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WEAK-TYPE INEQUALITIES FOR MAXIMAL OPERATORS ACTING ON LORENTZ SPACES

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Abstract. We prove sharp a priori estimates for the distribution function of the dyadic maximal function $\mathcal{M}\phi$, when ϕ belongs to the Lorentz space $L^{p,q}$, $1<p<\infty$, $1\leq q<\infty$. The approach rests on a precise evaluation of the Bellman function corresponding to the problem. As an application, we establish refined weak-type estimates for the dyadic maximal operator: for p, q as above and $r\in[1,p]$, we determine the best constant $C_{p,q,r}$ such that for any $\phi\in L^{p,q}$,

$$||\mathcal{M}\phi||_{r,\infty}\leq C_{p,q,r}||\phi||_{p,q}.$$

1. Introduction. The dyadic maximal operator on $\mathbb{R}^n$ is given by the formula

$$\mathcal{M}\phi(x)=\sup\left\{\frac{1}{|Q|}\int_Q|\phi(u)|\mathrm{d}u : x\in Q,\, Q\subset\mathbb{R}^n \text{ is a dyadic cube}\right\},$$

where ϕ is a locally integrable function on $\mathbb{R}^n$ and the dyadic cubes are those formed by the grids $2^{-N}\mathbb{Z}^n$, $N=0,\,1,\,2,\,\ldots$. It is well known that the maximal operator satisfies the weak-type $(1,1)$ inequality

$$|\{x\in\mathbb{R}^n:\mathcal{M}\phi(x)\geq\lambda\}|\leq\frac{1}{\lambda}\int_{\{\mathcal{M}\phi\geq\lambda\}}|\phi(u)|\mathrm{d}u \tag{1}$$

for any $\phi\in L^1(\mathbb{R}^n)$ and any $\lambda>0$. By integration, this leads to the L^p bound

$$||\mathcal{M}\phi||_{L^p(\mathbb{R}^n)}\leq\frac{p}{p-1}||\phi||_{L^p(\mathbb{R}^n)},\qquad 1<p\leq\infty, \tag{2}$$

in which the constant $p/(p-1)$ is the best possible. There is a powerful method which reduces the problem of proving a given inequality for the maximal operator to that of

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deriving of the corresponding special object, the so-called Bellman function. Not only does this technique allow to determine the best constants involved in the estimate under investigation, but it also provides some additional insight into the structure and the behavior of the maximal operator. For example, in order to study (2), Nazarow and Treil [NT] introduced the function

$$\begin{aligned}&\mathcal{B}_p(f,F,L)\\ &\quad=\sup\left\{\frac{1}{|Q|}\int_Q(\mathcal{M}\phi)^p:\frac{1}{|Q|}\int_Q\phi=f,\ \frac{1}{|Q|}\int_Q\phi^p=F,\ \sup_{R:Q\subseteq R}\frac{1}{|R|}\int_R\phi=L\right\},\end{aligned}$$

where Q is a fixed dyadic cube (in fact, one may take $Q=[0,1]^n$), f, F, L satisfy $0\leq f\leq L$, $f^p\leq F$ and the supremum is taken over all nonnegative functions $\phi\in L^p(Q)$ and all dyadic cubes R containing Q. Furthermore, Nazarov and Treil established the so-called "main inequality", which codifies the martingale-like dynamics of the function. Unfortunately, they did not find the function explicitly, settling instead for what they called a *supersolution*. It was Melas [M1] who discovered the explicit formula for $\mathcal{B}_p$, actually in a slightly more general setting of trees (see below):

$$\mathcal{B}_p(f,F,L)=\begin{cases}Fw_p\left(\frac{pL^{p-1}f-(p-1)L^p}{F}\right)^p & \text{if } L<\frac{p}{p-1}f,\\ L^p+\left(\frac{p}{p-1}\right)^p(F-f^p) & \text{if } L\geq\frac{p}{p-1}f,\end{cases}\tag{3}$$

where $w_p:[0,1]\to[1,p/(p-1)]$ is the inverse function of $z\mapsto-(p-1)z^p+pz^{p-1}$. Since $\mathcal{B}_p(f,F,L)\leq(p/(p-1))^pF$ and the equality can be attained, (2) holds true and the constant $p/(p-1)$ cannot be decreased. However, it is clear that Melas' result is a significant improvement of (2): the formula (3) brings much more information about the action of $\mathcal{M}$ on L^p.

A few remarks concerning the proof of the equality (3) are in order. Melas' approach is combinatorial in nature; the key step is to narrow down the class of functions among which the optimizers of the underlying extremal problem are found. Roughly speaking, in this line of reasoning one finds the Bellman function as the appropriate integral of the optimizer. This approach does not use the martingale dynamics of the problem and is specific to the discrete maximal operator. In particular, it does not directly apply to other dyadic operators, nor does it seem to work for other maximal functions. This technique should be contrasted with a relatively general PDE- and geometry-based method first used by Slavin, Stokolos and Vasyunin [SSV]. There, the "main inequality" of Nazarov and Treil was turned into a Monge-Ampère PDE on a plane domain, whose solution turned out to be Melas' function. The optimizers were then built along the straight-line characteristics of the PDE. Monge-Ampère equations are now found in many Bellman applications. Typically, they arise in settings with integral norms, such as L^p; and those where the main inequality can be interpreted as a convexity/concavity statement. However, one can also get a differential equation in other cases, as long as the main inequality is infinitesimally non-trivial. This approach has its roots in the work of Burkholder [B].

There are several other problems of this type which were successfully treated by the above methods (see e.g. [M2], [M3], [MN], [MN2] and [N]). We shall only mention here two of them, which are closely related to the results obtained in this paper. First, Melas and

Nikolidakis [MN] studied various extensions of the weak-type estimate (1) for $1 \leq p < \infty$, and, in particular, derived the explicit formula for the corresponding Bellman function

$$\mathcal{B}(f, F, L) = \sup \left\{ \frac{1}{|Q|} ||\mathcal{M}\phi||^p_{L^{p,\infty}(Q)} \right\}.$$

Here the supremum is taken over the same parameters as previously and, as usual, $||\mathcal{M}\phi||_{L^{p,\infty}(Q)} = \sup\{\lambda |\{x \in Q : \mathcal{M}\phi(x) \geq \lambda\}|^{1/p} : \lambda > 0\}$ denotes the weak p-th norm of $\mathcal{M}\phi$ restricted to Q. The second result is that of Nikolidakis [N], who established the sharp estimate

$$||\mathcal{M}\phi||_{L^{p,\infty}(\mathbb{R}^n)} \leq \frac{p}{p-1} ||\phi||_{L^{p,\infty}(\mathbb{R}^n)}, \qquad 1 < p < \infty.$$

This was accomplished by deriving the formula for

$$\mathcal{B}_p(f, F) = \sup \left\{ \frac{1}{|Q|} ||\mathcal{M}\phi||_{L^{p,\infty}(Q)} : \frac{1}{|Q|} \int_Q \phi = f,\ \frac{1}{|Q|} ||\phi||_{L^{p,\infty}(Q)} = F \right\},$$

with f, F satisfying $f \leq \frac{p}{p-1} F$ and the supremum taken over all $\phi \geq 0$.

The objective of this paper is to generalize and unify the above two statements by comparing the $L^{r,\infty}$-norm of $\mathcal{M}\phi$ to the Lorentz $L^{p,q}$-norm of ϕ, where $1 < p < \infty$, $1 \leq q \leq \infty$ and $1 \leq r \leq p$. Actually, we shall work in a more general setting and investigate the maximal operator acting on a nonatomic probability space (X, μ) equipped with a tree structure $\mathcal{T}$ (see Definition 2.1 below).

