SHARP WEAK-TYPE ESTIMATES FOR THE DYADIC-LIKE MAXIMAL OPERATORS

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Abstract. We provide the explicit formula for the Bellman function corresponding to the weak-type $L^{p,\infty} \to L^{q,\infty}$ estimates for the dyadic maximal operator on $\mathbb{R}^n$. Actually, we do this in the more general setting of maximal operators associated with a tree-like structure on a nonatomic probability space. The approach rests on a clever combination of some novel optimization and combinatorial arguments.

1. Introduction

The results of this paper are motivated by certain basic estimates for the dyadic maximal operator on $\mathbb{R}^n$. Recall that this operator is given by

$$\mathcal{M}_d\phi(x) = \sup\left\{\frac{1}{|Q|}\int_Q |\phi(u)|\mathrm{d}u : x \in Q,\, Q \subset \mathbb{R}^n \text{ is a dyadic cube}\right\},$$

where ϕ is a locally integrable function on $\mathbb{R}^n$, $|Q|$ denotes the Lebesgue measure of Q and the dyadic cubes are those formed by the grids

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$2^{-N}\mathbb{Z}^n$, $N = 0, 1, 2, \ldots$. This maximal operator plays a crucial role in analysis and PDEs, and it is of considerable interest to obtain optimal, or at least good bounds for its norms. For instance, $\mathcal{M}_d$ satisfies the weak-type $(1,1)$ inequality

$$\lambda \big| \{x \in \mathbb{R}^n : \mathcal{M}_d\phi(x) \geq \lambda\} \big| \leq \int_{\{\mathcal{M}_d\phi \geq \lambda\}} |\phi(u)| \mathrm{d}u \tag{1.1}$$

for any $\phi \in L^1(\mathbb{R}^n)$ and any $\lambda > 0$. As one easily verifies, this bound is sharp: for each λ there is a nonzero ϕ for which both sides are equal. The dyadic maximal operator enjoys Hardy-Littlewood-Doob L^p estimate

$$||\mathcal{M}_d\phi||_{L^p(\mathbb{R}^n)} \leq \frac{p}{p-1} ||\phi||_{L^p(\mathbb{R}^n)}, \qquad 1 < p \leq \infty, \tag{1.2}$$

in which the constant $p/(p-1)$ is also the best. The above two estimates are of fundamental importance and have served as a motivation for numerous further extensions. The literature on the subject is very large, and in what follows we will only focus on some statements which are closely related to the contents of this paper. The first extension we would like to mention is that both (1.1) and (1.2) hold in the more general setting of maximal operators $\mathcal{M}_\mathcal{T}$ associated with tree-like structure $\mathcal{T}$. Let us provide the necessary definition.

Definition 1.1. Assume that (X, μ) is a nonatomic probability space. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

(i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.

(ii) For every $I \in \mathcal{T}$ there is a finite subset $C(I) \subset \mathcal{T}$ containing at least two elements such that

(a) the elements of $C(I)$ are pairwise almost disjoint subsets of I (i.e., the intersection of any two of them has μ-measure 0),

(b) $I = \bigcup C(I)$.

(iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}^m$, where $\mathcal{T}^0 = \{X\}$ and $T^{m+1} = \bigcup_{I \in \mathcal{T}^m} C(I)$.

(iv) We have $\lim_{m \to \infty} \sup_{I \in \mathcal{T}^m} \mu(I) = 0$.

Given a probability space and a tree $\mathcal{T}$ of measurable subsets, one defines the associated maximal operator $\mathcal{M}_{\mathcal{T}}$ by the formula

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup \left\{ \frac{1}{\mu(I)} \int_I |\phi(u)| \mathrm{d}\mu(u) : x \in I, I \in \mathcal{T} \right\}.$$

It is clear that this new setting generalizes the dyadic case considered at the beginning, at least if we restrict ourselves there to functions supported on the cube $[0,1]^n$ (actually, this is not restrictive at all; the case of general ϕ follows from some standard localization and dilation arguments). Indeed, the class of dyadic cubes contained in $[0,1]^n$ forms a tree, and the associated maximal operator coincides with the dyadic maximal operator. We would also like to mention here the probabilistic context in which one can consider the above notions: the setting of trees is closely related to the theory of martingales.

There is a number of estimates for the operator $\mathcal{M}_{\mathcal{T}}$ (closely related to (1.1) and (1.2)), which have been studied in depth in the literature by several authors. For instance, Melas [8] studied the norms of $\mathcal{M}_{\mathcal{T}}$, considered as an operator from $L^p(X)$ to $L^q(X)$ (for $1 \leq q < p$). More precisely, he determined the best constant $C_{p,q}$ in the following local estimate: for any measurable E,

$$\left(\int_E (\mathcal{M}_\mathcal{T}\phi)^q \mathrm{d}\mu\right)^{1/q} \leq C_{p,q}\left(\int_X |\phi|^p \mathrm{d}\mu\right)^{1/p} \mu(E)^{1/q-1/p}$$

(see also [7] for the related LlogL bound). The paper [10] by Melas and Nikolidakis continues the research in this direction and treats the sharp version of the corresponding Kolmogorov's inequality. Namely, for any $0 < q < 1$ and any measurable set E, we have

$$\left(\int_E |\mathcal{M}_\mathcal{T}\phi|^q \mathrm{d}\mu\right)^{1/q} \leq \left(\frac{1}{1-q}\right)^{1/q}\left(\int_X |\phi| \mathrm{d}\mu\right)\mu(E)^{1/q-1},$$

and the constant $(1-q)^{-1/q}$ cannot be improved. Finally, let us mention here four papers devoted to weak-type estimates. First, Melas and Nikolidakis [9] investigated various sharp extensions of the inequality

$$||\mathcal{M}_\mathcal{T}\phi||_{p,\infty} \leq ||\phi||_p, \qquad 1 \leq p < \infty,$$

where $||\phi||_{p,\infty} = \sup_{\lambda>0} \lambda \left[\mu(\{x \in X : |\phi(x)| \geq \lambda\})\right]^{1/p}$ is the usual weak p-th quasinorm. The subsequent work [12] of Nikolidakis contains the sharp comparison of weak quasinorms:

$$||\mathcal{M}_\mathcal{T}\phi||_{p,\infty} \leq \frac{p}{p-1}||\phi||_{p,\infty}, \qquad 1 < p < \infty. \tag{1.3}$$

Consult also the recent Nikolidakis' paper [13] for the further refinement of this result. In [15], the author studied the action of $\mathcal{M}_\mathcal{T}$ as an operator from $L^{p,\infty}$ to L^q, $1 \leq q < p < \infty$. One of the main results of that paper is the estimate

$$||\mathcal{M}_{\mathcal{T}}\phi||_q \leq \left(\frac{p}{p-q}\right)^{1/q} \frac{p}{p-1}||\phi||_{p,\infty},$$

in which the constant on the right is the best possible.

