A sharp weak-type (∞, ∞) inequality for the Hilbert transform

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Abstract. The paper is devoted to sharp weak type (∞, ∞) estimates for $\mathcal{H}^{\mathbb{T}}$ and $\mathcal{H}^{\mathbb{R}}$, the Hilbert transforms on the circle and real line, respectively. Specifically, it is proved that

$$||\mathcal{H}^{\mathbb{T}} f||_{W(\mathbb{T})} \leq ||f||_{L^\infty(\mathbb{T})}$$

and

$$||\mathcal{H}^{\mathbb{R}} f||_{W(\mathbb{R})} \leq ||f||_{L^\infty(\mathbb{R})},$$

where $W(\mathbb{T})$ and $W(\mathbb{R})$ stand for the weak-L^∞ spaces introduced by Bennett, DeVore and Sharpley. In both estimates, the constant 1 on the right is shown to be the best possible.

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1. Introduction

Our motivation comes from a very basic question about the Hilbert transform $\mathcal{H}^{\mathbb{T}}$ on the unit circle $\mathbb{T} \simeq (-\pi, \pi]$ equipped with a normalized uniform measure m. Recall that this operator is given by the singular integral

$$\mathcal{H}^{\mathbb{T}} f(x) = \text{p.v.} \int_{-\pi}^{\pi} f(t) \cot \frac{x-t}{2} m(\mathrm{d}t), \qquad x \in \mathbb{T},$$

when $f \in L^1(\mathbb{T})$. A classical result of M. Riesz [13] states that for any $1 < p < \infty$ there is a finite universal constant C_p such that

$$||\mathcal{H}^{\mathbb{T}} f||_{L^p(\mathbb{T})} \leq C_p ||f||_{L^p(\mathbb{T})}, \qquad f \in L^p(\mathbb{T}). \tag{1.1}$$

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For $p = 1$ the above estimate does not hold with any $C_1 < \infty$, but, as Kolmogorov showed in [11], there is an absolute $c_1 < \infty$ such that

$$||\mathcal{H}^{\mathbb{T}} f||_{L^{1,\infty}(\mathbb{T})} := \sup_{\lambda>0} \left[\lambda\, m(\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f(x)| \geq \lambda\}) \right] \leq c_1 ||f||_{L^1(\mathbb{T})}, \quad (1.2)$$

whenever $f \in L^1(\mathbb{T})$. The optimal values of the constants C_p and c_1 were determined in 1970s: Pichorides [12] and Cole (unpublished: see Gamelin [9]) proved that the best constant in (1.1) equals $\cot \frac{\pi}{2p^*}$, where $p^* = \max\{p, p/(p-1)\}$, and Davis [6] showed that the optimal choice for the constant c_1 in (1.2) is

$$\left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\left|\frac{2}{\pi} \log |t|\right|}{t^2+1} \mathrm{d}t \right)^{-1} = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots} = 1.347\dots.$$

The above results are of fundamental importance to harmonic analysis. Furthermore, the methods developed by Riesz [13] have had a profound influence on the shape of the contemporary mathematics. For numerous extensions and applications of the above statements, consult e.g. the works of Burkholder [3], Calderón and Zygmund [5], Essén [8], Gohberg and Krupnik [10], Stein [14] and Zygmund [15], and many more.

We will continue the research in this direction. We will be interested in a "dual" version of Kolmogorov's result, i.e., in a weak-L^∞ estimate for $\mathcal{H}^{\mathbb{T}}$. To explain what the weak-L^∞ space is, we need more notation. For a given measurable function $f : \mathbb{T} \to \mathbb{R}$, we define f^*, the decreasing rearrangement of f, by

$$f^*(t) = \inf \left\{ \lambda \geq 0 : m(\{x \in \mathbb{T} : |f(x)| > \lambda\}) \leq t \right\}.$$

Then $f^{**} : (0, 1] \to [0, \infty)$, the maximal function of f^*, is given by the formula

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \mathrm{d}s, \qquad t \in (0, 1].$$

One easily verifies that f^{**} can alternatively be defined by

$$f^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |f| \mathrm{d}m \,:\, E \subseteq \mathbb{T},\, m(E) = t \right\}.$$

