Weak Φ-inequalities for the Haar system and differentially subordinated martingales

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Abstract: For a wide class of Young functions $\Phi : [0,\infty) \to [0,\infty)$, we determine the best constant C_Φ such that the following holds. If $(h_k)_{k\geq 0}$ is the Haar system on $[0,1]$, then for any vectors a_k from a separable Hilbert space $\mathcal{H}$ and $\varepsilon_k \in \{-1,1\}$, $k = 0, 1, 2, \ldots$, we have

$$\left|\left\{x \in [0,1] : \left|\sum_{k=0}^n \varepsilon_k a_k h_k(x)\right| \geq 1\right\}\right| \leq C_\Phi \int_0^1 \Phi\left(\left|\sum_{k=0}^n a_k h_k(x)\right|\right) \mathrm{d}x, \qquad n = 0, 1, 2, \ldots.$$

This is generalized to the sharp weak-Φ inequality

$$\mathbb{P}\left(\sup_{t\geq 0} |Y_t| \geq 1\right) \leq C_\Phi \sup_{t\geq 0} \mathbb{E}\Phi(|X_t|),$$

where X, Y stand for $\mathcal{H}$-valued martingales such that Y is differentially subordinate to X. These statements complement and generalize the results of Burkholder, Suh, the author and others.

Key words: Haar system; martingale; weak-Φ inequality; best constant.

1. Introduction Our motivation comes from a very basic question about $(h_k)_{k\geq 0}$, the Haar system on $[0,1]$. A classical result of Marcinkiewicz [8] (see also Paley [13]) states that if $1 < p < \infty$, then there is a universal finite constant c_p such that

$$c_p^{-1} \left\|\sum_{k=0}^n a_k h_k\right\|_p \leq \left\|\sum_{k=0}^n \varepsilon_k a_k h_k\right\|_p \leq c_p \left\|\sum_{k=0}^n a_k h_k\right\|_p \tag{1.1}$$

for any n and any $a_k \in \mathbb{R}$, $\varepsilon_k \in \{-1,1\}$, $k = 0, 1, 2, \ldots, n$. In other words, this means that the Haar system is an unconditional basis of $L^p([0,1])$, $1 < p < \infty$. This result was extended by Burkholder [4] to the martingale setting. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_k)_{k\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$. Assume that $f = (f_k)_{k\geq 0}$ is a real-valued martingale with the difference sequence $(df_k)_{k\geq 0}$ given by $df_0 = f_0$ and $df_k = f_k - f_{k-1}$ for $k \geq 1$. Let g be a transform of f by a real predictable sequence $v = (v_k)_{k\geq 0}$ bounded in absolute value by 1: that is, $dg_k = v_k df_k$ for all $k \geq 0$ and by predictability we mean that each term v_k is measurable with respect to $\mathcal{F}_{(k-1)\vee 0}$. Then for $1 < p < \infty$ there is an absolute constant c_p' for which

$$\sup_{n\geq 0} ||g_n||_p \leq c_p' \sup_{n\geq 0} ||f_n||_p. \tag{1.2}$$

Let $c_p(1.1)$, $c_p'(1.2)$ denote the optimal constants in (1.1) and (1.2), respectively. The Haar system is a martingale difference sequence with respect to its natural filtration (on the probability space being the Lebesgue's unit interval) and hence so is $(a_k h_k)_{k\geq 0}$, for given fixed real numbers a_0, a_1, a_2, ... Therefore, $c_p(1.1) \leq c_p'(1.2)$ for all $1 < p < \infty$. In fact, by the results of Burkholder [4] and Maurey [9], these constants coincide: $c_p(1.1) = c_p'(1.2)$ for all $1 < p < \infty$. The precise value of $c_p(1.1)$ was identified by Burkholder in [4]: $c_p(1.1) = p^* - 1$ (where $p^* = \max\{p, p/(p-1)\}$) for $1 < p < \infty$. Furthermore, the constant does not change if we allow the martingales and the coefficients a_k to take values in a separable Hilbert space $\mathcal{H}$. In fact, (1.2) can be studied under the less restrictive assumption of differential subordination in the continuous-time setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete and equip it with a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted cadlag martingales taking values in $\mathcal{H}$ which, as we may and do assume from now on, is equal to ℓ^2. Following Wang [16], we say that Y is *differentially subordinate* to X, if the process