It should be stressed here that each of the works cited above is devoted to much more detailed analysis of the underlying estimate. More precisely, all the papers contain explicit formulas for the associated so-called Bellman functions. In general, such an explicit derivation provides much more information about the action of maximal operators on the corresponding spaces. For the necessary definitions and the interplay between the estimates and the associated Bellman functions, see Section 2 below.

In this paper, we will be interested in the norm of $\mathcal{M}_{\mathcal{T}}$ as an operator from $L^{p,\infty}(X)$ to $L^{q,\infty}(X)$, $1 < q \leq p < \infty$. Furthermore, in contrast with (1.3), we will work under a different norming of the weak spaces:

$$|||\phi|||_{p,\infty} = \sup\left\{\frac{1}{\mu(E)^{1-1/p}}\int_E |\phi|\mathrm{d}\mu \ : \ E \text{ measurable}, \, \mu(E) > 0\right\}.$$

Then $|||\cdot|||_{p,\infty}$ is actually a norm, and in general, it is more difficult to handle than the quasinorm $||\cdot||_{p,\infty}$ considered above. One of our main results can be stated as follows.

Theorem 1.2. *Suppose that $1 < q \leq p < \infty$ are fixed parameters. Then for any μ-integrable function ϕ on X we have*

$$|||\mathcal{M}_{\mathcal{T}}\phi|||_{q,\infty} \leq \frac{p}{p-1}|||\phi|||_{p,\infty}\,, \tag{1.4}$$

$$|||\mathcal{M}_{\mathcal{T}}\phi|||_{q,\infty} \leq \left(\frac{p}{p-1}\right)^2 ||\phi||_{p,\infty}\,, \tag{1.5}$$

and the constants $p/(p-1)$, $(p/(p-1))^2$ are the best possible.

Note that in (1.5), we have the quasinorm $||\cdot||_{p,\infty}$ on the right.

Let us state here the probabilistic analogue of the above result. It can be expressed in the language of martingales, and follows from Theorem 1.2 by straightforward approximation. Though we will not go any further in this direction, we find the version worth stating as a separate theorem. For the necessary definitions and related results, we refer the reader to the classical monograph of Doob [5].

Theorem 1.3. *Suppose that $f = (f_n)_{n\geq 0}$ is a martingale on a certain probability space (with respect to its natural filtration). Then for any $1 < q \leq p < \infty$ and any $n \geq 0$ we have*

$$\left|\left|\left|\sup_{0\leq k\leq n}|f_k|\right|\right|\right|_{q,\infty} \leq \frac{p}{p-1}|||f_n|||_{p,\infty}\,,$$

$$\left|\left|\left|\sup_{0\leq k\leq n}|f_k|\right|\right|\right|_{q,\infty} \leq \left(\frac{p}{p-1}\right)^2 ||f_n||_{p,\infty}\,,$$

and the constants on the right-hand sides are the best possible.

A few words about the proof of Theorem 1.2. In analogy with the papers cited above, we will actually show much more. Namely, we will identify the explicit formula for the Bellman function associated with the estimates (1.4) and (1.5). We would also like to stress that our approach differs significantly from that appearing in [6]-[10] and [12, 13]. Instead of combinatorial analysis of the extremal functions, our reasoning rests on a novel unification of Bellman induction and optimization techniques. This approach is quite general in nature and, as we believe, can be applied in many other results of this type. In particular, some version of this reasoning has been successfully applied in [15] (see Section 2 below for more information), but the use of the norms $|||\cdot|||_{p,\infty}$ requires an additional complication of the argument.

We have organized the paper as follows. The next section explains the methodology which leads to our main results. In Section 3 we apply the method and obtain an upper bound for the Bellman function; this, in particular, allows us to establish the estimates (1.4) and (1.5). The final part of the paper is devoted to the lower bound for the Bellman function (which, on the level of (1.4) and (1.5), means the sharpness). This is done by constructing appropriate extremal examples there.

2. The associated Bellman function

There has been a considerable interest in the literature concerning inequalities for maximal operators, and various effective tools have been invented (e.g., covering theorems, optimizing procedures, Calderón-Zygmund-type decompositions, etc.). As we have announced in the previous section, we will particularly focus on yet another technique, the so-called Bellman function method. The underlying concept of this approach is to relate the problem of proving a given inequality for the maximal operator to the existence of a special function enjoying certain concavity and majorization properties. In general, such a special function carries much more information on the estimate under investigation, e.g., the best constants, extremal functions, and often provides further insight into the structure and the behavior of the maximal operator. A convenient reference is Section 2 in [15], where this is explained in detail. For related Bellman-function problems, we refer the interested reader to the works [1, 11, 16, 17, 18, 19, 20, 21]. There is a stochastic branch of this subject, concerning sharp estimates for semimartingales; see e.g. [2, 3, 4] and [14] and references therein.

We turn our attention to the weak-type inequalities (1.4) and (1.5). A key element in the study of these estimates is the identification of the explicit formula for the associated Bellman function:

$$\mathfrak{B}_p(f,F,L,s)=\sup\bigg\{\int_E \max\{\mathcal{M}_\mathcal{T}\phi,L\}\,\mathrm{d}\mu:\phi\geq 0,\\ \int_X\phi\,\mathrm{d}\mu=f,\,|||\phi|||_{p,\infty}\leq F,\,\mu(E)=s\bigg\}.$$

Here the supremum is taken over all ϕ and E satisfying the displayed conditions. We will see in a moment how the knowledge of this function yields the validity of the weak-type estimates.

Let us start the analysis from determining the appropriate natural domain $\mathcal{D}$ of the Bellman function. Here this set consists of all (f,F,L,s) satisfying $s\in[0,1]$, $f\leq L$ and $f\leq F$. Indeed, we have $\mathcal{M}_\mathcal{T}\phi\geq\int_X\phi\,\mathrm{d}\mu=f$, so there is no point in considering $L<f$ (because then we have $\mathfrak{B}_p(f,F,L,s)=\mathfrak{B}_p(f,F,f,s)$). Furthermore, $f=\int_X\phi\mathrm{d}\mu\leq F\mu(X)^{1-1/p}=F$, which gives the third inequality defining the domain $\mathcal{D}$. Actually, it is not difficult to show that for any $f\leq F$ there is a function $\phi:X\to[0,\infty)$ satisfying $\int_X\phi\,\mathrm{d}\mu=f$ and $|||\phi|||_{p,\infty}=F$ (e.g., see Section 4 below). Thus, $\mathfrak{B}_p$ is well-defined on $\mathcal{D}$: the supremum is taken over nonempty class of functions.