2. Preliminaries. Let (X, μ) be a nonatomic probability space. Two measurable subsets A, B of X will be called *almost disjoint* if $\mu(A \cap B) = 0$. We start with the following

Definition 2.1. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

- (i) $X \in \mathcal{T}$ and for every $I \in \mathcal{T}$ we have $\mu(I) > 0$.
- (ii) For every $I \in \mathcal{T}$ there is a subset $\mathcal{C}(I) \subset \mathcal{T}$ containing at least two elements such that
  - (a) the elements of $\mathcal{C}(I)$ are pairwise almost disjoint,
  - (b) $I = \bigcup \mathcal{C}(I)$.
- (iii) $\mathcal{T} = \bigcup_{m \geq 0} \mathcal{T}_{(m)}$, where $\mathcal{T}_0 = \{X\}$ and $\mathcal{T}_{(m+1)} = \bigcup_{I \in \mathcal{T}_{(m)}} \mathcal{C}(I)$.
- (iv) $\lim_{m \to \infty} \sup_{I \in \mathcal{T}_{(m)}} \mu(I) = 0$.

The elements of a tree $\mathcal{T}$ have similar behavior to that of the dyadic cubes; for example, if the intersection of two elements of $\mathcal{T}$ has positive measure, then one is contained in the other. For more details and for the proof of the following statement, see [M1].

Lemma 2.2. *Let (X, μ) be a nonatomic probability space equipped with a tree $\mathcal{T}$. For every $I \in \mathcal{T}$ and every $\alpha \in [0, 1]$ there exists a subfamily $F(I) \subset \mathcal{T}$ consisting of pairwise almost disjoint subsets of I such that*

$$\mu\left(\bigcup F(I)\right) = \sum_{J \in F(I)} \mu(J) = \alpha \mu(I).$$

In other words, the elements of a tree can form a set of an arbitrary measure. In fact, as we shall prove now, they can also be used to build functions of an arbitrary distribution. Let ϕ be a nonnegative and measurable function on X. Recall that ϕ^*, the decreasing rearrangement of ϕ, is given by the formula

$$\phi^*(t) = \inf\{s > 0 : \mu(\phi > s) \leq t\}$$

$\big($here and below, we use the notation $\mu(\phi > s) = \mu(\{x \in X : \phi(x) > s\})\big)$.

LEMMA 2.3. *Let (X, μ) be a nonatomic probability space equipped with a tree $\mathcal{T}$. For any nondecreasing, right-continuous function $g : (0, 1] \to [0, \infty)$ there is a function $\phi : X \to [0, \infty)$, measurable with respect to the σ-algebra generated by $\mathcal{T}$, satisfying $\phi^* = g$ on $(0, 1]$.*

Proof. We construct inductively an appropriate sequence $(\phi_n)_{n\geq 0}$ of step functions on X. We start with $\phi_0 = g(1)1_X$. For $n \geq 0$, define

$$\phi_{n+1} = \sum_{k=1}^{2^{n+1}} g(k2^{-n-1})1_{A_{k,n+1}},$$

where $\{A_{k,n+1}\}_{k=1}^{2^{n+1}}$ is a family of pairwise almost disjoint elements of $\mathcal{T}$ such that $A_{k,n} = A_{2k-1,n+1} \cup A_{2k,n+1}$ and $\mu(A_{2k-1,n+1}) = \mu(A_{2k,n+1}) = \mu(A_{k,n})/2$ for each k. The existence of such a family is guaranteed by the previous lemma. Directly from the construction, we see that

$$\phi_n^* = \sum_{k=1}^{2^n} g(k2^{-n})1_{[(k-1)2^{-n},k2^{-n})}.$$

Furthermore, the sequence $(\phi_n)_{n\geq 0}$ is nondecreasing and hence the pointwise limit $\phi = \lim_{n\to\infty} \phi_n$ exists. It remains to note that ϕ is $\sigma(\mathcal{T})$-measurable and

$$\phi^* = \lim_{n\to\infty} \phi_n^* = g \qquad \text{on } (0, 1],$$

which completes the proof. ∎

We conclude this section with the definition of the maximal operator associated to the tree $\mathcal{T}$.

DEFINITION 2.4. Let (X, μ) be a probability space equipped with a tree structure $\mathcal{T}$. We define the maximal operator $\mathcal{M}_\mathcal{T}$ acting on integrable functions $\phi : X \to [0, \infty)$ by the formula

$$\mathcal{M}_\mathcal{T}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I \phi\,\mathrm{d}\mu : x \in I \in \mathcal{T}\right\}.$$

3. A Bellman function. Throughout the paper, p, q are fixed numbers satisfying $1 < p < \infty$ and $1 \leq q < \infty$. The key role of the paper is played by the function

$$\mathbb{B}_{p,q}(\lambda, f, F) = \sup\left\{\mu(\mathcal{M}_\mathcal{T}\phi \geq \lambda) : \int_X \phi\,\mathrm{d}\mu = f,\ ||\phi||_{p,q} = F\right\}, \tag{4}$$

where the supremum is taken over nonnegative and integrable functions ϕ on X and $||\phi||_{p,q}$ denotes the Lorentz norm of ϕ, given by

$$||\phi||_{p,q} = p^{1/q}\left(\int_0^\infty \mu(\phi \geq s)^{q/p}s^{q-1}\mathrm{d}s\right)^{1/q}.$$

Our main goal will be to find the explicit formula for $\mathbb{B}_{p,q}$. Before we provide the precise statement, let us determine the domain of this function: for which f and F the supremum above is taken over an nonempty set? To answer this question, introduce the constant $C_{p,q}$, given by

$$C_{p,q} = \begin{cases} \left(\frac{q}{p}\right)^{1/q} & \text{if } q \le p, \\ \left(\frac{p(q-1)}{q(p-1)}\right)^{1-1/q} & \text{if } q > p. \end{cases}$$

THEOREM 3.1. *Suppose that f, $F > 0$. There is a function $\phi : X \to [0, \infty)$ satisfying $\int_X \phi d\mu = f$ and $||\phi||_{p,q} = F$ if and only if*

$$f \le C_{p,q} F. \tag{5}$$

Proof. By homogeneity, it suffices to prove the equivalence for $f = 1$. We consider the cases $q \le p$ and $q > p$ separately.

The case $q \le p$. Suppose that ϕ satisfies $\int_X \phi = 1$ and $||\phi||_{p,q} = F$. We have

$$\mu(\phi \ge s) \le \mu(\phi \ge s)^{q/p} s^{q-1} \qquad \text{for } s \ge 1$$

and

$$\mu(\phi \ge s) \le \mu(\phi \ge s)^{q/p} s^{q-1} + 1 - s^{q-1} \qquad \text{for } 0 < s < 1$$

(to see the latter bound, simply check it for $\mu(\phi > s) \in \{0, 1\}$ and observe that the function $t \mapsto t^{q/p} s^{q-1} + 1 - s^{q-1} - t$ is concave on $[0, 1]$). Consequently,

$$1 = \int_X \phi \, \mathrm{d}\mu = \int_0^\infty \mu(\phi \ge s) \mathrm{d}s \le \frac{1}{p} ||\phi||^q_{L^{p,q}(X)} + 1 - \frac{1}{q},$$

which is equivalent to (5). To see the reverse implication, consider the family $\{\phi_\alpha\}_{\alpha \in (0,1]}$ of functions on X, with the distribution determined by

$$\phi^*_\alpha(s) = \begin{cases} c_\alpha \alpha^{-1/p} & \text{if } s < \alpha, \\ c_\alpha s^{-1/p} & \text{if } s \in [\alpha, 1], \end{cases}$$

where $c_\alpha = (p-1)/(p - \alpha^{1-1/p})$. Such functions exist due to Lemma 2.3. It is easy to verify that $\int_X \phi_\alpha = 1$ and

$$||\phi_\alpha||^q_{p,q} = \left(\frac{p-1}{p - \alpha^{1-1/p}}\right)^q \left(\frac{p}{q} - \log \alpha\right)$$

for every $\alpha \in (0, 1]$. It remains to note that the expression on the right is equal to p/q when $\alpha = 1$, and tends to infinity as $\alpha \downarrow 0$.