2010 Mathematics Subject Classification. Primary 60G44; Secondary 60G42, 46E30.

$([X,X]_t - [Y,Y]_t)_{t\geq 0}$ is nondecreasing and nonnegative as a function of t. Here $[X,Y] = \sum_{j=0}^{\infty}[X^j,Y^j]$, where X^j, Y^j stand for the j-th coordinates of X and Y, respectively, and $[X^j,Y^j]$ is the quadratic covariance process of X^j and Y^j (see e.g. Dellacherie and Meyer [6]). If we treat the discrete-time martingales $f = (f_k)_{k=0}^{\infty}$, $g = (g_k)_{k=0}^{\infty}$ as continuous-time processes (via $X_t = f_{\lfloor t\rfloor}$ and $Y_t = g_{\lfloor t\rfloor}$ for $t \geq 0$), then the above condition reads

$$|dg_k| \leq |df_k| \qquad \text{for } k \geq 0,$$

which is the original definition of the differential subordination due to Burkholder [4]. This domination is satisfied in the setting of martingale transforms studied above; thus the following result, proved by Wang [16] (see also the earlier paper [5] by Burkholder), extends (1.1) and (1.2): for $1 < p < \infty$,

$$\sup_{t\geq 0} ||Y_t||_p \leq (p^* - 1) \sup_{t\geq 0} ||X_t||_p, \tag{1.3}$$

and the constant $p^* - 1$ is the best possible. This result have found many applications, in particular to the study of the L^p-boundedness of wide classes of Fourier multipliers (cf. [1], [2] and [7]). See also [10] and [11] for related extensions of (1.3).

For $p = 1$ the inequalities (1.1), (1.2) and (1.3) do not hold with any finite constant, but one can show an appropriate weak-type $(1,1)$ bound. In fact a much more general weak Φ-estimate is valid. Suppose that $\Phi : [0,\infty) \to [0,\infty)$ is an increasing convex function such that Φ is twice differentiable on $(0,\infty)$, Φ' is concave and $\Phi(0) = \Phi'(0+) = 0$. Then, as shown by Burkholder [4] and Wang [16], if Y is differentially subordinate to X, then

$$\mathbb{P}(\sup_{t\geq 0}|Y_t| \geq 1) \leq 2\left(\int_0^{\infty}\Phi(t)e^{-t}\mathrm{d}t\right)^{-1} \sup_{t\geq 0}\mathbb{E}\Phi(|X_t|)$$

and the inequality is sharp. In particular, if we take $\Phi(t) = t^p$, $1 \leq p \leq 2$, then we obtain a weak-type (p,p) estimate with the best constant $2/\Gamma(p+1)$. A natural problem arises: what happens for other functions Φ, say, for which Φ' is convex? This question turns out to be much more difficult. A partial answer to it was given by Suh [15], as many as twenty years after Burkholder's paper [4]. She showed that if $\Phi(t) = t^p$, $p > 2$, then the best constant C_Φ in

$$\mathbb{P}(\sup_{t\geq 0}|Y_t| \geq 1) \leq C_\Phi \sup_{t\geq 0}\mathbb{E}\Phi(|X_t|) \tag{1.4}$$

for *real-valued* X, Y is equal to $p^{p-1}/2$. The purpose of this note is to extend this inequality to a much wider class of functions. Denote by $\mathcal{C}$ the class of all strictly convex functions $\Phi : [0,\infty) \to [0,\infty)$ which are C^2 and satisfy

(a) $\Phi(0) = \lim_{x\downarrow 0}\Phi'(x)/x = 0$,
(b) $\left|\int_0^1 \log\Phi'(s)\mathrm{d}s\right| < \infty$,
(c) $\Phi''(x)x \geq \Phi'(x)$ for $x > 0$

(for example, $\Phi(t) = t^p$, $p > 2$; or $\Phi(t) = e^{t^p} - 1$, $p > 2$; see Section 4). Our result can be formulated as follows.