We are ready to introduce the weak-L^∞ space. Following Bennett, DeVore and Sharpley [1], we let

$$||f||_{W(\mathbb{T})} = \sup_{t \geq 0} (f^{**}(t) - f^*(t))$$

and define $W(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{R} : ||f||_{W(\mathbb{T})} < \infty\}$. Some words explaining the meaning of this space are in order. For each $1 \leq p < \infty$, the usual weak space $L^{p,\infty}$ properly contains L^p, but for $p = \infty$, the two spaces coincide. Thus, there is no Marcinkiewicz interpolation theorem between L^1 and L^∞ for operators which are unbounded on L^∞. The reason for introducing the space W was to fill this gap. It can be verified that this space contains L^∞, can be understood as an appropriate limit of $L^{p,\infty}$ as $p \to \infty$, and enjoys the required interpolation property: if T is bounded as an operator from L^1 to $L^{1,\infty}$ and from L^∞ to W, then it has an extension which is bounded on

L^p spaces, $1 < p < \infty$. See [1] for details. There is a further evidence, again rooted in the interpolation theory, that the space W can serve as a substitute for weak-L^∞. Namely, the Peetre K-functional for the pair (L^1, L^∞) (cf. [4, p.184]) is explicitly given by

$$K(f, t; L^1, L^\infty) = \int_0^t f^*(s)\mathrm{d}s = tf^{**}(t), \qquad t \in (0, 1].$$

Thus, the weak-L^1 norm can be expressed in terms of the K-functional by

$$||f||_{L^{1,\infty}(\mathbb{T})} = \sup_{t\in(0,1]} tf^*(t) = \sup_{t\in(0,1]} t\frac{d}{dt}K(f, t; L^1, L^\infty). \tag{1.3}$$

Now if we reverse the roles of L^1 and L^∞, and make use of the identity $K(f, t; L^\infty, L^1) = tK(f, t^{-1}; L^1, L^\infty)$, we see that the expression on the right of (1.3) is precisely $\sup_{t\in(0,1]}[f^{**}(t) - f^*(t)]$. Hence this number can be understood as a substitute for the norm in the weak-L^∞. For more on this interplay, the connections between W and BMO, as well as other interesting properties of W, we refer the reader to [1] and the monograph [2] by Bennett and Sharpley.

One of our main results is the identification of the norm of $\mathcal{H}^\mathbb{T}$ as an operator acting from $L^\infty(\mathbb{T})$ to $W(\mathbb{T})$. Here is the precise statement.

Theorem 1.1. *For any $f \in L^\infty(\mathbb{T})$ we have*

$$||\mathcal{H}^\mathbb{T} f||_{W(\mathbb{T})} \leq ||f||_{L^\infty(\mathbb{T})}. \tag{1.4}$$

The inequality is sharp: for any $c < 1$ there is a function $f \in L^\infty(\mathbb{T})$ such that $||\mathcal{H}^\mathbb{T} f||_{W(\mathbb{T})} > c||f||_{L^\infty(\mathbb{T})}$.

We will also study an analogue of the above result in the nonperiodic case. Recall that the Hilbert transform $\mathcal{H}^\mathbb{R}$ on the real line is defined by the principal value integral

$$\mathcal{H}^\mathbb{R} f(x) = \frac{1}{\pi}\text{p.v.}\int_\mathbb{R} \frac{f(t)}{x - t}\mathrm{d}t, \qquad x \in \mathbb{R},$$

when $f \in L^1(\mathbb{R})$. The above strong and weak-type inequalities (1.1), (1.2) can be extended to analogous statements for $\mathcal{H}^\mathbb{R}$ and the optimal constants remain unchanged (see e.g. [13], [15]). It is natural to ask about a sharp weak-type (∞, ∞) inequality in this setting. To study this problem, define the weak space $W(\mathbb{R})$ in the same manner as above:

$$W(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \,:\, ||f||_{W(\mathbb{R})} := \sup_{t>0} \left[f^{**}(t) - f^*(t)\right] < \infty \right\},$$

where, as previously, f^* denotes the decreasing rearrangement of f and f^{**} stands for the maximal function of f^*. Here is the nonperiodic version of Theorem 1.1. It is well known that some technical problems arise when one defines the action of the Hilbert transform on $L^\infty(\mathbb{R})$; to avoid these, we impose a slightly stronger integrability on functions.