The case $q > p$. Suppose first that ϕ satisfies $\int_X \phi = 1$ and $||\phi||_{p,q} = F$. Let $s_0 = \frac{q(p-1)}{p(q-1)}$. By Young's inequality, we have, for $s \ge s_0$,

$$\mu(\phi \ge s) \le \frac{p}{q} \left(\frac{s}{s_0}\right)^{q-1} \mu(\phi \ge s)^{q/p} + \left(1 - \frac{p}{q}\right) \left(\frac{s_0}{s}\right)^{(q-1)p/(q-p)}. \tag{6}$$

On the other hand, if $0 < s < s_0$, then

$$\mu(\phi \ge s) \le \frac{p}{q} \left(\frac{s}{s_0}\right)^{q-1} \mu(\phi \ge s)^{q/p} + 1 - \frac{p}{q} \left(\frac{s}{s_0}\right)^{q-1}. \tag{7}$$

Indeed, $\mu(\phi \geq s) \in [0,1]$, both sides above are equal for $\mu(\phi \geq s) = 1$, and the function $x \mapsto x - \frac{p}{q}\left(\frac{s}{s_0}\right)^{q-1} x^{q/p}$ is increasing on $[0,1]$. Combining (6) and (7) yields

$$\begin{aligned} 1 &= \int_0^\infty \mu(\phi \geq s)\mathrm{d}s \\ &\leq \frac{p}{qs_0^{q-1}} \int_0^\infty \mu(\phi \geq s)^{q/p} s^{q-1}\mathrm{d}s + \left(\frac{q-p}{q}\right)^2 \frac{s_0}{p-1} + s_0 - \frac{ps_0}{q^2}, \end{aligned}$$

which is (5), after a little computation. To get the reverse implication, consider the family $\{\phi_\alpha\}_{\alpha > -p^{-1}}$ of functions on X such that

$$\phi_\alpha^*(t) = (\alpha+1)t^\alpha, \qquad t \in [0,1],\ \alpha > -p^{-1}.$$

The existence of such objects follows from Lemma 2.3. We easily check that for any $\alpha > p^{-1}$ the function ϕ_α has integral 1 and

$$||\phi_\alpha||_{p,q}^q = \int_0^1 t^{q/p-1}(\phi_\alpha^*(t))^q \mathrm{d}t = \frac{p}{q(p\alpha+1)}(\alpha+1)^q.$$

A straightforward analysis shows that the minimum of the right-hand side is precisely $C_{p,q}^{-q}$ and the supremum is equal to ∞. The proof is complete. ∎

We turn to the explicit formula for $\mathbb{B}_{p,q}(\lambda, f, F)$. Here is the main result of this paper.

THEOREM 3.2. *For any $\lambda > 0$ and $f \leq C_{p,q}F$ we have*

$$\mathbb{B}_{p,q}(\lambda, f, F) = \begin{cases} 1 & \text{if } \lambda \leq f, \\ f\lambda^{-1} & \text{if } \lambda > f,\ F \geq C_{p,q}^{-1}\lambda^{1-1/p}f^{1/p}, \\ G_{p,q,\lambda/f}(F^q/f^q) & \text{if } \lambda > f,\ F < C_{p,q}^{-1}\lambda^{1-1/p}f^{1/p}. \end{cases} \tag{8}$$

Here, for $\lambda > 1$, the function $G_{p,q,\lambda} : [C_{p,q}^{-q}, C_{p,q}^{-q}\lambda^{q-q/p}] \to \mathbb{R}$ is the inverse to $D_{p,q,\lambda}$, given as follows: if $q \leq p$, then

$$D_{p,q,\lambda}(t) = C_{p,q}^{-q}\left[\left(\frac{1-\lambda t}{1-t}\right)^q (1-t^{q/p}) + t^{q/p}\lambda^q\right], \qquad t \in [0, \lambda^{-1}],$$

while for $q > p$,

$$D_{p,q,\lambda}(t) = C_{p,q}^{-q}\left[\frac{(1-\lambda t)^q}{\left(1 - t^{q(p-1)/(p(q-1))}\right)^{q-1}} + t^{q/p}\lambda^q\right], \quad t \in [\lambda^{p(q-1)/(p-q)}, \lambda^{-1}].$$

This result will be proved in the subsequent sections.

4. Derivation of $\mathbb{B}_{p,q}(\lambda, f, F)$ for small λ. This section contains some initial analysis of $\mathbb{B}_{p,q}$ as well as the proof of (8) on the sets $\{\lambda \leq f\}$ and $\{\lambda > f,\ F \geq C_{p,q}^{-1}\lambda^{1-1/p}f^{1/p}\}$. We start with the observation that $\mathbb{B}_{p,q}$ satisfies the homogeneity property

$$\mathbb{B}_{p,q}(c\lambda, cf, cF) = \mathbb{B}_{p,q}(\lambda, f, F) \quad \text{for all } c > 0,$$

which is clear from the very definition. Therefore, it suffices to focus of finding the formula for

$$B_{p,q}(\lambda, F) := \mathbb{B}_{p,q}(\lambda, 1, F).$$

Next, any nonnegative function ϕ on X with integral 1 satisfies $\mathcal{M}\phi(x) \geq 1$ for all $x \in X$. This yields $B_{p,q}(\lambda, F) = 1$ if $\lambda \leq 1$, and gives (8) for $\lambda \leq f$.

Thus, from now on, we assume that $\lambda > 1$. The key fact in our further considerations is the alternative definition of $B_{p,q}$, which does not involve the maximal operator. To formulate this definition, we require some additional notation. For a fixed $F \geq C_{p,q}^{-1}$, consider the class $\mathcal{K}(F)$ which consists of those nonincreasing and right-continuous functions $\psi : (0, 1] \to [0, \infty)$, which satisfy $\int_0^1 \psi = 1$ and $||\psi||_{p,q} \leq F$. The class is nonempty, which follows directly from Theorem 3.1. For any $\psi \in \mathcal{K}(F)$, let

$$t = t(\psi) := \sup \left\{ r \in [0, 1] : \int_0^r \psi(u)\mathrm{d}u \geq \lambda r \right\}$$

and define

$$s_0 = s_0(\psi) = \begin{cases} \lim_{r\uparrow t} \psi(r) & \text{if } t(\psi) > 0, \\ +\infty & \text{if } t(\psi) = 0. \end{cases}$$

Clearly, we have $\int_0^{t(\psi)} \psi = \lambda t(\psi)$ and hence $t(\psi) \leq \lambda^{-1}$.