Theorem 1.1. *Let $\Phi \in \mathcal{C}$. Assume that X, Y are Hilbert-space-valued martingales such that Y is differentially subordinate to X. Then*

$$\mathbb{P}\big(\sup_{t\geq 0}|Y_t| \geq 1\big) \leq (2b\Phi'(b))^{-1}\sup_{t\geq 0}\mathbb{E}\Phi(|X_t|), \tag{1.5}$$

where b is the unique solution to the equation

$$\int_0^b \frac{\Phi''(s)s}{\Phi'(s)}ds = 1 - b.$$

The inequality is sharp even in the setting of the Haar system. Precisely, for any $C < (2b\Phi'(b))^{-1}$ there is an integer n and the numbers a_0, a_1, ..., $a_n \in \mathbb{R}$, ε_1, ε_2, ..., $\varepsilon_n \in \{-1,1\}$ for which

$$\begin{aligned}\left|\left\{x \in [0,1] : \sum_{k=0}^{n}\varepsilon_k a_k h_k(x) \geq 1\right\}\right| \\ > C\Phi\int_0^1\left(\left|\sum_{k=0}^{n}a_k h_k(x)\right|\right)dx.\end{aligned} \tag{1.6}$$

Let us stress here that on the left-hand side of (1.6), we have the one-sided estimate, i.e., the series $\sum_{k=0}^{n}\varepsilon_k a_k h_k$ is not in absolute values.

While Suh's proof is very technical and involves the analysis of complicated differential equations, our approach is much simpler, works in the vector case and yields much more general statement. It rests on the properties of an appropriate special function and uses the so-called integration method (see e.g. [12]).

The remainder of this note is split into three parts. In the next section we present the proof of (1.5), and in Section 3 we deal with the sharpness of this estimate for the Haar system. The final part contains some examples. Throughout Sections 2 and 3, we assume that Φ is a fixed element of $\mathcal{C}$.

2. Proof of (1.5).

We start with the following straightforward fact.

Lemma 2.1. *There exist α, $\beta > 0$ such that*

$$\Phi(x) \geq \alpha x^2 - \beta \qquad \textit{for } x > 0. \tag{2.1}$$

Proof. We may restrict ourselves to $x > 1$, replacing β with $\max\{\alpha, \beta\}$, if necessary. By (c), the function $x \mapsto \Phi'(x)/x$ is nondecreasing on $(0, \infty)$, so $\Phi'(x) \geq \Phi'(1)x$ for $x \geq 1$ and thus

$$\Phi(x) = \Phi(1) + \int_1^x \Phi'(s)\mathrm{d}s \geq \Phi(1) + \frac{\Phi'(1)(x^2-1)}{2}.$$

This yields (2.1). □

Introduce the function $\gamma : [0, \infty) \to [0, \infty)$ by

$$\gamma(x) = \int_0^x \frac{\Phi''(s)s - \Phi'(s)}{\Phi'(s)}\mathrm{d}s.$$

The finiteness of γ follows from (b) and the integration by parts; this also shows that $\lim_{x\to 0}\gamma(x) = 0$. Furthermore, by (c), γ is nondecreasing and hence there is a unique $b \in (0, 1/2]$ satisfying $\gamma(b) + 2b = 1$. Define a function $a : (0, \infty) \to \mathbb{R}$ by the formula

$$a(\gamma(x) + x) = (\gamma(x) + x)^2 \cdot \frac{\Phi''(x)x - \Phi'(x)}{2x^3\Phi''(x)} \cdot \Phi'(x).$$

Let us gather some properties of these objects.