We will prove that $\mathfrak{B}_p$ admits the following explicit formula. Introduce $\mathcal{B}_p : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{B}_p(f, F, L, s)$$
$$= \begin{cases} \dfrac{pF}{p-1} s^{1-1/p} & \text{if } s < \left(\frac{f}{F}\right)^{p/(p-1)} \text{ and } Ls^{1/p} \leq F, \\ \dfrac{F^p}{(p-1)L^{p-1}} + Ls & \text{if } s < \left(\frac{f}{F}\right)^{p/(p-1)} \text{ and } Ls^{1/p} > F, \\ \dfrac{pf}{p-1} + f \ln \left[\left(\dfrac{F}{f} \right)^{p/(p-1)} s \right] & \text{if } s \geq \left(\frac{f}{F}\right)^{p/(p-1)} \text{ and } Ls \leq f, \\ Ls + \dfrac{f}{p-1} + f \ln \left[\dfrac{F^{p/(p-1)}}{L f^{1/(p-1)}} \right] & \text{if } s \geq \left(\frac{f}{F}\right)^{p/(p-1)} \\ & \quad \text{and } \frac{f}{s} < L \leq F^{p/(p-1)}/f^{1/(p-1)}, \\ Ls + \dfrac{F^p}{(p-1)L^{p-1}} & \text{if } s \geq \left(\frac{f}{F}\right)^{p/(p-1)} \\ & \quad \text{and } L > F^{p/(p-1)}/f^{1/(p-1)}. \end{cases}$$

Our main result is the following.

Theorem 2.1. *We have $\mathfrak{B}_p = \mathcal{B}_p$ for any $1 < p < \infty$.*

Now, let us describe (informally) the idea behind our approach to the above theorem. We have $\max\{\mathcal{M}_\mathcal{T}\phi, L\} \geq L$ and hence, by the very definition,

$$\mathfrak{B}_p(f, F, L, s) \geq Ls. \tag{2.1}$$

It is not difficult to check that $\mathcal{B}_p$ also satisfies this lower bound. Now, in a typical situation, a key ingredient of the analysis is to show that $\mathfrak{B}_p$ and $\mathcal{B}_p$ satisfy a certain structural condition (called "the main inequality", in the terminology introduced in [11]). In our case, it *would* look as follows: for any positive $f_\pm$, $F_\pm$, L, $s_\pm$ satisfying $f_\pm^p \leq F_\pm$ and $s_- + s_+ \leq 1$, we have

$$\begin{aligned}\mathfrak{B}_p&\left(\frac{f_-+f_+}{2}, \frac{F_-+F_+}{2}, \max\left\{\frac{f_-+f_+}{2}, L\right\}, s_- + s_+\right)\\ &\geq \frac{\mathfrak{B}_p(f_-, F_-, \max\{f_-, L\}, s_-) + \mathfrak{B}_p(f_+, F_+, \max\{f_+, L\}, s_+)}{2},\end{aligned}$$

and similarly for $\mathcal{B}_p$. The combination of this inequality and (2.1) is a main step of the estimate $\mathfrak{B}_p \leq \mathcal{B}_p$. The reverse bound is obtained with the use of appropriate examples. See [11] for the full description of the method.

Unfortunately, here the approach does not work: the main inequality is not valid, since the norm $|||\cdot|||_{p,\infty}$ is not of integral type. To overcome this difficulty, we will consider a certain majorant of $\mathfrak{B}_p$ which does enjoy the main inequality (a similar approach turned out to be successful in [15]). More precisely, we propose the following two-step procedure. Fix $\alpha > 0$, a function $R : [0,\infty) \to [0,\infty)$ and consider the following auxiliary object:

$$\begin{aligned}\mathfrak{B}^R(f, F, L, s) = \sup\bigg\{ \int_E &\max\{\mathcal{M}_\mathcal{T}\phi, L\}\,\mathrm{d}\mu : \phi \geq 0,\\ &\int_X \phi \mathrm{d}\mu = f, \int_X R(\phi)\,\mathrm{d}\mu \leq \alpha F^p,\ \mu(E) = s\bigg\}.\end{aligned}$$

This function can be studied with the use of "main inequality": the non-integral condition $|||\phi|||_{p,\infty} \leq F$ has been replaced by an integral assumption $\int_X R(\phi)\mathrm{d}\mu \leq \alpha F^p$. Furthermore, if R satisfies the additional property

$$\int_X R(\phi)\mathrm{d}\mu \leq \alpha|||\phi|||_{p,\infty}^p \tag{2.2}$$

for any $\phi \geq 0$, then we have $\mathfrak{B}_p \leq \mathfrak{B}^R$. The crucial observation is the following: if for any $(f, F, L, s) \in \mathcal{D}$ there is a function ϕ for which both sides of (2.2) are equal and which is an optimizer of $\mathfrak{B}^R(f, F, L, s)$, then in fact we have $\mathfrak{B}_p(f, F, L, s) = \mathfrak{B}^R(f, F, L, s)$ and we are done. Thus, we have reduced the problem to that of finding an appropriate R, and then identifying the corresponding $\mathfrak{B}^R$. We refer the reader to the paper [15] for a similar approach to the study of $L^{p,\infty} \to L^q$ estimates. Let us also make an important comment here. For a given R, the function $\mathfrak{B}^R$ depends on four variables; in a typical situation, one tries to decrease the dimension of the problem. Actually, as we will see in Section 3 below, we will manage to reduce the number of variables to two, and then come back to the original four-dimensional setting with the use of appropriate optimization argument (see Theorem 3.2). This novel reasoning significantly reduces the technicalities involved in the study of the Bellman function $\mathfrak{B}^R$.

Before we proceed to the proof of Theorem 2.1, let us show how it yields the assertion of Theorem 1.2. By the definition of $\mathfrak{B}_p$, we may write

$$\begin{aligned}&\sup\Big\{|||\mathcal{M}_{\mathcal{T}}\phi|||_{q,\infty}\,:\,|||\phi|||_{p,\infty} = F\Big\}\\&\qquad= \sup\Big\{s^{1/q-1}\mathfrak{B}_p\left(f, F, f, s\right)\,:\,s \in (0,1],\ f \in (0, F]\Big\}.\end{aligned}$$

However, we have

$$\mathfrak{B}_p(f,F,f,s)=\begin{cases}\dfrac{pF}{p-1}s^{1-1/p} & \text{if } s<\left(\frac{f}{F}\right)^{p/(p-1)},\\ sf+\dfrac{f}{p-1}+\dfrac{pf}{p-1}\ln\dfrac{F}{f} & \text{if } s\geq\left(\frac{f}{F}\right)^{p/(p-1)}.\end{cases}$$

But for fixed s and F, the function $\xi(f)=sf+\frac{f}{p-1}+\frac{pf}{p-1}\ln\frac{F}{f}$ is nondecreasing on $(0,Fs^{1-1/p}]$. Indeed, we compute that

$$\xi'(f)=s-1+\ln\left(\frac{F}{f}\right)^{p/(p-1)}\geq s-1-\ln s\geq 0.$$

This shows that $\mathfrak{B}_p(f,F,f,s)\leq\frac{pF}{p-1}s^{1-1/p}$ and hence

$$\begin{aligned}\sup\Big\{|||\mathcal{M}\phi|||_{q,\infty}\,:\,|||\phi|||_{p,\infty}=F\Big\}&=\frac{pF}{p-1}\sup_{s\in(0,1]}s^{1/q-1/p}\\&=\frac{pF}{p-1}.\end{aligned}$$

This establishes the first inequality of Theorem 1.2 as well as its sharpness. The estimate (1.5) follows at once from (1.4) and the well-known easy bound $|||\phi|||_{p,\infty}\leq\frac{p}{p-1}||\phi||_{p,\infty}$. To prove that the constant $(p/(p-1))^2$ cannot be improved, see Remark 4.2 below.