Theorem 1.2. *If f belongs to $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $1 \leq p < \infty$, then we have the sharp bound*

$$||\mathcal{H}^{\mathbb{R}} f||_{W(\mathbb{R})} \leq ||f||_{L^\infty(\mathbb{R})}. \tag{1.5}$$

The paper is organized as follows. In the next section we establish Theorem 1.1. In the proof of (1.4) we make use of Bellman function method: the estimate is deduced from the existence of a certain special superharmonic function. In the final part of the paper we present the proof of Theorem 1.2, which follows from Theorem 1.1 by certain transference-type arguments.

2. Periodic case

For any $c \geq 0$, define the function $V^{(c)} : [-1,1] \times [0,\infty) \to \mathbb{R}$ by $V^{(c)}(x,y) = (y-c)\chi_{\{y>0\}}$ (here and below, χ_A denotes the indicator function of a set A). Furthermore, let $U^{(c)} : (-1,1) \times (0,\infty) \to \mathbb{R}$ be given by the formula

$$\begin{aligned} U^{(c)}(x,y) = y - c &+ \frac{2c}{\pi} \arctan\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan\frac{\pi x}{2}\right) \\ &+ \frac{2c}{\pi} \arctan\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan\frac{\pi x}{2}\right). \end{aligned}$$

It is easy to check that $U^{(c)}$ is a harmonic function. Actually, it can be regarded as a harmonic lift of $V^{(c)}$, in the sense explained in the first part of the lemma below.

Lemma 2.1. *The function $U^{(c)}$ has the following properties.*

(i) If $Y > 0$, then $\lim_{(x,y)\to(\pm 1,Y)} U^{(c)}(x,y) = V^{(c)}(\pm 1, Y)$; if $X \in (-1,1)$, then $\lim_{(x,y)\to(X,0)} U^{(c)}(x,y) = V^{(c)}(X,0)$.

(ii) For any $x \in (-1,1)$, we have

$$\lim_{y \downarrow 0} U^{(c)}(x,y)/y = 1 - c\left(\cos\frac{\pi x}{2}\right)^{-1}.$$

(iii) For any $(x,y) \in (-1,1) \times (0,\infty)$, we have $U^{(c)}(x,y) \geq V^{(c)}(x,y)$.

Proof. The properties (i) and (ii) are straightforward and left to the reader. The majorization (iii) is also easy: we must show that

$$\arctan\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan\frac{\pi x}{2}\right) + \arctan\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan\frac{\pi x}{2}\right) \geq 0.$$

This follows from the estimate

$$\left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} - \tan\frac{\pi x}{2}\right) + \left(\frac{e^{-\pi y/2}}{\cos(\pi x/2)} + \tan\frac{\pi x}{2}\right) \geq 0$$

and the fact that the arctangent function is odd and increasing on the real line. □

It will be convenient for us to extend $U^{(c)}$ to the halfstrip $[-1,1]\times[0,\infty)$ by the requirement that $U^{(c)}$ and $V^{(c)}$ match at the boundary of this set. Then $U^{(c)}$ becomes a harmonic majorant of $V^{(c)}$ on the whole $[-1,1]\times[0,\infty)$, and it is continuous except for the points $(\pm 1, 0)$. In addition, part (ii) of the above lemma implies that for $c\geq 1$, the one-sided partial derivative $U^{(c)}_{y+}$ satisfies $U^{(c)}_{y+}(x,0)\leq 0$ for all $x\in(-1,1)$.

The above function $U^{(c)}$ is a "building block" for a larger class of superharmonic functions. For a fixed parameter $\lambda\geq 0$, introduce the functions $\mathcal{U}^{(c)}_\lambda$, $\mathcal{V}^{(c)}_\lambda$ on the strip $[-1,1]\times\mathbb{R}$ by the formulas

$$\mathcal{U}^{(c)}_\lambda(x,y)=U^{(c)}(x,(|y|-\lambda)_+)=\begin{cases} U^{(c)}(x,y-\lambda) & \text{if } y\geq\lambda,\\ 0 & \text{if } |y|<\lambda,\\ U^{(c)}(x,-\lambda-y) & \text{if } y<-\lambda\end{cases}$$

and $\mathcal{V}^{(c)}_\lambda(x,y)=V^{(c)}(x,(|y|-\lambda)_+)=(|y|-\lambda)_+-c\chi_{\{|y|>\lambda\}}$.

