \geq \mathbb{E}U_p(W_0, Z_0) \geq 0,$$

since $Z_0 = 0$ and $U_p(w,0) = 0$ for any w. Now use the majorization (4.1) and let $L \to \infty$, $t \to \infty$ to obtain

$$\mathbb{P}(|W_t| \geq 1 \text{ for some } t) \leq ||W + \overline{Z}||_p,$$

which gives (4.2). The sharpness of the estimate is clear: take W to be a standard two-dimensional Brownian motion stopped at the unit circle $\mathbb{T}$ and $Z \equiv 0$ to obtain that both sides are equal to 1. □

Proof of Theorem 1.2. We proceed as in the proof of Theorem 1.1 and show that for any trigonometric polynomial $f(\zeta) = \sum_{n=-k}^{\ell} \hat{f}(n)\zeta^n$ we have

$$|\{\theta \in [-\pi,\pi] : |P_+ f(e^{i\theta})| \geq 1\}|/(2\pi) \leq ||f||_p^p, \tag{4.3}$$

$$|\{\theta \in [-\pi,\pi] : |P_- f(e^{i\theta})| \geq 1\}|/(2\pi) \leq ||f||_p^p. \tag{4.4}$$

These estimates follow from (4.2) applied to the conformal martingales $(W_t, Z_t)_{t\geq 0} = (f_+(B_{\tau\wedge t}), f_-(B_{\tau\wedge t}))_{t\geq 0}$ and $(W_t, Z_t)_{t\geq 0} = (f_-(B_{\tau\wedge t}), f_+(B_{\tau\wedge t}))_{t\geq 0}$, respectively. Here, as above, f_+ and f_- are given by (3.15), B is a standard two-dimensional Brownian motion and τ is the exit time of B from the unit disc.

As in the probabilistic setting, the sharpness of the estimate is trivial: take $f(\zeta) = \zeta$ to obtain equality in (4.3), and $f(\zeta) = \zeta^{-1}$ to get equality in (4.4). □

5. The weak norms in the nonperiodic case

In this section we will apply the results above in order to derive the corresponding norms in the nonperiodic case. It will be convenient to split the proof into two parts.

Upper bounds for the norms. To deduce the estimates

$$||\mathcal{H}^{\mathbb{R}}||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})} = ||\mathcal{S}^{\mathbb{R}}||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})} \leq K_p$$

and

$$||P_\pm||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}})\to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})} \leq 1,$$

from their counterparts in the periodic setting, we use a standard argument known as "blowing up the circle", which is due to Zygmund ([13], Chapter XVI, Theorem 3.8). For the reader's convenience, we sketch the proof of the weak type estimate for the Hilbert transform. Let $f = (f^1, f^2, \ldots)$ and let $u^j = \mathrm{Re} f^j$, $v^j = \mathrm{Im} f^j$, $j = 1, 2, \ldots$. Introduce the functions $g_n = (g_n^1, g_n^2, \ldots)$, $h_n = (h_n^1, h_n^2, \ldots)$ by

$$g_n^j(x) = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} u^j(t) \cot\frac{x-t}{2n}\mathrm{d}t,$$

$$h_n^j(x) = \frac{1}{2\pi n}\int_{-\pi n}^{\pi n} v^j(t) \cot\frac{x-t}{2n}\mathrm{d}t,$$

for j, $n \geq 1$. As shown by Zygmund [13], for any fixed j we have $g_n^j \to \mathcal{H}^{\mathbb{R}} u^j$ and $h_n^j \to \mathcal{H}^{\mathbb{R}} v^j$ a.e. as $n \to \infty$. On the other hand, the function $x \mapsto g_n^j(nx) + ih_n^j(nx)$

is precisely the periodic Hilbert transform of the function $x \mapsto f^j(nx)$, $|x| \leq \pi$ (see (1.1)). Therefore, by Theorem 1.1, for any $\varepsilon > 0$,

$$\begin{aligned} &|\{x \in [-\pi n, \pi n] : |g_n(x) + ih_n(x)| \geq 1-\varepsilon\}| \\ &= n \left|\{|x| \in [-\pi,\pi] : |\mathcal{H}^{\mathbb{T}} f(nx)| \geq 1-\varepsilon\}\right| \\ &\leq \frac{nK_p^p}{(1-\varepsilon)^p} \int_{-\pi}^{\pi} |f(nx)|^p \mathrm{d}x = \frac{K_p^p}{(1-\varepsilon)^p} \int_{-\pi n}^{\pi n} |f(x)|^p \mathrm{d}x \leq \frac{K_p^p}{(1-\varepsilon)^p} ||f||_p^p. \end{aligned}$$

Now let $n \to \infty$ to obtain

$$|\{x \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} f(x)| \geq 1\}| \leq \frac{K_p^p}{(1-\varepsilon)^p} ||f||_p^p,$$

and since $\varepsilon > 0$ was arbitrary, we obtain $||\mathcal{H}^{\mathbb{R}}||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}}) \to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})} \leq K_p$.