Lemma 2.2. *For any $x > 0$ we have*

$$\int_0^{\gamma(x)+x} \frac{a(r)}{r^2}\,dr = \frac{\Phi'(x)}{2x}, \tag{2.2}$$

$$\int_0^{\gamma(x)+x} \frac{a(r)}{r}\,dr = \frac{\Phi'(x)\gamma(x)}{2x} \tag{2.3}$$

and

$$\int_0^{\gamma(x)+x} a(r)\,dr = \frac{\Phi'(x)(\gamma(x)^2 + x^2)}{2x} - \Phi(x). \tag{2.4}$$

Proof. By the definition of γ, we have

$$\gamma'(x) + 1 = \Phi''(x)x/\Phi'(x), \qquad x > 0. \tag{2.5}$$

To show (2.2), we make the substitution $r = \gamma(s) + s$ and use (2.5) to obtain the equivalent identity

$$\int_0^x \frac{\Phi''(s)s - \Phi'(s)}{2s^2}\,\mathrm{d}s = \frac{\Phi'(x)}{2x},$$

which holds true, because of the condition (a). To check (2.3) and (2.4), note that the expressions on the left and on the right tend to 0 as $x \to 0$. Thus it suffices to verify whether the corresponding derivatives are equal. A direct differentiation of both sides of (2.3) leads to the equality

$$\frac{a(\gamma(x)+x)(\gamma'(x)+1)}{\gamma(x)+x} = \frac{(\Phi''(x)x - \Phi'(x))\gamma(x)}{2x^2} + \frac{\Phi'(x)\gamma'(x)}{2x}.$$

Plugging the formula for the function a and using (2.5) we obtain, after some straightforward calculations, that both sides above are equal to

$$(\gamma(x) + x)(\Phi''(x)x - \Phi'(x))/(2x^2).$$

For the equation (2.4) the verification is similar; we leave the details to the reader. □

The next step is to define an auxiliary special function $u : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$. It is given by

$$u(x, y) = \begin{cases} (|y| - 1)^2 - |x|^2 & \text{if } |x| + |y| \geq 1, \\ 0 & \text{if } |x| + |y| < 1. \end{cases}$$

The key property of this object is stated in Lemma 2.3 below. Note that if X is square-integrable and Y is differentially subordinate to X, then, by (1.3), Y also belongs to L^2. Consequently, the pointwise limits $X_\infty = \lim_{t\to\infty} X_t$, $Y_\infty = \lim_{t\to\infty} Y_t$ exist almost surely.

Lemma 2.3. *Assume that X, Y are martingales such that Y is differentially subordinate to X and $X \in L^2$. Then $\mathbb{E}u(X_\infty, Y_\infty) \leq 0$.*

For the proof, see Lemma 2.2 in [12].

We move to the central object of this note. Introduce the special function $U : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$U(x, y) = \int_0^{1-b} a(r)u(x/r, y/r)\mathrm{d}r.$$

It is easy to check that

$$\begin{aligned} U(x, y) = (|y|^2 - |x|^2) &\int_0^{(1-b)\wedge(|x|+|y|)} \frac{a(r)}{r^2}\mathrm{d}r \\ &- 2|y| \int_0^{(1-b)\wedge(|x|+|y|)} \frac{a(r)}{r}\mathrm{d}r \\ &+ \int_0^{(1-b)\wedge(|x|+|y|)} a(r)\mathrm{d}r. \end{aligned}$$

Let us show the crucial property of U.

Lemma 2.4. *For any x, $y \in \mathcal{H}$ we have*

$$U(x, y) \geq 2\Phi'(b)b\,1_{\{|y|\geq 1\}} - \Phi(|x|). \tag{2.6}$$

Proof. Since U depends on x and y through their norms, it suffices to show the majorization for $\mathcal{H} = \mathbb{R}$ and x, $y \geq 0$. We will first deal with the case $x + y \leq 1 - b$. Fix $u \in (0, b]$ and let

$$F(s) = U(u - s, \gamma(u) + s), \quad G(s) = -\Phi(u - s)$$

for $s \in [-\gamma(u), u]$. Clearly, F is linear and G is concave. Furthermore, $F'(0)$ is given by