3. Proof of the inequality $\mathfrak{B}_p\leq\mathcal{B}_p$

Let κ be a fixed positive parameter and let $1<p<\infty$. Define the function $R_\kappa:[0,\infty)\to[0,\infty)$ by

$$R_\kappa(x)=\begin{cases}0 & \text{if } x\leq\dfrac{p-1}{p}\kappa,\\ -px+px\ln\left[\dfrac{px}{(p-1)\kappa}\right]+(p-1)\kappa & \text{if } x>\dfrac{p-1}{p}\kappa.\end{cases}\tag{3.1}$$

It is easy to check that the function R_κ is convex and of class C^1 on $(0,\infty)$. Next, for any $0<\lambda\leq\kappa$, introduce the special functions $B_{\lambda,\kappa}:\{(x,y):0\leq x\leq y\}\to\mathbb{R}$ by the formula

$$
\begin{aligned}
&B_{\lambda,\kappa}(x,y)\\
&=\begin{cases}
0 & \text{if } y\leq\lambda,\\
y-\lambda+x\ln\dfrac{\lambda}{y} & \text{if } \lambda<y<\kappa,\\
y-\lambda-R_\kappa(x)+x\ln\dfrac{\lambda}{\kappa} & \text{if } y\geq\kappa,\ x\leq\frac{p-1}{p}y,\\
p\left(y-\kappa+x\ln\dfrac{\kappa}{y}\right)+x\ln\dfrac{\lambda}{\kappa}+\kappa-\lambda & \text{if } y\geq\kappa,\ x\geq\frac{p-1}{p}y.
\end{cases}
\end{aligned}
\tag{3.2}
$$

In the lemma below, we study two properties of $B_{\lambda,\kappa}$ which can be regarded as appropriate versions of (2.1) and the "main inequality".

Lemma 3.1. *Let $0<\lambda\leq\kappa$ be fixed.*

(i) For any $0\leq x\leq y$ we have

$$B_{\lambda,\kappa}(x,y)\geq(y-\lambda)_+-R_\kappa(x)+x\ln\frac{\lambda}{\kappa}. \tag{3.3}$$

(ii) For any $0\leq x\leq y$ and any $d\geq -x$ we have

$$B_{\lambda,\kappa}(x+d,y\vee(x+d))\leq B_{\lambda,\kappa}(x,y)+\frac{\partial B_{\lambda,\kappa}(x,y)}{\partial x}d. \tag{3.4}$$

Proof. (i) Fix $y\geq 0$ and consider the function

$$\xi(x)=B_{\lambda,\kappa}(x,y)-(y-\lambda)_++R_\kappa(x)-x\ln\frac{\lambda}{\kappa},\qquad x\in[0,y].$$

If $y\leq\kappa$, then $\xi(x)=x\ln(\kappa/\max\{y,\lambda\})+R_k(x)$ is a sum of two nondecreasing functions on $[0,y]$, which vanish for $x=0$; hence $\xi\geq 0$

and (3.3) is established. If $y > \kappa$, we easily verify that ξ is convex and vanishes, along with its derivative, at the point $x = (p-1)y/p$. This yields the majorization.

(ii) The inequality is clear when $x + d \leq y$, since for any positive y, the function $x \mapsto B_{\lambda,\kappa}(x,y)$ is concave on $[0,y]$. Suppose then that $x + d > y$ and consider the function $\zeta(s) = B_{\lambda,\kappa}(s,s)$. We have

$$\zeta'(s) = \frac{\partial B_{\lambda,\kappa}(s,s)}{\partial x} + \frac{\partial B_{\lambda,\kappa}(s,s)}{\partial y} = \frac{\partial B_{\lambda,\kappa}(s,s)}{\partial y},$$

which combined with the aforementioned concavity of $x \mapsto B_{\lambda,\kappa}(x,y)$ gives

$$\begin{aligned} B_{\lambda,\kappa}(x,y) + \frac{\partial B_{\lambda,\kappa}(x,y)}{\partial x} d &\geq B_{\lambda,\kappa}(y,y) + \frac{\partial B_{\lambda,\kappa}(x,y)}{\partial x}(x+d-y) \\ &\geq B_{\lambda,\kappa}(y,y) + \frac{\partial B_{\lambda,\kappa}(y,y)}{\partial x}(x+d-y) \\ &= \zeta(y) + \zeta'(y)(x+d-y). \end{aligned}$$

Thus, we will be done if we show that ζ is concave. But this is evident: ζ is of class C^1 on $(0,\infty)$ and admits the formula

$$\zeta(y) = \begin{cases} 0 & \text{if } y \leq \lambda, \\ y - \lambda + y\ln(\lambda/y) & \text{if } \lambda < y < \kappa, \\ p\left(y - \kappa + y\ln(\kappa/y)\right) + y\ln(\lambda/\kappa) + \kappa - \lambda & \text{if } y \geq \kappa. \end{cases}$$

The proof is complete. $\square$

The next statement can be understood as the upper bound for the function $\mathfrak{B}^{R_\kappa}$, introduced in the previous section. Actually, it can be also shown that the estimate is sharp, but we will not need this.

Theorem 3.2. *Let $0 < \lambda \leq \kappa$ be fixed. For any nonnegative ϕ on X, any $L > 0$ and any measurable $E \subset X$ we have*

$$\begin{aligned} \int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\, d\mu \leq \lambda\mu(E) + \int_X \left(R_\kappa(\phi) - \phi \cdot \ln \frac{\lambda}{\kappa}\right) d\mu \\ + B_{\lambda,\kappa}\left(\int_X \phi\, d\mu, \max\left\{\int_X \phi\, d\mu, L\right\}\right). \end{aligned} \tag{3.5}$$

Proof. The reasoning splits naturally into two parts.