Theorem 4.1. *For any $\lambda > 1$ and $F \geq C_{p,q}^{-1}$ we have*

$$B_{p,q}(\lambda, F) = \sup \{t(\psi) : \psi \in \mathcal{K}(F)\}. \tag{9}$$

Proof. The inequality "$\leq$" is straightforward: for any ϕ as in the definition of $B_{p,q}(\lambda, F)$ we have $\phi^* \in \mathcal{K}(F)$ and $\mu(\mathcal{M}_\mathcal{T}\phi \geq \lambda) \leq t(\phi^*)$. Indeed, the latter estimate is due to Doob's weak-type bound:

$$\lambda\mu(\mathcal{M}_\mathcal{T}\phi \geq \lambda) \leq \int_{\{\mathcal{M}_\mathcal{T}\phi\geq\lambda\}} \phi \leq \int_0^{\mu(\mathcal{M}_\mathcal{T}\phi\geq\lambda)} \phi^*,$$

combined with the definition of $t(\phi^*)$. To prove the reverse estimate, pick $\bar{\psi} \in \mathcal{K}(F)$ with $t(\bar{\psi}) > 0$ and let $\varepsilon > 0$. It is easy to modify $\bar{\psi}$ on the interval $[0, t(\bar{\psi})]$ so that its $L^{p,q}$-norm increases to F; more precisely, the new, modified function ψ satisfies $\int_0^1 \psi = 1$, $||\psi||_{p,q} = F$ and $t(\psi) > t(\bar{\psi}) - \varepsilon$. Using Lemma 2.2, we construct a family $\{A_j\}_{j\in J} \subset \mathcal{T}$ of pairwise almost disjoint sets such that $\mu(\bigcup_{j\in J} A_j) = t(\psi)$. We have $\int_0^{t(\psi)} \psi = \lambda t(\psi)$, so using a straightforward induction argument, we find a family $\{B_j\}_{j\in J}$ of pairwise almost disjoint Borel subsets of $[0, t(\psi)]$ such that $|B_j| = \mu(A_j)$ and $\int_{B_j} \psi = \lambda|B_j|$ for each $j \in J$. By Lemma 2.3 (applied on each A_j, $j \in J$, and on $X \setminus \bigcup_{j\in J} A_j$) there is a function $\phi : X \to [0, \infty)$ with $\phi^* = \psi$ such that for each $j \in J$, $\phi|_{A_j}$ has the same distribution as $\psi|_{B_j}$, and $\phi|_{X\setminus\bigcup_{j\in J} A_j}$ has the same distribution as $\psi|_{[0,1]\setminus\bigcup_{j\in J} B_j}$. Thus ϕ satisfies the conditions listed in the definition of $B_{p,q}(\lambda, F)$ and for each $x \in A_j$,

$$\mathcal{M}_\mathcal{T}\phi(x) \geq \frac{1}{\mu(A_j)} \int_{A_j} \phi = \frac{1}{|B_j|} \int_{B_j} \psi = \lambda.$$

Therefore $\mu(\mathcal{M}_\mathcal{T}\phi \geq \lambda) \geq \mu(\bigcup_{j\in J} A_j) = t(\psi) > t(\bar{\psi}) - \varepsilon$. This finishes the proof, since $\bar{\psi} \in \mathcal{K}(F)$ and $\varepsilon > 0$ were arbitrary. ∎

We are ready to prove the validity of (8) on the second part of the domain of $\mathbb{B}_{p,q}$. This follows from the statement below.

Theorem 4.2. *If $\lambda > 1$ and $F \geq C_{p,q}^{-1}\lambda^{1-1/p}$, then $B_{p,q}(\lambda, F) = \lambda^{-1}$.*

Proof. We have $t(\psi) \leq \lambda^{-1}$ for any $\psi \in \mathcal{K}(F)$ (see the sentence right below the definitions of $t(\psi)$ and $s_0(\psi)$) and thus $B_{p,q}(\lambda, F) \leq \lambda^{-1}$, by virtue of (9). To get the inequality in the reverse direction, let $\phi : X \to [0,\infty)$ be a function of integral 1, satisfying $||\phi||_{p,q} = C_{p,q}^{-1}$. The existence of such an object is guaranteed by Lemma 2.3 and Theorem 3.1. Let $\psi : (0,1] \to [0,\infty)$ be given by

$$\psi(r) = \begin{cases} \lambda\phi^*(\lambda r) & \text{if } 0 < r \leq \lambda^{-1}, \\ 0 & \text{if } \lambda^{-1} < r \leq 1. \end{cases}$$

We see that $\int_0^1 \psi = 1$ and $|\psi \geq s| = \lambda^{-1}|\phi^* \geq s\lambda^{-1}|$ for any $s \geq 0$, so

$$\begin{aligned} ||\psi||_{p,q} &= p^{1/q} \left(\int_0^\infty |\psi \geq s|^{q/p} s^{q-1} \mathrm{d}s \right)^{1/q} \\ &= p^{1/q} \left(\int_0^\infty \lambda^{q-q/p} |\phi^* \geq \lambda|^{q/p} s^{q-1} \mathrm{d}s \right)^{1/q} \\ &= \lambda^{1-1/p} ||\phi||_{p,q} \leq F \end{aligned}$$

and hence ψ belongs to the class $\mathcal{K}(F)$. It remains to note that

$$\int_0^{\lambda^{-1}} \psi(u)\mathrm{d}u = \int_0^1 \phi^*(u)\mathrm{d}u = 1,$$

which implies $t(\psi) \geq \lambda^{-1}$. The use of Theorem 4.1 completes the proof. ∎

Therefore, all that is left is to determine the function $B_{p,q}$ on the set $\{(\lambda, F) : C_{p,q}^{-1} \leq F < C_{p,q}\lambda^{1-1/p}\}$. This task is much more elaborate, the details will be presented in the next two sections.

5. The case $q \leq p$. The idea is to exhibit extremals of (9), i.e., to narrow the class of functions over which we take the supremum. Let ψ be an arbitrary function from $\mathcal{K}(F)$ and put $t = t(\psi)$, $s_0 = s_0(\psi)$. Let

$$\rho = \frac{1 - \lambda t}{1 - t}$$

be the average of ψ over the interval $[t, 1]$; clearly, we have $0 < \rho \leq s_0 \leq \lambda$. Introduce $\varphi = \varphi_{\lambda,t} : [0,1] \to \mathbb{R}$ by the formula

$$\varphi_{\lambda,t}(r) = \begin{cases} \lambda & \text{if } 0 \leq r \leq t, \\ \rho & \text{if } t < r \leq 1. \end{cases} \tag{10}$$

It is easy to see that $\int_0^t \psi = \int_0^t \varphi_{\lambda,t}$, $\int_t^1 \psi = \int_t^1 \varphi_{\lambda,t}$ and hence

$$\int_0^{s_0} |\psi \geq s|\mathrm{d}s = \int_t^1 \psi(u)\mathrm{d}u + s_0 t = \int_t^1 \varphi_{\lambda,t}(u)\mathrm{d}u + s_0 t = \int_0^{s_0} |\varphi_{\lambda,t} \geq s|\mathrm{d}s \tag{11}$$

and, similarly,

$$\int_{s_0}^\infty |\psi \geq s|\mathrm{d}s = \int_{s_0}^\infty |\varphi_{\lambda,t} \geq s|\mathrm{d}s. \tag{12}$$

We have the following fact.

LEMMA 5.1. *We have $t(\psi) = t(\varphi_{\lambda,t(\psi)})$ and $\varphi_{\lambda,t(\psi)} \in \mathcal{K}(||\psi||_{p,q})$.*

Proof. The first equality follows directly from the definition of the parameter t. To prove the inclusion, let

$$\alpha = \frac{1-t}{\rho^{q-1}(1-t^{q/p})}$$

and write down the estimates

$$\begin{aligned}
&|\psi \geq s| \leq t^{1-q/p}\lambda^{1-q}|\psi \geq s|^{q/p}s^{q-1} && \text{if } s \geq \lambda,\\
&|\psi \geq s| \leq t^{1-q/p}\lambda^{1-q}|\psi \geq s|^{q/p}s^{q-1} + t - t\lambda^{1-q}s^{q-1} && \text{if } s_0 < s < \lambda,\\
&|\psi \geq s| \leq \alpha|\psi \geq s|^{q/p}s^{q-1} + t - \alpha t^{q/p}s^{q-1} && \text{if } \rho < s \leq s_0,\\
&|\psi \geq s| \leq \alpha|\psi \geq s|^{q/p}s^{q-1} + 1 - \alpha s^{q-1} && \text{if } 0 \leq s \leq \rho.
\end{aligned}$$