$$2(\gamma(u) + u)\int_0^{\gamma(u)+u} \frac{a(r)}{r^2}\mathrm{d}r - 2\int_0^{\gamma(u)+u} \frac{a(r)}{r}\mathrm{d}r,$$

which, by (2.2) and (2.3), equals $\Phi'(u) = G'(0)$. Similarly, (2.2)-(2.4) imply that $F(0) = G(0)$. In consequence, we have $F(s) \geq G(s)$ for all s and hence the substitution $x = u - s$, $y = \gamma(u) + s$ gives (2.6) on the set $\{(x, y) : x + y \leq 1 - b\}$.

Now, suppose that $x + y > 1 - b$. Then the majorization takes the form

$$(2.7)\qquad \begin{aligned} &(y^2 - x^2)\int_0^{1-b} \frac{a(r)}{r^2}\mathrm{d}r - 2y\int_0^{1-b}\frac{a(r)}{r}\mathrm{d}r \\ &+ \int_0^{1-b} a(r)\mathrm{d}r - 2\Phi'(b)b\,1_{\{y\geq 1\}} + \Phi(x) \geq 0. \end{aligned}$$

In fact, this bound holds true for *all* x, $y \geq 0$. Indeed, for a fixed x, the left-hand side is a convex function of y, which attains its minimum at

$$y_0 = \int_0^{1-b} \frac{a(r)}{r}\mathrm{d}r \left(\int_0^{1-b}\frac{a(r)}{r^2}\mathrm{d}r\right)^{-1} = \gamma(b) < 1$$

(to see the second equality, apply (2.2) and (2.3) with $x = b$). In consequence, it suffices to verify (2.7) for $y = \gamma(b)$ and $y = 1$ only. If the first possibility occurs, then both sides are equal for $x = b$ (by virtue of (2.2), (2.3) and (2.4)). Moreover, if we differentiate the left hand side of (2.7) over x, we obtain the expression

$$(2.8)\qquad 2x\left(-\int_0^{1-b}\frac{a(t)}{t^2}\,\mathrm{d}t + \frac{\Phi'(x)}{2x}\right).$$

By (2.2), this is zero for $x = b$, and since $x \mapsto \Phi'(x)/x$ is nondecreasing (see (c)), we deduce that (2.8) is nonpositive for $x \leq b$ and nonnegative for $x \geq b$. This gives (2.7) for $x \geq 0$ and $y = \gamma(b)$. When $y = 1$, we argue similarly: both sides of (2.7) are equal for $x = b$, and the partial derivative with respect to x (which is again given by (2.8)) has the appropriate behavior. This completes the proof. □

We are ready to establish our main inequality.

Proof of (1.5). We start with some reductions. First, we may assume that $\sup_{t\geq 0}\mathbb{E}\Phi(|X_t|) < \infty$, since otherwise there is nothing to prove. By (2.1), this assumption gives that X is bounded in L^2 and hence, by Burkholder's inequality (1.3), so is Y. The second observation is that it suffices to show that

$$(2.9)\qquad 2b\Phi'(b)\mathbb{P}(|Y_\infty| \geq 1) \leq \mathbb{E}\Phi(|X_\infty|).$$

To see this, let us introduce the stopping time $\tau = \inf\{t \geq 0 : |Y_t| \geq 1\}$ (with the usual convention $\inf\emptyset = \infty$) and the stopped martingales $X^\tau = (X_{\tau\wedge t})_{t\geq 0}$, $Y^\tau = (Y_{\tau\wedge t})_{t\geq 0}$. Obviously, Y^τ is differentially subordinate to X^τ, $\mathbb{E}\Phi(|X^\tau_\infty|) \leq \mathbb{E}\Phi(|X_\infty|) = \sup_{t\geq 0}\mathbb{E}\Phi(|X_t|)$ and

$$\{\sup_{t\geq 0}|Y_t| \geq 1\} = \{Y_\tau \geq 1\} = \{Y^\tau_\infty \geq 1\}.$$

Therefore, if we succeed in proving (2.9), we will apply it to the pair X^τ, Y^τ and obtain the stronger bound (1.5).