Step 1. First we will show the intermediate estimate

$$\begin{aligned} \int_X (\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} - \lambda)_+\, \mathrm{d}\mu \leq \int_X \left(R_\kappa(\phi) - \phi \cdot \ln \frac{\lambda}{\kappa}\right) \mathrm{d}\mu \\ + B_{\lambda,\kappa}\left(\int_X \phi\, \mathrm{d}\mu, \max\left\{\int_X \phi\, \mathrm{d}\mu, L\right\}\right). \end{aligned} \tag{3.6}$$

To accomplish this, consider the associated sequence $(\phi_n)_{n\geq 0}$ of conditional expectations of ϕ with respect to $(\mathcal{T}^n)_{n\geq 0}$. That is, for any $x \in X$ and any nonnegative integer n, put

$$\phi_n(x) = \frac{1}{\mu(E)} \int_E \phi\, \mathrm{d}\mu, \tag{3.7}$$

where E is the element of $\mathcal{T}^n$ which contains x (since the elements of $\mathcal{T}^n$ are pairwise almost disjoint, such a set E is determined uniquely for μ-almost all x). We will also use the notation

$$\mathcal{M}_{\mathcal{T}^n}\phi(x) = \sup\left\{\frac{1}{\mu(I)} \int_I |\phi(u)| \mathrm{d}\mu(u) : x \in I, I \in \mathcal{T}^k \text{ for some } k \leq n\right\}.$$

Next, pick an integer $n \geq 0$, $E \in \mathcal{T}^n$ and let E_1, E_2, $\ldots$, E_m be the elements of $\mathcal{T}^{n+1}$ whose union is E. We will prove that

$$\begin{aligned}(3.8)\qquad &\int_E B_{\lambda,\kappa}\left(\phi_{n+1}, \max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\}\right) \mathrm{d}\mu \\ &\qquad\qquad \leq \int_E B_{\lambda,\kappa}\left(\phi_n, \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}\right) \mathrm{d}\mu.\end{aligned}$$

To do this, note that both ϕ_n and $\max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}$ are constant on E: denote the values of these functions by x and y, respectively. On the other hand, we have $\max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\} = \max\left\{\mathcal{M}_{\mathcal{T}^n}\phi, L, \phi_{n+1}\right\}$ and the function ϕ_{n+1} is constant on E_1, E_2, $\ldots$, E_m. Letting $d_j = (\phi_{n+1} - \phi_n)|_{E_j} = \phi_{n+1}|_{E_j} - x$, it follows directly from (3.7) that

$$(3.9)\qquad d_j \geq -x \quad \text{and} \quad \sum_{j=1}^m \mu(E_j) d_j = 0.$$

Now, apply (3.4) to x, y and $d = d_j$, and multiply both sides by $\mu(E_j)$, $j = 1$, 2, $\ldots$, m. If we sum up the obtained inequalities, we get

$$\sum_{j=1}^m \mu(E_j) B_{\lambda,\kappa}(\phi_{n+1}|_{E_j}, \max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\}|_{E_j}) \leq \mu(E) B_{\lambda,\kappa}(x, y),$$

which is precisely (3.8). Summing these estimates over all $E \in \mathcal{T}^n$, we get the bound

$$\begin{aligned}&\int_X B_{\lambda,\kappa}\left(\phi_{n+1}, \max\{\mathcal{M}_{\mathcal{T}^{n+1}}\phi, L\}\right) \mathrm{d}\mu \\ &\qquad\qquad \leq \int_X B_{\lambda,\kappa}\left(\phi_n, \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}\right) \mathrm{d}\mu\end{aligned}$$

and hence, by induction,

$$\int_X B_{\lambda,\kappa}\left(\phi_n, \max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\}\right) \mathrm{d}\mu \leq \int_X B_{\lambda,\kappa}\left(\phi_0, \max\{\mathcal{M}_{\mathcal{T}^0}\phi, L\}\right) \mathrm{d}\mu.$$

However, we have $\phi_0 = \mathcal{M}_{\mathcal{T}^0}\phi = \int_X \phi \mathrm{d}\mu$ and hence the right hand side is equal to $B_{\lambda,\kappa}\left(\int_X \phi \mathrm{d}\mu, \max\left\{\int_X \phi \mathrm{d}\mu, L\right\}\right)$. To deal with the left-hand side, we make use of the majorization (3.3) and, as the result, obtain the bound

$$\begin{aligned}
&\int_X \left(\max\{\mathcal{M}_{\mathcal{T}^n}\phi, L\} - \lambda\right)_+ \mathrm{d}\mu \\
&\leq \int_X \left(R_\kappa\left(\phi_n\right) - \phi_n \cdot \ln\frac{\lambda}{\kappa}\right)\mathrm{d}\mu + B_\lambda\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right) \\
&\leq \int_X \left(R_\kappa\left(\phi\right) - \phi \cdot \ln\frac{\lambda}{\kappa}\right)\mathrm{d}\mu + B_\lambda\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right).
\end{aligned}$$

Here the latter estimate follows from Jensen inequality: $x \mapsto R_\kappa(x) - x\ln(\lambda/\kappa)$ is a convex function. It remains to observe that if we let $n \to \infty$, then $\mathcal{M}_{\mathcal{T}^n}\phi$ increases to $\mathcal{M}_{\mathcal{T}}\phi$; therefore, (3.6) follows from Lebesgue's monotone convergence theorem.

Step 2. Now we deduce (3.5). Pick a set E as in the statement and decompose it into

$$E^+ = E \cap \{\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} \geq \lambda\}, \qquad E^- = E \cap \{\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} < \lambda\}.$$

By (3.6), we have

$$\begin{aligned}
&\int_{E^+} \big(\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} - \lambda\big)\mathrm{d}\mu \\
&\leq \int_X \left(\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} - \lambda\right)_+ \mathrm{d}\mu \\
&\leq \int_X \left(R_\kappa(\phi) - \phi \cdot \ln\frac{\lambda}{\kappa}\right)\mathrm{d}\mu + B_{\lambda,\kappa}\left(\int_X \phi\,\mathrm{d}\mu, \max\left\{\int_X \phi\,\mathrm{d}\mu, L\right\}\right)
\end{aligned}$$

and, obviously,

$$\int_{E^-}\big(\max\{\mathcal{M}_{\mathcal{T}}\phi, L\} - \lambda\big)\mathrm{d}\mu \leq 0.$$

It remains to add the two inequalities above to get the claim. □

In our further considerations, we will need the following auxiliary well-known statement.

Lemma 3.3. *Suppose that* ξ, $\zeta : [0,1] \to [0,\infty)$ *are two nonincreasing integrable functions such that*

$$\int_0^t \xi(s)\,ds \leq \int_0^t \zeta(s)\,ds \qquad \text{for all } t \in (0,1].$$

Then for any convex increasing function $R : [0,\infty) \to [0,\infty)$ *we have*

$$\int_0^1 R(\xi(s))\,ds \leq \int_0^1 R(\zeta(s))\,ds.$$

Proof. This is straightforward. Any convex function R as in the statement can be approximated by linear combinations, with positive coefficients, of a constant and the functions of the form $t \mapsto (t-a)_+$, for $a \in [0,\infty)$. Thus, it is enough to show that $\int_0^1(\xi(s)-a)_+\mathrm{d}s \leq \int_0^1(\zeta(s)-a)_+\mathrm{d}s$ for any nonnegative a. For such a there is $u \in [0,1]$ for which

$$\int_0^1(\xi(s)-a)_+\mathrm{d}s = \int_0^u(\xi(s)-a)\mathrm{d}s \leq \int_0^u(\zeta(s)-a)\mathrm{d}s \leq \int_0^1(\zeta(s)-a)_+\mathrm{d}s.$$

This completes the proof. □

We are ready to establish the first half of Theorem 2.1.