We shall only prove the first bound, the remaining ones can be established in a similar manner. Since $s \geq \lambda$, we have $s^{q-1}\lambda^{1-q} \geq 1$ and it suffices to prove that $|\psi \geq s| \leq t^{1-q/p}|\psi \geq s|^{q/p}$, or $|\psi \geq s| \leq t$; however, this follows from the definition of $t(\psi)$, since

$$\int_0^{|\psi\geq s|}\psi \leq \int_0^{|\psi\geq\lambda|}\psi \geq \lambda|\psi \geq \lambda|.$$

Furthermore, if in the system of the four estimates above we replace ψ by $\varphi_{\lambda,t}$, the bounds turn into equalities. Consequently, using (11), (12) and integrating the above four inequalities, we see that

$$\int_0^{s_0}|\psi \geq s|^{q/p}s^{q-1}\mathrm{d}s \geq \int_0^{s_0}|\varphi_{\lambda,t} \geq s|^{q/p}s^{q-1}\mathrm{d}s$$

and

$$\int_{s_0}^{\infty}|\psi \geq s|^{q/p}s^{q-1}\mathrm{d}s \geq \int_{s_0}^{\infty}|\varphi_{\lambda,t} \geq s|^{q/p}s^{q-1}\mathrm{d}s.$$

Therefore $||\varphi_{\lambda,t}||_{p,q} \leq ||\psi||_{p,q}$ and the claim follows. ∎

The above lemma implies that in the formula on the right-hand side of (9) we may restrict ourselves to the functions of the form (10). We derive that

$$||\varphi_{\lambda,t}||_{p,q}^q = D_{p,q,\lambda}(t) = \frac{p}{q}\left[\left(\frac{1-\lambda t}{1-t}\right)^q(1-t^{q/p}) + t^{q/p}\lambda^q\right] \tag{13}$$

is the function we have already introduced in the statement of Theorem 3.2. We shall require the following property of this object.

Lemma 5.2. *For a fixed $\lambda > 1$, the function $D_{p,q,\lambda}$ is strictly increasing on* $[0,\lambda^{-1}]$.

Proof. First we shall show that the function is nondecreasing. If we calculate the derivative, we see that it suffices to prove that

$$q\left(\frac{1-\lambda t}{1-t}\right)^{q-1}\cdot\frac{(1-\lambda)(1-t^{q/p})}{(1-t)^2} - \frac{q}{p}\left(\frac{1-\lambda t}{1-t}\right)^q t^{(q-p)/p} + \frac{q}{p}t^{(q-p)/p}\lambda^q \geq 0$$

for $t \in (0,\lambda^{-1})$. Divide throughout by $qp^{-1}t^{(q-p)/p}((1-\lambda t)/(1-t))^q$ to obtain the equivalent estimate

$$\left(\frac{\lambda-\lambda t}{1-\lambda t}\right)^q \geq 1 + p\frac{(\lambda-1)(1-t^{q/p})}{(1-t)(1-\lambda t)t^{(q-p)/p}}. \tag{14}$$

However, by the mean-value theorem we have

$$\left(\frac{\lambda-\lambda t}{1-\lambda t}\right)^q=\left(1+\frac{\lambda-1}{1-\lambda t}\right)^q\geq 1+q\frac{\lambda-1}{1-\lambda t} \tag{15}$$

and we will be done if we show that

$$q\geq\frac{p(1-t^{q/p})}{(1-t)t^{(q-p)/p}}.$$

Using the substitution $x=q/p\in(0,1]$, the latter estimate transforms into

$$xt^{x-1}+(1-x)t^x\geq 1. \tag{16}$$

As a function of $t\in(0,1)$, the left-hand side is strictly decreasing (the derivative is $x(x-1)(1-t)t^{x-2}<0$) and tends to 1 as $t\uparrow 1$. This proves (16) and hence the function $D_{p,q,\lambda}$ is nondecreasing. To get the strict monotonicity, simply note that (15) is strict for $q\neq 1$, and (16) is strict for $x\neq 1$ (i.e., for $p\neq q$). ∎

Thus, the function $D_{p,q,\lambda}:[0,\lambda^{-1}]\to[C_{p,q}^{-q},C_{p,q}^{-q}\lambda^{q-q/p}]$ is invertible and the inverse $G_{p,q,\lambda}:[\frac{p}{q},\frac{p}{q}\lambda^{q-q/p}]\to[0,\lambda^{-1}]$ is also strictly increasing. We are ready to complete the proof of Theorem 3.2 in the case $q\leq p$.

THEOREM 5.3. *Suppose that $q\leq p$ and $C_{p,q}^{-1}\leq F<C_{p,q}^{-1}\lambda^{1-1/p}$. Then*

$$B_{p,q}(\lambda,F)=G_{p,q,\lambda}(F^q).$$

Proof. It suffices to gather all the facts proved in this section. We have

$$\begin{aligned}B_{p,q}(\lambda,F)&=\sup\left\{t(\psi):\psi\in\mathcal{K}(F)\right\}\\&=\sup\left\{t\in[0,\lambda^{-1}]:||\varphi_{\lambda,t}||_{p,q}\leq F\right\}\\&=\sup\left\{t\in[0,\lambda^{-1}]:D_{p,q,\lambda}(\lambda,t)\leq F^q\right\}\\&=\sup\left\{t\in[0,\lambda^{-1}]:t\leq G_{p,q,\lambda}(F^q)\right\}=G_{p,q,\lambda}(F^q).\end{aligned}$$

This finishes the proof. ∎

As an application, we shall establish the following sharp weak-type estimate for the maximal operator.

THEOREM 5.4. *Assume that $1<p<\infty$ and let q, r be two numbers from the interval $[1,p]$. Then for any $\phi:X\to[0,\infty)$ we have*

$$||\mathcal{M}_{\mathcal{T}}\phi||_{r,\infty}\leq C_{p,q}||\phi||_{p,q} \tag{17}$$

and the constant is the best possible.

Proof. In the proof of (17) we may assume that $\int_X\phi=1$, due to the homogeneity. Then the bound amounts to saying that $\lambda B_{p,q}(\lambda,F)^{1/r}\leq C_{p,q}F$ for any $\lambda>0$ and $F\geq C_{p,q}^{-1}$. The left-hand side, as a function of λ, is nondecreasing on $[0,(C_{p,q}F)^{p/(p-1)}]$; thus it suffices to prove the estimate for $\lambda\geq(C_{p,q}F)^{p/(p-1)}$. For these λ's, the inequality reads

$$G_{p,q,\lambda}(F^q)\leq\left(\frac{C_{p,q}F}{\lambda}\right)^r. \tag{18}$$

If $(C_{p,q}F/\lambda)^r\geq\lambda^{-1}$, the bound holds true, since $\sup G_{p,q,\lambda}=\lambda^{-1}$. If we have the reverse estimate $(C_{p,q}F/\lambda)^r<\lambda^{-1}$, then (18) can be rewritten in the form $F^q\leq$

$D_{p,q,\lambda}\left((C_{p,q}F/\lambda)^r\right)$, or

$$x^{q/r} \leq \left(\frac{\lambda^{-1}-x}{1-x}\right)^q (1-x^{q/p}) + x^{q/p},$$

where $x = (C_{p,q}F/\lambda)^r$. Since $x < \lambda^{-1} \leq 1$ and $r \leq p$, we have $x^{q/r} \leq x^{q/p}$ and the desired estimate is valid. To see that $C_{p,q}$ in (17) is the best possible, note that by Theorem 3.1, there is a function ϕ on X satisfying $||\phi||_{p,q} = C_{p,q}^{-1}$ and $\int_X \phi = 1$. The latter equality implies $\mu(\mathcal{M}_\mathcal{T}\phi \geq 1) = 1$ and thus

$$||\mathcal{M}_\mathcal{T}\phi||_{r,\infty} \geq 1 = C_{p,q}||\phi||_{p,q},$$

and we are done. ∎

6. The case $q > p$. The underlying concept is similar to that from the previous section: we narrow the class of functions ψ over which the supremum in (9) has to be taken. However, here the calculations are much more involved and, for the sake of convenience, we have decided to split the optimization procedure into several intermediate steps.