Lower bounds for the norms. We will use Davis' argument from [1]. Consider a function $K(z) = (1+z)^2/4z$, which maps the half disc $\mathbb{D} \cap H$ onto H, and the boundary of $\mathbb{D} \cap H$ onto $\mathbb{R}$. Let L be the inverse of K. Then L maps $[0,1]$ onto the half circle $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$, and $\mathbb{R} \setminus [0,1]$ onto $(-1,1)$. Let d_n be the density of $L^n([0,1])$ on $\mathbb{T}$ with respect to the normalized Lebesgue's measure, i.e. for any $-\pi < \alpha < \beta < \pi$,

$$\int_{\alpha}^{\beta} d_n(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} = \left|\{r \in [0,1] : L_n(r) \in \{e^{i\theta} : \alpha < \theta < \beta\}\}\right|.$$

Then (see Lemma 3 in [1]) $d_n \to 1$ uniformly on $\mathbb{T}$.

Recall the map F be given by (3.16) and set $M_n = F(L^n(z))$ and $m_n(r) = \mathrm{Re} M_n(r)$ for $z \in H$ and $r \in \mathbb{R}$. It can be easily verified that M_n maps $\mathbb{R} \setminus [0,1]$ onto $\{ai : a \in (-1,1)\}$, which implies that $m_n(r) = 0$ for $r \notin [0,1]$ and, in consequence,

$$\begin{aligned} ||m_n||_p^p &= \int_{[0,1]} |m_n(r)|^p \mathrm{d}r = \int_{-\pi}^{\pi} |\mathrm{Re} F(e^{i\theta})|^p d_n(e^{i\theta}) \frac{\mathrm{d}\theta}{2\pi} \\ &\to \int_{-\pi}^{\pi} |\mathrm{Re} F(e^{i\theta})|^p \frac{\mathrm{d}\theta}{2\pi} = K_p^{-p}, \end{aligned}$$

in view of (3.17). It remains to observe that $\mathcal{S}^{\mathbb{R}} m_n = \mathcal{H}^{\mathbb{R}} m_n = \mathrm{Im} M_n$ and M_n maps $[0,1]$ onto the boundary of S, so

$$||\mathcal{S}^{\mathbb{R}} m_n||_{p,\infty} = ||\mathcal{H}^{\mathbb{R}} m_n||_{p,\infty} \geq |\{r \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} m_n(r)| \geq 1\}| \geq |[0,1]| = 1,$$

which completes the proof of (1.6). To deal with the Riesz projection, let $h_n(s) = L^n(s)$ for $s \in \mathbb{R}$ and $n = 1, 2, \ldots$. One easily derives that

$$h(s) = \begin{cases} \frac{s}{(s+\sqrt{s^2-s})^2} & \text{if } s \in (1,\infty), \\ \frac{s}{(s-\sqrt{s^2-s})^2} & \text{if } s \in (-\infty,0). \end{cases} \tag{5.1}$$

Since L^n is analytic, we have $P_+^{\mathbb{R}} h_n = h_n$. In addition,

$$||h_n||_p^p = \int_{[0,1]} |h_n(s)|^p \mathrm{d}s + \int_{\mathbb{R}\setminus[0,1]} |h_n(s)|^p \mathrm{d}s \xrightarrow{n \to \infty} 1,$$

because, by (5.1), the second integral converges to 0. Finally,

$$|\{s \in \mathbb{R} : |h_n(s)| \geq 1\}| \geq |[0,1]| = 1.$$

Thus $||P_+||_{L^p(\mathbb{R},\ell^2_{\mathbb{C}}) \to L^{p,\infty}(\mathbb{R},\ell^2_{\mathbb{C}})} \geq 1$. The bound for P_- is proved in the same manner, using the functions $\overline{h_n}$.

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