Theorem 3.4. *We have* $\mathfrak{B}_p \leq \mathcal{B}_p$.

Proof. Fix $(f, F, L, s) \in \mathcal{D}$ and a function ϕ as in the definition of $\mathfrak{B}_p(f, F, L, s)$. Pick parameters $0 < \lambda \leq \kappa$ (to be specified later) and apply (3.5) to get

$$\begin{aligned} &\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \\ &\qquad \leq \lambda\mu(E) + \int_X R_\kappa(\phi)\mathrm{d}\mu - f\ln\frac{\lambda}{\kappa} + B_{\lambda,\kappa}(f, L). \end{aligned} \tag{3.10}$$

Since $|||\phi|||_{p,\infty} \leq F$, the nonincreasing rearrangement ϕ^* of ϕ satisfies

$$\int_0^s \phi^*(t)\mathrm{d}t \leq Fs^{1-1/p} = \int_0^s \frac{pFt^{-1/p}}{p-1}\mathrm{d}t \qquad \text{for } s \in (0, 1],$$

and hence Lemma 3.3 gives

$$\int_X R_\kappa(\phi)\mathrm{d}\mu = \int_0^1 R_\kappa(\phi^*(t))\mathrm{d}t \leq \int_0^1 R_\kappa\left(\frac{pFt^{-1/p}}{p-1}\right)\mathrm{d}t.$$

Some tedious, but straightforward calculations show that the latter integral is equal to $F^p\kappa^{1-p}/(p-1)$; thus we arrive at the following version of (2.2):

$$\int_X R_\kappa(\phi)\mathrm{d}\mu \leq F^p\kappa^{1-p}/(p-1).$$

Hence, coming back to (3.10), we get the estimate

$$\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \leq \lambda\mu(E) + \frac{F^p\kappa^{1-p}}{p-1} - f\ln\frac{\lambda}{\kappa} + B_{\lambda,\kappa}(f, L). \tag{3.11}$$

We complete the proof of $\mathfrak{B}_p \leq \mathcal{B}_p$ by picking appropriate values of the parameters λ and κ. If $\mu(E) < (f/F)^{p/(p-1)}$ and $L\mu(E)^{1/p} \leq F$, then we take $\lambda = \kappa = F\mu(E)^{-1/p}$: then $L \leq \lambda$, so $B_{\lambda,\kappa}(f, L) = 0$ and the right-hand side of (3.11) reduces to $pF\mu(E)^{1-1/p}/(p-1)$. If $\mu(E) <$

$(f/F)^{p/(p-1)}$ and $L\mu(E)^{1/p} > F$, then we put $\lambda = \kappa = L$: again, we have $B_{\lambda,\kappa}(f, L) = 0$ and (3.11) yields the claim. If $\mu(E) \geq (f/F)^{p/(p-1)}$ and $L\mu(E) \leq f$, then we obtain the assertion by taking $\lambda = f/\mu(E)$ and $\kappa = F^{p/(p-1)}/f^{1/(p-1)}$ (the condition $\lambda \leq \kappa$ is satisfied, which is due to the assumed lower bound for $\mu(E)$). If $\mu(E) \geq (f/F)^{p/(p-1)}$ and $f/\mu(E) < L \leq F^{p/(p-1)}/f^{1/(p-1)}$, then the choice $\lambda = L$ and $\kappa = F^{p/(p-1)}/f^{1/(p-1)}$ does the job. Finally, if $\mu(E) \geq (f/F)^{p/(p-1)}$ and $L > F^{p/(p-1)}/f^{1/(p-1)}$, then one picks $\lambda = \kappa = L$ and the claim follows. The proof is complete. □

4. Sharpness

4.1. **An example.** We will require the following fact, which appears in [6, Lemma 1].

Lemma 4.1. *For every $I \in \mathcal{T}$ and every $\alpha \in (0, 1)$ there is a subfamily $F(I) \subset \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\left(\bigcup_{J \in F(I)} J\right) = \sum_{J \in F(I)} \mu(J) = \alpha\mu(I).$$

Fix $1 < p < \infty$, $p' > p$ and $0 < f \leq F$. If $f = F$, pick $N = 0$; otherwise, let N be a large positive integer. Then there exists a positive number δ which satisfies the equation $(1 + \delta)^N = (F/f)^{p/(p-1)}$. This number can be made arbitrarily small: indeed, there is nothing to prove for $f = F$, while for $f < F$ it is enough to pick sufficiently large N. By an inductive use of Lemma 4.1, there is a sequence $X = A_0 \supset A_1 \supset A_2 \supset \ldots$ of measurable subsets of X, satisfying

(i) For each k, A_k is a union of certain pairwise disjoint subsets from $\mathcal{T}$: we have $A_k = \bigcup F_k$ for some $F_k \subset \mathcal{T}$.

(ii) For any $n = 0, 1, 2, \ldots, N-1$ and any $I \in F_n$, we have

$$\frac{\mu(A_{n+1} \cap I)}{\mu(I)} = (1+\delta)^{-1}.$$

(iii) For any $n = N, N+1, N+2, \ldots$ and any $I \in F_n$, we have

$$\frac{\mu(A_{n+1} \cap I)}{\mu(I)} = (1+p'\delta)^{-1}.$$

In particular, (ii) and (iii) imply

$$\mu(A_n) = \begin{cases} (1+\delta)^{-n} & \text{if } n = 0, 1, \ldots, N-1, \\ (f/F)^{p/(p-1)} (1+p'\delta)^{N-n} & \text{if } n \geq N. \end{cases} \tag{4.1}$$

Introduce the function ϕ on X by

$$\phi = \sum_{n=N}^{\infty} \frac{(p'-1)F^{p/(p-1)}}{p' f^{1/(p-1)}} (1+\delta)^{n-N} \chi_{A_n \setminus A_{n+1}}.$$

This gives a well-defined function on X, since $\mu(A_n) \to 0$ as $n \to \infty$. We easily compute that for any $k \geq N$,

$$\begin{aligned} \int_{A_k} \phi \mathrm{d}\mu &= \sum_{n=k}^{\infty} \frac{(p'-1)F^{p/(p-1)}}{p' f^{1/(p-1)}} (1+\delta)^{n-N} (\mu(A_n) - \mu(A_{n+1})) \\ &= \left(\frac{1+\delta}{1+p'\delta}\right)^{k-N} f \end{aligned}$$

and hence, in particular,

$$\int_X \phi \mathrm{d}\mu = \int_{A_0} \phi \mathrm{d}\mu = \int_{A_N} \phi \mathrm{d}\mu = f.$$

Moreover, if δ is sufficiently small, then $(1+\delta)^p \leq (1+p'\delta)$, so for $k \geq N$,

$$\int_{A_k} \phi \mathrm{d}\mu \leq f \cdot (1+p'\delta)^{(k-N)(1/p-1)} = F\mu(A_k)^{1-1/p}. \tag{4.2}$$