Thus, all we need is to establish (2.9). Note that the auxiliary function u satisfies

$$u(x, y) \leq (|y| - 1)^2 + |x|^2 \leq |x|^2 + |y|^2 + 1$$

for all x, $y \in \mathcal{H}$, and hence

$$\begin{aligned} &\mathbb{E}\int_0^{1-b} a(r)|u(X_\infty/r, Y_\infty/r)|\mathrm{d}r \\ &\leq \mathbb{E}\int_0^{1-b}\frac{a(r)}{r^2}(|X_\infty|^2 + |Y_\infty|^2 + r^2)\mathrm{d}r < \infty. \end{aligned}$$

Therefore, we are permitted to apply Fubini's theorem and obtain, by Lemma 2.3,

$$\mathbb{E}U(X_\infty, Y_\infty) = \int_0^\infty a(r)\mathbb{E}u(X_\infty/r, Y_\infty/r)\mathrm{d}r \leq 0,$$

because for any $r \geq 0$, the martingale Y/r is differentially subordinate to X/r. It remains to use (2.6) to obtain (2.9). □

3. Sharpness. Let C_Φ be the least number, depending only on Φ, such that for all n, all real numbers a_0, a_1, …, a_n and any sequence ε_0, ε_1, …, ε_n of signs we have

$$\left|\left\{x : \sum_{k=0}^n \varepsilon_k a_k h_k(x) \geq 1\right\}\right| \leq C_\Phi\int_0^1 \Phi\left(\left|\sum_{k=0}^n a_k h_k\right|\right).$$

It follows from the results of Maurey [9] or Burkholder (see Section 10 in [4]), that C_Φ coincides with the optimal constant C'_Φ in the estimate

$$\mathbb{P}(\sup_{n\geq 0} g_n \geq 1) \leq C'_\Phi \sup_{n\geq 0}\mathbb{E}\Phi(|f_n|),$$

valid for all real martingales f and their transforms g by predictable sequences bounded in absolute value by 1. Passing to the continuous-time setting and using some standard approximation, we see that C'_Φ is precisely the best constant in the inequality

$$\mathbb{P}(\sup_{t\geq 0} Y_t \geq 1) \leq C'_\Phi \sup_{t\geq 0}\mathbb{E}\Phi(|X_t|),$$

in which X is a real martingale and Y is the stochastic integral, with respect to X, of a certain predictable process $H = (H_t)_{t\geq 0}$ taking values in $[-1, 1]$. For the precise justification of this step, see

Bichteler [3]. Summarizing, the optimality of C_Φ will follow if we construct a pair (X,Y) as above, for which the ratio $\mathbb{P}(\sup_{t\geq 0} Y_t \geq 1)/\sup_{t\geq 0}\mathbb{E}\Phi(|X_t|)$ is arbitrarily close to $(2b\Phi'(b))^{-1}$.

Fix $\varepsilon \in (0,1-b)$. Let $B=(B_t)_{t\geq 0}$ be a standard, one-dimensional Brownian motion starting from 0 and let $\beta=(\beta_t)_{t\geq 0}$ be given by

$$\beta_t = \varepsilon - \int_0^t \operatorname{sgn} B_s \, \mathrm{d}B_s, \qquad t \geq 0$$