Step 1. Let $\lambda > 1$, $\psi \in \mathcal{K}(F)$ be given and put $t = t(\psi)$, $s_0 = s_0(\psi)$. Introduce a nonincreasing, right-continuous $\varphi : (0, 1] \to \mathbb{R}$ with the distribution given by

$$|\varphi \geq s| = \begin{cases} 1 & \text{if } s \leq s_1, \\ \max\{\beta s^{-p(q-1)/(q-p)}, t\} & \text{if } s_1 < s \leq s_0, \\ \min\{\alpha s^{-p(q-1)/(q-p)}, t\} & \text{if } s > s_0, \end{cases} \tag{19}$$

where α, β and s_1 are uniquely determined by the equations

$$\int_0^t \varphi = \lambda t, \qquad \int_t^1 \varphi = 1 - \lambda t \quad \text{ and } \quad \lim_{s \to s_1} |\varphi \geq s| = 1. \tag{20}$$

See Figure 1 below which illustrates four possible types of the graph of φ.

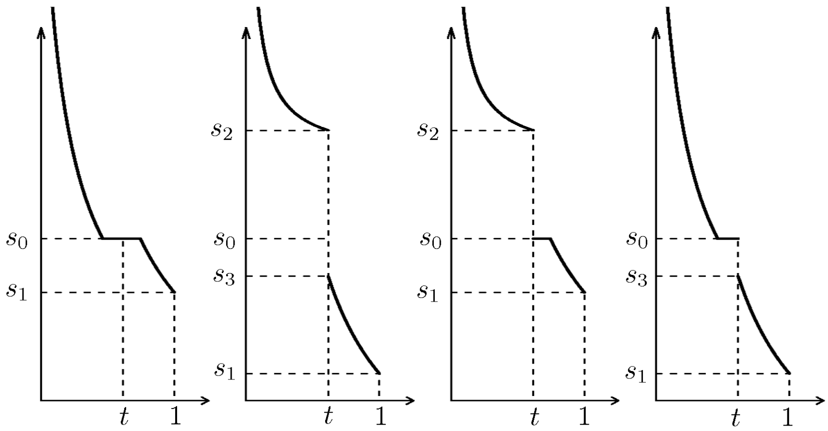

Fig. 1. Four possible types of graphs of the function φ. For example, the first picture corresponds to the case in which $\beta s^{-p(q-1)/(q-p)} \geq t$ for all $s \in [s_1, s_0]$ and $\alpha s^{-p(q-1)/(q-p)} \leq t$ for all $s > s_0$.

Here is the analogue of Lemma 5.1.

LEMMA 6.1. *We have $t(\psi) \leq t(\varphi)$ and $\varphi \in \mathcal{K}(||\psi||_{p,q})$.*

Proof. The inequality follows from the first equation in (20). To prove the inclusion, we proceed as in the proof of Lemma 5.1 and write down a system of appropriate inequalities. If $s \leq s_1$, then

$$|\psi \geq s| \leq \frac{p}{q}\beta^{1-q/p}|\psi \geq s|^{q/p}s^{q-1} + 1 - \frac{p}{q}\beta^{1-q/p}s^{q-1}. \tag{21}$$

Indeed, both sides are equal when $|\psi \geq s| = 1$, and the function $t \mapsto t - \frac{p}{q}\beta^{1-q/p}t^{q/p}s^{q-1}$ is nondecreasing on $[0,1]$ for $s \leq s_1$ (here we use the equality $\beta = s_1^{p(q-1)/(q-p)}$, which is due to the last condition in (20)). Next, for $s \in [s_1, s_0]$ such that $\beta s^{-p(q-1)/(q-p)} \geq t$ we have

$$|\psi \geq s| \leq \frac{p}{q}\beta^{1-q/p}|\psi \geq s|^{q/p}s^{q-1} + \left(1 - \frac{p}{q}\right)\beta s^{-p(q-1)/(q-p)}, \tag{22}$$

which follows directly from Young's inequality. For remaining points s from the interval $[s_1, s_0]$,

$$|\psi \geq s| \leq \frac{p}{q}\beta^{1-q/p}|\psi \geq s|^{q/p}s^{q-1} + t - \frac{p}{q}\beta^{1-q/p}t^{q/p}s^{q-1}, \tag{23}$$

which can be proved by a reasoning similar to that above. It is easy to check that if we replace ψ by φ, (21)-(23) become equalities. Since $\int_t^1 \psi = \int_t^1 \varphi$, we have $\int_0^{s_0} |\psi \geq s| \mathrm{d}s = \int_0^{s_0} |\varphi \geq s| \mathrm{d}s$ and hence we obtain

$$\int_0^{s_0} |\psi \geq s|^{q/p}s^{q-1}\mathrm{d}s \geq \int_0^{s_0} |\varphi \geq s|^{q/p}s^{q-1}\mathrm{d}s. \tag{24}$$

The inequality

$$\int_{s_0}^{\infty} |\psi \geq s|^{q/p}s^{q-1}\mathrm{d}s \geq \int_{s_0}^{\infty} |\varphi \geq s|^{q/p}s^{q-1}\mathrm{d}s \tag{25}$$

is proved analogously: for $s > s_0$ with $\alpha s^{-p(q-1)/(q-p)} \leq t$, we have

$$|\psi \geq s| \leq \frac{p}{q}\alpha^{1-q/p}|\psi \geq s|^{q/p}s^{q-1} + t - \frac{p}{q}\alpha^{1-q/p}t^{q/p}s^{q-1},$$

while for remaining $s > s_0$,

$$|\psi \geq s| \leq \frac{p}{q}\alpha^{1-q/p}|\psi \geq s|^{q/p}s^{q-1} + \left(1 - \frac{p}{q}\right)\alpha s^{-p(q-1)/(q-p)}.$$

Again, both estimates become equalities when $\psi = \varphi$, and hence (25) follows. Adding it to (24) yields $||\psi||_{p,q} \geq ||\varphi||_{p,q}$ and completes the proof of the lemma. ■

Thus, in the calculation of the supremum in (9), we may restrict ourselves to the functions φ of the form (19): that is,

$$B_{p,q}(\lambda, F) = \sup\{t(\varphi) : \varphi \text{ as in (19) with } ||\varphi||_{p,q} \leq F\}.$$

Step 2. In fact, we may exclude the last two possibilities from Figure 1. We shall present the detailed explanation in the case when the graph of φ is as the third one on

Figure 1; the fourth type of graph is dealt with similarly. So, suppose that

$$|\varphi \geq s| = \begin{cases} 1 & \text{if } s \leq s_1, \\ \beta s^{-p(q-1)/(q-p)} & \text{if } s_1 \leq s \leq s_0, \\ t & \text{if } s_0 < s \leq s_2, \\ \alpha s^{-p(q-1)/(q-p)} & \text{if } s \geq s_2, \end{cases}$$

for appropriate α, β, s_1 and s_2. Introduce another nonincreasing and right-continuous function $\overline{\varphi}$, which has the distribution

$$|\overline{\varphi} \geq s| = \begin{cases} 1 & \text{if } s \leq \overline{s_1}, \\ \max\{\overline{\beta} s^{-p(q-1)/(q-p)}, t\} & \text{if } \overline{s_1} \leq s \leq s_2, \\ \alpha s^{-p(q-1)/(q-p)} & \text{if } s \geq s_2, \end{cases}$$

where $\overline{\beta}$ and $\overline{s_1}$ are such that

$$\int_0^t \overline{\varphi} = \lambda t, \quad \text{and} \quad \int_t^1 \overline{\varphi} = 1 - \lambda t.$$