Lemma 2.2. *For each $\lambda\geq 0$ and $c\geq 1$, the function $\mathcal{U}^{(c)}_\lambda$ is a superharmonic majorant of $\mathcal{V}^{(c)}_\lambda$.*

Proof. Assume first that $c>1$. The inequality $\mathcal{U}^{(c)}_\lambda\geq\mathcal{V}^{(c)}_\lambda$ follows immediately from the majorization $U^{(c)}\geq V^{(c)}$ established above; hence all we need is the superharmonicity of $\mathcal{U}^{(c)}_\lambda$. Observe that this function is harmonic on each of the domains $(-1,1)\times(-\infty,-\lambda)$, $(-1,1)\times(-\lambda,\lambda)$ and $(-1,1)\times(\lambda,\infty)$. Consequently, it is enough to check that $\mathcal{U}^{(c)}_\lambda$ satisfies the mean value property at each point of the form $(x,\pm\lambda)$. But this follows at once from the inequality $U^{(c)}_{y+}(x,0)<0$ (here the strictness is due to $c>1$). To get the claim for $c=1$, note that $U^{(1)}$ is a pointwise limit of $U^{(c)}$ as $c\downarrow 1$. □

In the next lemma we establish an intermediate result which is of its own interest.

Lemma 2.3. *For any $f:\mathbb{T}\to[-1,1]$ and any $\lambda\geq 0$, we have*

$$\int_{\mathbb{T}}(|\mathcal{H}^{\mathbb{T}}f|-\lambda)_+\,dm\leq m\big(\{x\in\mathbb{T}:|\mathcal{H}^{\mathbb{T}}f(x)|>\lambda\}\big). \tag{2.1}$$

Proof. Let u, v denote the harmonic extensions of f and $\mathcal{H}^{\mathbb{T}}f$ to the unit disc, obtained via the Poisson kernel. Then u, v satisfy Cauchy-Riemann equations and $v(0,0)=0$ (cf. Riesz [13]). Consequently, the function $\mathcal{U}^{(1)}_\lambda(u,v)$ is superharmonic (being the composition of a superharmonic $\mathcal{U}^{(1)}$ and the analytic $u+iv$) and it majorizes $\mathcal{V}^{(1)}_\lambda(u,v)$. Therefore, by the mean value property,

$$\begin{aligned}\int_{\mathbb{T}}\mathcal{V}^{(1)}_\lambda(u,v)\mathrm{d}m &\leq \int_{\mathbb{T}}\mathcal{U}^{(1)}_\lambda(u,v)\mathrm{d}m\\ &\leq \mathcal{U}^{(1)}_\lambda(u(0,0),v(0,0))=\mathcal{U}^{(1)}_\lambda(u(0,0),0)=0.\end{aligned}$$

This is precisely (2.1). □

We turn our attention to Theorem 1.1.

Proof of (1.4). By homogeneity, we may and do assume that $||f||_{L^\infty(\mathbb{T})}=1$. By the definition of $(\mathcal{H}^\mathbb{T} f)^{**}$, we may write

$$\begin{aligned}(\mathcal{H}^\mathbb{T} f)^{**}(t)&-(\mathcal{H}^\mathbb{T} f)^*(t)\\ &=\sup\left\{\frac{1}{m(E)}\int_E\big[|\mathcal{H}^\mathbb{T} f(x)|-(\mathcal{H}^\mathbb{T} f)^*(t)\big]m(\mathrm{d}x)\,:\,E\subseteq\mathbb{T},\,m(E)=t\right\}.\end{aligned}$$

It is clear that when computing this supremum, we may restrict ourselves to those E, which satisfy

$$\{x\in\mathbb{T}:|\mathcal{H}^\mathbb{T} f(x)|>\lambda\}\subseteq E\subseteq\{x\in\mathbb{T}:|\mathcal{H}^\mathbb{T} f(x)|\geq\lambda\}$$

for some $\lambda\geq 0$. Actually, since $m(E)=t$, this λ must be equal to $(\mathcal{H}^\mathbb{T} f)^*(t)$. For such E, it is clear that

$$\begin{aligned}&\frac{1}{m(E)}\int_E\big[|\mathcal{H}^\mathbb{T} f(x)|-(\mathcal{H}^\mathbb{T} f)^*(t)\big]m(\mathrm{d}x)\\ &\leq\frac{1}{m(\{x\in\mathbb{T}:|\mathcal{H}^\mathbb{T} f(x)|>\lambda\})}\int_{\{|\mathcal{H}^\mathbb{T} f|>\lambda\}}\big[|\mathcal{H}^\mathbb{T} f|-\lambda\big]\mathrm{d}m\leq 1,\end{aligned}$$