(here $\operatorname{sgn} x = 1$ if $x>0$ and $\operatorname{sgn} x = -1$ if $x \leq 0$). By Itô-Tanaka's formula (see e.g. Revuz and Yor [14]), we have $\beta_t = \varepsilon + L_t - |B_t|$, where $L=(L_t)_{t\geq 0}$ denotes the local time of B at 0. In consequence, we see that the maximum process $(\sup_{s\leq t}\beta_s)_{t\geq 0}$ increases on the set $\{t : B_t = 0\}$. Next, introduce the stopping time $\sigma = \inf\{t : \beta_t = 1-b \text{ or } \beta_t = \gamma(B_t)\}$. Furthermore, if $\beta_\sigma = \gamma(B_\sigma)$, we put $\tau=\sigma$; if $\beta_\sigma = 1-b$, then, as we have already observed above, we have $B_\sigma = 0$ and in this case we let $\tau = \inf\{t \geq \sigma : B_t \in \{-b-\varepsilon, b\}\}$. It is easy to see that τ is exponentially integrable, since $\tau \leq \inf\{t : |B_t| \geq b+\varepsilon\}$.

Define the martingales X, Y by $X_t = B_{\tau\wedge t}$ and $Y_t = \int_0^t H_s \mathrm{d}X_s$, where H is a predictable process given by

$$H_s = \begin{cases} -\operatorname{sgn} B_s & \text{if } s \leq \sigma, \\ -1 & \text{if } s > \sigma. \end{cases}$$

To gather some intuition about the behavior of the pair $(X, \varepsilon+Y)$, let us make the following observations. The pair starts from the point $(0,\varepsilon)$ and takes values in the set $\{(x,y) : y \geq \gamma(|x|)\}$; when it is in the first quadrant, it moves along a line segment of slope -1 until it reaches the y-axis or the curve $\{(x,y) : y = \gamma(x)\}$; if it belongs to the second quadrant, it moves along the line segment of slope 1 until it reaches the y-axis or the curve $y=\gamma(-x)$; when it is on the y-axis and $Y < 1-b$, then it makes "an infinitely small martingale move" along the line segment of slope 1. Finally, if the pair ever reaches the point $(0,1-b)$, then it starts moving along the line segment of slope -1, until it reaches the point $(b,\gamma(b))$ or $(-b-\varepsilon, 1+\varepsilon)$.

To compute the lower bound for the ratio $\mathbb{P}(\sup_{t\geq 0}|Y_t| \geq 1)/\sup_{t\geq 0}\mathbb{E}\Phi(|X_t|)$, we will again use the special functions u and U. Fix $r \in (0,1-b)$ and introduce the stopping time $\eta = \inf\{t \geq 0 : \varepsilon + Y_t \geq r\}$. Of course, we have

$$u(X_{\tau\wedge\eta}/r, (\varepsilon+Y_{\tau\wedge\eta})/r) = u(X_0/r, (\varepsilon+Y_0)/r),$$

since both sides are 0 if $\varepsilon \leq r$, and $\eta \equiv 0$ if $\varepsilon > r$. In addition, the above analysis of (X,Y) shows that for $t \in (\eta,\tau]$, the rescaled pair $(X_t/r, (\varepsilon+Y_t)/r)$ belongs to the set $\{(x,y) : y \geq 0,\ |x|+y \geq 1\}$. However, u coincides on this set with the smooth function $(x,y) \mapsto (y-1)^2 - x^2$. Thus we are allowed to apply Itô's formula to $u(X_t/r, (\varepsilon+Y_t)/r)$ and obtain

$$u(X_\tau/r, (\varepsilon+Y_\tau)/r) = u(0,\varepsilon/r) + I_1 + I_2, \tag{3.1}$$

where

$$I_1 = -\int_{\tau\wedge\eta}^{\tau} \frac{2X_s}{r^2}\,\mathrm{d}X_s + \int_{\tau\wedge\eta}^{\tau} 2\left(\frac{\varepsilon+Y_s}{r} - 1\right)\frac{\mathrm{d}Y_s}{r},$$

$$I_2 = -\frac{1}{r^2}\int_\eta^\tau \mathrm{d}[X,X]_s + \frac{1}{r^2}\int_\eta^\tau \mathrm{d}[Y,Y]_s = 0.$$