Then the graph of $\overline{\varphi}$ is as on the first or the second picture on Figure 1. Furthermore, φ and $\overline{\varphi}$ coincide on $[0, t]$, so $t(\varphi)$ and $t(\overline{\varphi})$ are equal, and

$$\int_{s_2}^{\infty} |\varphi \geq s|^{q/p} s^{q-1} \mathrm{d}s = \int_{s_2}^{\infty} |\overline{\varphi} \geq s|^{q/p} s^{q-1} \mathrm{d}s.$$

Finally, we have

$$\int_0^{s_2} |\varphi \geq s|^{q/p} s^{q-1} \mathrm{d}s \geq \int_0^{s_2} |\overline{\varphi} \geq s|^{q/p} s^{q-1} \mathrm{d}s, \tag{26}$$

so $||\varphi||_{p,q} \geq ||\overline{\varphi}||_{p,q}$. To see this, repeat (21), (22) and (23), with s_0, s_1 and β replaced by s_2, $\overline{s_1}$ and $\overline{\beta}$, respectively. Then the estimates hold true for $\psi := \varphi$, while for $\psi := \overline{\varphi}$ we obtain three equalities. This yields (26).

Step 3. Let us study the first possibility illustrated on Figure 1. That is, let φ be a nonincreasing and right-continuous function on $(0, 1]$, with the distribution

$$|\varphi \geq s| = \begin{cases} 1 & \text{if } 0 \leq s \leq s_1, \\ \beta s^{-(q-1)p/(q-p)} & \text{if } s_1 \leq s \leq s_0, \\ \alpha s^{-(q-1)p/(q-p)} & \text{if } s > s_0, \end{cases} \tag{27}$$

such that $\int_0^1 \varphi = 1$ and $t(\varphi) \in \big[|\varphi > s_0|, |\varphi \geq s_0|\big]$. We shall prove that among the class of such functions, the function $\overline{\varphi}$ determined by

$$|\overline{\varphi} \geq s| = \begin{cases} 1 & \text{if } 0 \leq s \leq \overline{s_1}, \\ (s/\overline{s_1})^{-(q-1)p/(q-p)} & \text{if } s > \overline{s_1}, \end{cases}$$

has the smallest Lorentz norm and the largest parameter t.

Lemma 6.2. *We have $t(\varphi) \leq t(\overline{\varphi})$ and $\overline{\varphi} \in \mathcal{K}(||\varphi||_{p,q})$.*

Proof. First, note that $\int_0^1 \overline{\varphi} = 1$ implies $\overline{s_1} = \frac{q(p-1)}{p(q-1)}$ and hence

$$||\overline{\varphi}||_{p,q} = C_{p,q}^{-1} \quad \text{and} \quad t(\overline{\varphi}) = \lambda^{-p(q-1)/(q-p)}.$$

Now, turn to the analysis of φ. Let $t_0 = |\varphi > s_0|$, $t = t(\varphi)$ and note that

$$\alpha = t_0 s_0^{p(q-1)/(q-p)} \quad \text{and} \quad \beta = s_1^{p(q-1)/(q-p)}. \tag{28}$$

By the definition of $t(\varphi)$,

$$\lambda t = \int_0^t \varphi = s_0 + \alpha \int_{s_0}^\infty s^{-p(q-1)/(q-p)} \mathrm{d}s = s_0 t + \frac{q-p}{q(p-1)} s_0 t_0, \tag{29}$$

where in the latter passage we have exploited (28). Since $\int_0^1 \varphi = 1$, we obtain

$$1 - \lambda t = \int_t^1 \varphi = s_1 + \beta \int_{s_1}^{s_0} s^{-p(q-1)/(q-p)} \mathrm{d}s - s_0 t,$$

which, by virtue of (28), can be transformed into the identity

$$t_0 = \frac{q(p-1)}{q-p} \frac{1}{s_0} - \frac{p(q-1)}{q-p} r + r^{p(q-1)/(q-p)}, \tag{30}$$

where $r = s_1/s_0 \in [0, 1]$. The inequality $t(\varphi) \leq t(\overline{\varphi})$ is equivalent to

$$\frac{q(p-1)}{q-p} \frac{1}{s_0} - \frac{p(q-1)}{q-p} r + r^{p(q-1)/(q-p)} \leq \lambda^{-p(q-1)/(q-p)}. \tag{31}$$

If we fix s_0, the left-hand side is a nonincreasing function of r. Furthermore, we have $\alpha \leq \beta$, so

$$1 = \int_0^\infty |\varphi \geq s| \mathrm{d}s \leq s_1 + \int_{s_1}^\infty \beta s^{-p(q-1)/(q-p)} \mathrm{d}s,$$

which is equivalent to

$$s_1 \geq \frac{q(p-1)}{p(q-1)} \tag{32}$$

and implies that r is not smaller than $q(p-1)/(p(q-1)s_0)$. Therefore, the left-hand side of (31) is not larger than $[q(p-1)/(p(q-1)s_0)]^{p(q-1)/(q-p)}$ and thus it suffices to show that

$$\lambda \leq \frac{p(q-1)s_0}{q(p-1)}.$$

This follows immediately from (29), combined with the estimate $t_0 \leq t$. Thus we have shown that $t(\varphi) \leq t(\overline{\varphi})$ and now we turn to the estimate $||\varphi||_{p,q} \geq ||\overline{\varphi}||_{p,q}$. Since

$$||\varphi||_{p,q}^q = \frac{p(q-p)}{q(p-1)} s_0^q \left[t_0^{q/p} - r^{q(q-1)/(q-p)} + \frac{q-1}{q-p} r^q \right], \tag{33}$$

we must prove that

$$\frac{p(q-p)}{q(p-1)} s_0^q \left[t_0^{q/p} - r^{q(q-1)/(q-p)} + \frac{q-1}{q-p} r^q \right] \geq \left(\frac{q(p-1)}{p(q-1)} \right)^{q-1}.$$

Dividing throughout by r^q and calculating a little bit, we arrive at the following equivalent form

$$\frac{t_0^{q/p}}{r^q} - r^{q(p-1)/(q-p)} \geq \frac{q-1}{q-p} \left[\left(\frac{q(p-1)}{p(q-1)} \right)^q s_1^{-q} - 1 \right]. \tag{34}$$

Fix s_1. The left-hand side can be rewritten in the form

$$F(R) = R^{1-q/p} \left(\frac{q(p-1)}{(q-p)s_1} - \frac{p(q-1)}{q-p} + R \right)^{q/p} - R,$$

where $R = r^{q(p-1)/(q-p)}$. We have that

$$F'(R) = \frac{q}{p}(t_0/R)^{q/p-1} - \frac{q-p}{p}(t_0/R)^{q/p} - 1,$$

which is nonpositive, by Young's inequality. Thus it suffices to prove (34) for the largest possible r, i.e. $r = 1$. For this value of r, the inequality becomes

$$\left(\frac{q(p-1)}{q-p}\right)^{q/p}(s_1^{-1}-1)^{q/p} - 1 \geq \frac{q-1}{q-p}\left[\left(\frac{q(p-1)}{p(q-1)}\right)^q s_1^{-q} - 1\right],$$

or

$$G(x) := \left(1 - \frac{p(q-1)}{q-p}x\right)^{q/p} - 1 - \frac{q-1}{q-p}\left[(1-x)^q - 1\right] \geq 0, \tag{35}$$

with $x = (p(q-1)s_1 - q(p-1))/(p(q-1)s_1)$; note that $x \geq 0$ by (32). The estimate (35) is straightforward: we have $G(0) = 0$ and

$$G'(x) = \frac{q(q-1)}{q-p}\left[(1-x)^{q-1} - \left(1 - \frac{p(q-1)}{q-p}x\right)^{q/p-1}\right] \geq 0$$

for $x \in [0, \frac{q-p}{p(q-1)}]$. Indeed, the latter estimate is equivalent to

$$(1-x)^{p(q-1)/(q-p)} \geq 1 - \frac{p(q-1)}{q-p}x,$$

which holds true by the mean-value property. This completes the proof of the lemma. ■