Since ϕ vanishes on $X \setminus A_N$, the above inequality holds also for $k = 0, 1, 2, \ldots, N-1$. These estimates imply that $|||\phi|||_{p,\infty} \leq F$, by the use of the following elementary and well-known argument. Pick an arbitrary set E with $\mu(E) > 0$ and take a look at the quantity

$$\frac{1}{\mu(E)^{1-1/p}} \int_E \phi \mathrm{d}\mu.$$

Let us try to maximize it, keeping $\mu(E)$ fixed. As n increases, the (constant) value of ϕ on $A_n \setminus A_{n+1}$ does not decrease, and therefore in the above optimization, we may assume that there is $k \geq 1$ such that $A_k \subset E \subset A_{k-1}$. Then $\mu(E) = s\mu(A_{k-1}) + (1-s)\mu(A_k)$ for some $s \in [0,1]$, so by (4.2) and Jensen's inequality,

$$\begin{aligned}
\int_E \phi \mathrm{d}\mu &= s\int_{A_{k-1}} \phi \mathrm{d}\mu + (1-s)\int_{A_k} \phi \mathrm{d}\mu \\
&\leq F\left[s\mu(A_{k-1})^{1-1/p} + (1-s)\mu(A_k)^{1-1/p}\right] \\
&\leq F\left[s\mu(A_{k-1}) + (1-s)\mu(A_k)\right]^{1-1/p} = F\mu(E)^{1-1/p}.
\end{aligned}$$

To analyze the behavior of $\mathcal{M}_\mathcal{T}\phi$, note that for any $n \geq N$,

$$\frac{1}{\mu(A_n)} \int_{A_n} \phi \mathrm{d}\mu = \frac{F^{p/(p-1)}}{f^{1/(p-1)}}(1+\delta)^{n-N}.$$

This implies a slightly stronger statement. Namely, for any $n \geq N$ and any $I \in F_n$,

$$\frac{1}{\mu(I)} \int_I \phi \, \mathrm{d}\mu = \frac{F^{p/(p-1)}}{f^{1/(p-1)}} (1+\delta)^{n-N}.$$

Indeed, by (ii), (iii) and the definition of ϕ, the conditional distribution of ϕ is the same on each $I \in F_n$ (i.e., $\mu(\{x \in I : \phi(x) \geq \lambda\})/\mu(I)$ does not depend on I, but only on the "level" n to which I belongs). The latter equality, compared to the definition of ϕ, implies the pointwise bound

$$\mathcal{M}_{\mathcal{T}}\phi \geq \frac{p'}{p'-1}\phi \qquad \mu\text{-almost everywhere on } A_N. \tag{4.3}$$

Furthermore, if $n \in \{0, 1, \ldots, N-1\}$, then

$$\frac{1}{\mu(A_n)} \int_{A_n} \phi \mathrm{d}\mu = f(1+\delta)^n,$$

and, using the same argument as above, we see that the equality still holds true if we replace A_n by any set $I \in F_n$. Consequently, we get

$$\mathcal{M}_{\mathcal{T}}\phi \geq f(1+\delta)^n \qquad \text{on } A_n \setminus A_{n+1},\ n = 0, 1, 2, \ldots, N-1. \tag{4.4}$$

4.2. **Proof of the inequality $\mathfrak{B}_p \geq \mathcal{B}_p$.** We consider two major cases.

Case I: $0 < s \leq (f/F)^{p/(p-1)}$. Then there is $k = k(s,\delta) \geq N-1$ for which $s \in (\mu(A_{k+1}), \mu(A_k)]$: see (4.1). By Lemma 4.1, there is a measurable set E satisfying $A_{k+1} \subset E \subseteq A_k$ and $\mu(E) = s$. If $Ls^{1/p} \leq F$, we use (4.3) to get that

$$\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\} \mathrm{d}\mu \geq \frac{p'}{p'-1} \int_{A_{k+1}} \phi \mathrm{d}\mu = \frac{p'}{p'-1} \left(\frac{1+\delta}{1+p'\delta}\right)^{k-N+1} f.$$

Now we let δ go to 0. It will be convenient to write $A \approx B$ when $\lim_{\delta\to 0} A/B = 1$. Then $1+\delta \approx (1+p'\delta)^{1/p'}$ and $s \approx (f/F)^{p/(p-1)}(1+p'\delta)^{N-k}$, by the definition of k. Thus,

$$\begin{aligned}\int_E \mathcal{M}_{\mathcal{T}}\phi \mathrm{d}\mu &\approx \frac{p'f}{p'-1}(1+p'\delta)^{(N-k)(1-1/p')}\\ &\approx \frac{p'f}{p'-1}s^{1-1/p'}\left(\frac{F}{f}\right)^{p(p'-1)/(p'(p-1))},\end{aligned}$$

which, for p' sufficiently close to p, can be made arbitrarily close to $pFs^{1-1/p}/(p-1) = \mathcal{B}_p(f,F,L,s)$. If $Ls^{1/p} > F$, the calculations are slightly more complicated. By the definition of $k = k(s,\delta)$, we have

$$\begin{aligned}L > Fs^{-1/p} \geq F\mu(A_k)^{-1/p} &= \frac{F^{p/(p-1)}}{f^{1/(p-1)}}(1+p'\delta)^{(k-N+1)/p}\\ &\geq \frac{F^{p/(p-1)}}{f^{1/(p-1)}}(1+\delta)^{k-N+1},\end{aligned}$$

where the last bound holds true for sufficiently small δ. Thus, there is $\ell \geq k$ such that

$$\frac{F^{p/(p-1)}}{f^{1/(p-1)}}(1+\delta)^{\ell-N} \leq L < \frac{F^{p/(p-1)}}{f^{1/(p-1)}}(1+\delta)^{\ell+1-N}.$$

Consequently, by (4.3), we may write

$$\begin{aligned}
&\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \\
&\qquad \geq \int_{A_{k+1}} \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \\
&\qquad \geq \int_{A_{\ell+1}} \mathcal{M}_{\mathcal{T}}\phi\mathrm{d}\mu + L\mu(A_{k+1}\setminus A_{\ell+1}) \\
&\qquad \geq \frac{p'}{p'-1}\int_{A_{\ell+1}} \phi\mathrm{d}\mu + L(\mu(A_{k+1}) - \mu(A_{\ell+1})) \\
&\qquad = \frac{p'}{p'-1}\left(\frac{1+\delta}{1+p'\delta}\right)^{\ell-N+1} f \\
&\qquad\quad + L\cdot\left(\frac{f}{F}\right)^{p/(p-1)}(1+p'\delta)^{N-k-1}\big[1-(1+p'\delta)^{k-\ell}\big].
\end{aligned}$$

Letting $\delta\to 0$, we see that the latter expression converges to

$$\begin{aligned}
&\frac{p'}{p'-1}\left(L\frac{f^{1/(p-1)}}{F^{p/(p-1)}}\right)^{1-p'} f + sL - \frac{f^{(p-p')/(p-1)}F^{p(p'-1)/(p-1)}}{L^{p'-1}} \\
&= \frac{f^{(p-p')/(p-1)}F^{p(p'-1)/(p-1)}}{(p'-1)L^{p'-1}} + sL,
\end{aligned}$$

which can be made arbitrarily close to $F^pL^{1-p}/(p-1)+Ls = \mathcal{B}_p(f,F,L,s)$, by choosing p' sufficiently close to p. This completes the proof of the estimate $\mathfrak{B}_p \geq \mathcal{B}_p$ in the first case.