Since X and Y are bounded and τ is exponentially integrable, we have $\mathbb{E}I_1 = 0$, by the properties of stochastic integrals. Therefore, taking expectation of both sides of (3.1), we see that

$$\mathbb{E}u(X_\tau/r, (\varepsilon+Y_\tau)/r) = u(0,\varepsilon/r) \geq 0.$$

Consequently, by Fubini's theorem (which is applicable - repeat the reasoning from the previous section),

$$\begin{aligned} 0 &\leq \mathbb{E}U(X_\tau, \varepsilon+Y_\tau) \\ &= U(-b-\varepsilon, 1+\varepsilon)\mathbb{P}(\varepsilon+Y_\tau = 1+\varepsilon) \\ &\quad + \mathbb{E}U(X_\tau, \varepsilon+Y_\tau)1_{\{\varepsilon+Y_\tau<1+\varepsilon\}}. \end{aligned}$$

However, on $\{\varepsilon+Y_\tau < 1+\varepsilon\} = \{Y_\tau < 1\}$ we have $\varepsilon + Y_\tau = \gamma(|X_\tau|)$ and hence $U(X_\tau, \varepsilon+Y_\tau) = -\Phi(|X_\tau|)$. Thus, the preceding inequality implies

$$\mathbb{P}(Y_\tau = 1)\cdot\left(U(-b-\varepsilon, 1+\varepsilon) + \Phi(b+\varepsilon)\right) \geq \mathbb{E}\Phi(|X_\tau|)$$

and, in consequence,

$$\frac{\mathbb{P}(\sup_{t\geq 0} Y_t \geq 1)}{\sup_{t\geq 0}\mathbb{E}\Phi(|X_t|)} \geq \frac{1}{U(-b-\varepsilon,1+\varepsilon)+\Phi(b+\varepsilon)}.$$

It remains to let $\varepsilon \to 0$: then the right-hand side converges to $(2b\Phi'(b))^{-1}$. This proves the desired sharpness.

4. Examples. Finally, we present three families of functions Φ from $\mathcal{C}$, for which the corresponding weak-type constants C_Φ have a nice form.

4.1. Suh's estimate. We start with the choice $\Phi(t) = t^p$, $p > 2$. It is straightforward to check that Φ belongs to the class $\mathcal{C}$. Furthermore, all the parameters can be easily computed. Namely, we have $\gamma(x) = (p-2)x$, $b = 1/p$ and hence the weak (p,p) constant equals

$$C_\Phi = (2b\Phi'(b))^{-1} = \frac{p^{p-1}}{2}.$$

4.2. An exponential bound. Now take $\Phi(t) = e^{t^p} - 1$, $p > 2$. Then

$$\Phi'(t) = pt^{p-1}e^{t^p}, \quad \Phi''(t) = pt^{p-2}e^{t^p}(p - 1 + pt^p),$$

so it is evident that (a), (b) and (c) hold true. Next, we derive that

$$\gamma(x) = (p-2)x + \frac{p}{p+1}x^{p+1}, \qquad x \geq 0.$$

In consequence, the best weak-Φ constant equals

$$C_\Phi = (2pb^p e^{b^p})^{-1},$$

where b is the unique solution to the equation

$$pb + \frac{p}{p+1}b^{p+1} = 1.$$

4.3. Another exponential bound. Our final example is the following. Pick $p > 2$ and let Φ be given by

$$\Phi(t) = \int_0^t s^{p-1}e^s \mathrm{d}s, \qquad t \geq 0.$$

We have

$$\Phi'(t) = t^{p-1}e^t, \quad \Phi''(t) = t^{p-2}e^t(p - 1 + t),$$

so Φ belongs to the class $\mathcal{C}$. We compute that

$$\gamma(x) = (p-2)x + \frac{x^2}{2}, \qquad x \geq 0,$$

and hence the parameter b is the solution to

$$\frac{b^2}{2} + pb = 1,$$

i.e., it is given by

$$b = \sqrt{p^2+2} - p.$$

Thus, the best weak-Φ constant equals

$$C_\Phi = \frac{1}{2}\left(\frac{\sqrt{p^2+2}+p}{2}\right)^p \exp\left(p - \sqrt{p^2+2}\right).$$

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