Step 4. Now we consider the functions with a graph appearing as the second one on Figure 1. So, fix $\lambda > 1$ and suppose that φ is a nonincreasing and right-continuous function on $(0, 1]$, with the distribution

$$|\varphi \geq s| = \begin{cases} 1 & \text{if } 0 \leq s \leq s_1, \\ \beta s^{-p(q-1)/(q-p)} & \text{if } s_1 \leq s \leq s_3, \\ t & \text{if } s_3 \leq s \leq s_2, \\ \alpha s^{-p(q-1)/(q-p)} & \text{if } s \geq s_2, \end{cases} \tag{36}$$

satisfying

$$\int_0^t \varphi = \lambda t \quad \text{and} \quad \int_t^1 \varphi = 1 - \lambda t$$

(note that the function $\overline{\varphi}$ studied in the previous step is also of that form, with $s_2 = s_3$ and $\alpha = \beta$). A little computation yields

$$\frac{p(q-1)}{q(p-1)}s_2 = \lambda, \quad \alpha = ts_2^{p(q-1)/(q-p)}, \quad \beta = s_1^{p(q-1)/(q-p)} = ts_3^{p(q-1)/(q-p)}. \tag{37}$$

Furthermore, the condition $\int_0^1 \varphi = 1$ is equivalent to

$$\frac{q(p-1)}{p(q-1)} = s_1 + (s_2 - s_3)t. \tag{38}$$

Now we express $||\varphi||_{p,q}$ as a function of t and optimize. We derive that

$$\begin{aligned}||\varphi||_{p,q}^q = \frac{p(q-1)}{q(p-1)} s_2^q \Bigg[\left(\frac{s_1}{s_2}\right)^q - \frac{p-1}{q-1} t^{q/p} \left(\frac{s_3}{s_2}\right)^q + t^{q/p} \\ + \frac{q-p}{1-q} \left(\frac{s_1}{s_2}\right)^{q(q-1)/(q-p)} \left(\frac{s_3}{s_2}\right)^{q(1-p)/(q-p)} \Bigg].\end{aligned} \tag{39}$$

Let $r = s_1/s_2 < 1$. By the last inequality in (37) we get that

$$\frac{s_3}{s_2} = t^{(p-q)/(p(q-1))} r$$

and plugging this into (38) and (39) yields

$$\frac{q(p-1)}{p(q-1)} \frac{1}{s_2} = r + t - rt^{q(p-1)/(p(q-1))},$$

or

$$r = r(t) := \frac{\frac{q(p-1)}{p(q-1)} \frac{1}{s_2} - t}{1 - t^{q(p-1)/(p(q-1))}} = \frac{\lambda^{-1} - t}{1 - t^{q(p-1)/(p(q-1))}}, \tag{40}$$

and, using the first identity from (37), we obtain

$$||\varphi||_{p,q}^q = D_{p,q,\lambda}(t) = \left[\frac{q(p-1)}{p(q-1)}\right]^{q-1} \left[\frac{(1-\lambda t)^q}{\left(1 - t^{q(p-1)/(p(q-1))}\right)^{q-1}} + t^{q/p}\lambda^q \right] \tag{41}$$

(the function $D_{p,q,\lambda}$ has already appeared in the statement of Theorem 3.2 above). Since $\alpha \geq \beta$, we have $|\varphi \geq s| \leq \min\{\alpha s^{-p(q-1)/(q-p)}, 1\}$ and consequently,

$$1 \leq \int_0^\infty \min\{\alpha s^{-p(q-1)/(q-p)}, 1\} \mathrm{d}s,$$

which implies

$$\alpha \geq \left(\frac{q(p-1)}{p(q-1)}\right)^{p(q-1)/(q-p)}$$

and thus, by the first and the second identity in (37),

$$t \geq \left(\frac{q(p-1)}{p(q-1)} \frac{1}{s_2}\right)^{p(q-1)/(q-p)} = \lambda^{-p(q-1)/(q-p)}. \tag{42}$$

We derive that

$$\begin{aligned}r'(t) = \left(1 - t^{q(p-1)/(p(q-1))}\right)^2 \times \\ \times \left[-1 + \frac{q-p}{p(q-1)} t^{q(p-1)/(p(q-1))} + \left(\frac{q(p-1)}{p(q-1)}\right)^2 \frac{1}{s_2} t^{(p-q)/(p(q-1))} \right]\end{aligned}$$

and the expression in the square bracket is negative: the second term is smaller than $(q-p)/(p(q-1))$ and the third, in view of (42), does not exceed $q(p-1)/(p(q-1))$. Thus the function r is decreasing. Next, a little calculation gives

$$D'_{p,q,\lambda}(t) = \left[\frac{q(p-1)}{p(q-1)}\right]^{q-1} \lambda^q \left[\frac{p-1}{p} R^q(t) + \frac{1}{p} - R^{q-1}(t) \right],$$

where
$$R(t)=\frac{\lambda^{-1}t^{(p-q)/(p(q-1))}-t^{q(p-1)/(p(q-1))}}{1-t^{q(p-1)/(p(q-1))}}.$$
By (42), the first summand in the numerator is not larger than 1 and hence $R(t)\le 1$. This implies $D'_{p,q,\lambda}\ge 0$: indeed, since $q>p$, we get
$$\frac{p-1}{p}R^q(t)+\frac{1}{p}\ge\frac{q-1}{q}R^q(t)+\frac{1}{q}\ge R^{q-1}(t),$$
where the latter is due to Young's inequality. In fact, we easily see that $D_{p,q,\lambda}$ is strictly increasing, since the equality $D'_{p,q,\lambda}(t)=0$ is possible only for one value of t (for which both sides of (42) are equal).

Step 5. We are ready to provide the formula for the Bellman function $B_{p,q}$. The previous steps and Theorem 4.1 imply that in the computation of the supremum on the right-hand side of (9), we may restrict ourselves to the functions studied in Step 4 above. We have shown that the function $D_{p,q,\lambda}$ given by (41) is strictly increasing, furthermore, we check that
$$D_{p,q,\lambda}\left(\lambda^{-p(q-1)/(q-p)}\right)=C_{p,q}^{-q}$$
and
$$D_{p,q,\lambda}(\lambda^{-1})=C_{p,q}^{-q}\lambda^{q-q/p}.$$
In consequence, the function $D_{p,q,\lambda}:[\lambda^{-p(q-1)/(q-p)},\lambda^{-1}]\to[C_{p,q}^{-q},C_{p,q}^{-q}\lambda^{q-q/p}]$ is invertible, and the inverse $G_{p,q,\lambda}$ is also strictly increasing.

We are ready to complete the proof of Theorem 3.2.

THEOREM 6.3. *If $q>p$ and $C_{p,q}^{-1}\le F<C_{p,q}^{-1}\lambda^{1-1/p}$, then*
$$B_{p,q}(\lambda,F)=G_{p,q,\lambda}(F^q).$$

The proof is the same as that of Theorem 5.3 and is omitted. As a corollary, we establish the following sharp weak-type estimate for the maximal operator.

THEOREM 6.4. *Assume that $q>p$ and $r\in[1,p]$. Then for any $\phi:X\to[0,\infty)$,*
$$||\mathcal{M}_{\mathcal{T}}\phi||_{r,\infty}\le C_{p,q}||\phi||_{p,q} \tag{43}$$
and the constant is the best possible.

Proof. The argument is exactly the same as that used in the proof of Theorem 5.4. Since no additional technical difficulties arise, we omit the details, leaving them to the interested reader. ■

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