Case II: $s > (f/F)^{p/(p-1)}$. As previously, we consider the unique $k = k(s,\delta) < N$ for which $s\in(\mu(A_{k+1}),\mu(A_k)]$ and pick a measurable set E with $A_{k+1}\subset E\subseteq A_k$ and $\mu(E)=s$. If $L\leq f/s$, then we exploit

(4.3) and (4.4) to get

$$\begin{aligned}\int_E \max\{\mathcal{M}_\mathcal{T}\phi, L\}\mathrm{d}\mu &\geq \int_{A_{k+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu\\ &= \int_{A_{N+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu + \int_{A_{k+1}\setminus A_{N+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu\\ &\geq \frac{p'}{p'-1}\int_{A_{N+1}} \phi\mathrm{d}\mu + \sum_{n=k+1}^{N}\int_{A_n\setminus A_{n+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu\\ &\geq \frac{p'f}{p'-1} + \sum_{n=k+1}^{N} f(1+\delta)^n\mu(A_n\setminus A_{n+1})\\ &= \frac{p'f}{p'-1} + \sum_{k=n+1}^{N+1} f\big(1-(1+\delta)^{-1}\big)\\ &= \frac{p'f}{p'-1} + \frac{\delta f}{1+\delta}(N-k+1)\\ &\xrightarrow{\delta\to 0} \frac{p'f}{p'-1} + f\ln\left[\left(\frac{F}{f}\right)^{p/(p-1)} s\right],\end{aligned}$$

where in the last passage we have exploited the equality $(1+\delta)^N = (F/f)^{p/(p-1)}$ and the asymptotics $s \approx (1+\delta)^{-k}$. Since p' can be taken arbitrarily close to p, the bound $\mathfrak{B}_p \geq \mathcal{B}_p$ follows. Next, suppose that $f/s < L < F^{p/(p-1)}/f^{1/(p-1)}$. We have $fs^{-1} \geq f\mu(A_k)^{-1} = f(1+\delta)^k$ and $F^{p/(p-1)}/f^{1/(p-1)} = f(1+\delta)^N$, so there is $\ell \in \{k, k+1, \dots, N-1\}$ such that

$$f(1+\delta)^\ell \leq L \leq f(1+\delta)^{\ell+1}.$$

We derive that

$$\begin{aligned}\int_E \max\{\mathcal{M}_\mathcal{T}\phi, L\}\mathrm{d}\mu &\geq \int_{A_{k+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu\\ &= \int_{A_{\ell+1}} \mathcal{M}_\mathcal{T}\phi\mathrm{d}\mu + L\mu(A_{k+1}\setminus A_{\ell+1})\end{aligned} \tag{4.5}$$

By the above chain of inequalities, we have

$$\int_{A_{\ell+1}} \mathcal{M}_{\mathcal{T}}\phi \mathrm{d}\mu \geq \frac{p'f}{p'-1} + \frac{\delta f}{1+\delta}(N-\ell+1)$$

and hence

$$\begin{aligned}\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \geq &\frac{p'f}{p'-1} + \frac{\delta f}{1+\delta}(N-\ell+1) \\ &+ L((1+\delta)^{-k} - (1+\delta)^{-\ell}) \\ \xrightarrow{\delta\to 0}\ & Ls + \frac{f}{p'-1} + f\ln\left[\frac{F^{p/(p-1)}}{Lf^{1/(p-1)}}\right].\end{aligned}$$

It suffices to let $p' \downarrow p$ to see that $\mathfrak{B}_p \geq \mathcal{B}_p$. Finally, we turn to the case $L \geq F^{p/(p-1)}/f^{1/(p-1)}$. Then there is $\ell \geq N$ such that

$$f(1+\delta)^\ell \leq L \leq f(1+\delta)^{\ell+1}.$$

Since $\ell \geq k$, (4.5) holds true. We have already dealt with $\int_{A_{\ell+1}} \mathcal{M}_{\mathcal{T}}\phi \mathrm{d}\mu$ in Case I: this integral is not smaller than

$$\frac{p'}{p'-1}\left(\frac{1+\delta}{1+p'\delta}\right)^{\ell-N+1} f.$$

Consequently,

$$\begin{aligned}&\int_E \max\{\mathcal{M}_{\mathcal{T}}\phi, L\}\mathrm{d}\mu \\ &\geq \frac{p'f}{p'-1}\left(\frac{1+\delta}{1+p'\delta}\right)^{\ell-N+1} + L\Big[(1+\delta)^{-k} - (f/F)^{p/(p-1)}(1+p'\delta)^{N-\ell}\Big] \\ &\xrightarrow{\delta\to 0} \frac{F^{p(p'-1)/(p-1)}f^{(p-p')/(p-1)}}{(p'-1)L^{p'-1}} + Ls,\end{aligned}$$

and since p' is arbitrarily close to p, we obtain the desired bound.

This completes the proof of the lower bound for $\mathfrak{B}_p$ and hence Theorem 2.1 is established.

Remark 4.2. Finally, let us address the problem of optimality of the constant $(p/(p-1))^2$ in (1.5). We take the example ϕ of Subsection 4.1, corresponding to $f = F$ and let $s = 1$, $L = f$. Then $N = 0$ and, as we have proved in Case I of Subsection 4.2, the integral $\int_X \mathcal{M}_{\mathcal{T}}\phi \mathrm{d}\mu$ can be made arbitrarily close to $pF/(p-1)$ (by taking appropriately small δ). On the other hand, by the very definition of ϕ, we have

$$\begin{aligned} ||\phi||_{p,\infty} &= \sup_{n\geq 1}\left[\left(\operatorname*{essinf}_{A_n}\phi\right)^p \mu(A_n)\right]^{1/p} \\ &= \sup_{n\geq 1}\frac{(p'-1)F}{p'}(1+\delta)^{n-1}(1+p'\delta)^{-n/p} = \frac{(p'-1)F}{p'}. \end{aligned}$$

Letting $p' \downarrow p$, we see that the ratio $\int_X \mathcal{M}_{\mathcal{T}}\phi \mathrm{d}\mu/||\phi||_{p,\infty}$ can be made arbitrarily close to $(p/(p-1))^2$. This proves the desired sharpness of the estimate (1.5).

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