where the latter bound is due to (2.1). This establishes (1.4). □

Sharpness. Fix an arbitrary $\lambda\in(0,c/2)$. Consider the region $\mathcal{C}=[-1,1]\times[-\lambda,\infty)$ and let F be the conformal mapping which sends the unit disc $\mathbb{D}$ onto $\mathcal{C}$ and $(0,0)\in\mathbb{D}$ to $(0,0)\in\mathcal{C}$. Then F transports m, the harmonic measure on $\mathbb{T}$ with respect to $(0,0)$, to μ, the harmonic measure on $\partial\mathcal{C}$ with respect to $(0,0)$. Finally, put $u=\operatorname{Re}F$ and $v=\operatorname{Im}F$; clearly, the restrictions $f=u|_\mathbb{T}$ and $g=v|_\mathbb{T}$ satisfy the relation $g=\mathcal{H}^\mathbb{T} f$. The function $(x,y)\mapsto U^{(c)}(x,y+\lambda)$ is harmonic in the interior of $\mathcal{C}$, so by the mean-value property,

$$\begin{aligned}U^{(c)}(0,\lambda)&=\int_{\partial\mathcal{C}}U^{(c)}(x,y+\lambda)\mathrm{d}\mu(x,y)\\ &=\int_\mathbb{T} U^{(c)}(u,v+\lambda)\mathrm{d}m\\ &=\int_\mathbb{T}(v+\lambda-c)\chi_{\{v+\lambda>0\}}\mathrm{d}m\\ &\leq\int_\mathbb{T}(v+\lambda-c)\chi_{\{v-\lambda>0\}}\mathrm{d}m\\ &=\int_\mathbb{T}\mathcal{H}^\mathbb{T} f\chi_{\{\mathcal{H}^\mathbb{T} f>\lambda\}}\mathrm{d}m-(c-\lambda)m(\{x\in\mathbb{T}:\mathcal{H}^\mathbb{T} f(x)>\lambda\}).\end{aligned}$$

However, if λ is sufficiently close to 0, then $U^{(c)}(0,\lambda)>0$: this follows from Lemma 2.1 (ii). Hence, for such λ,

$$\frac{1}{m(\{x\in\mathbb{T}:\mathcal{H}^\mathbb{T} f(x)>\lambda\})}\int_{\{\mathcal{H}^\mathbb{T} f>\lambda\}}|\mathcal{H}^\mathbb{T} f(x)|m(\mathrm{d}x)\geq c-\lambda.$$

Now take $t=m(\{x\in\mathbb{T}:\mathcal{H}^\mathbb{T} f(x)>\lambda\})$. The above inequality implies that

$$(\mathcal{H}^\mathbb{T} f)^{**}(t)\geq c-\lambda. \tag{2.2}$$

In addition, since $\mathcal{H}^{\mathbb{T}} f \geq -\lambda$ on $\mathbb{T}$, we actually have $t = m(\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f(x)| > \lambda\})$. Hence, from the very definition of the decreasing rearrangement, we infer that $(\mathcal{H}^{\mathbb{T}} f)^*(t) \leq \lambda$. Combining this with (2.2), we obtain

$$||\mathcal{H}^{\mathbb{T}} f||_{W(\mathbb{T})} \geq c - 2\lambda.$$

It remains to observe that the right-hand side can be made arbitrarily close to 1, by choosing c appropriately close to 1 and then picking λ sufficiently small. This proves that the constant 1 cannot be replaced in (1.4) by a smaller number. □

3. The non-periodic case

Proof of (1.5). To deduce the weak-type estimate for the Hilbert transform on the real line, we use a standard argument known as "blowing up the circle", which is due to Zygmund ([15], Chapter XVI, Theorem 3.8). Let $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be a fixed function. For a given positive integer n and $x \in \mathbb{R}$, put

$$g_n(x) = \frac{1}{2\pi n} \text{p.v.} \int_{-\pi n}^{\pi n} f(t) \cot \frac{x-t}{2n} \mathrm{d}t.$$

As shown in [15], we have $g_n \to \mathcal{H}^{\mathbb{R}} f$ almost everywhere as $n \to \infty$. On the other hand, the function

$$x \mapsto g_n(nx) = \text{p.v.} \int_{-\pi}^{\pi} f(nt) \cot \frac{x-t}{2} m(\mathrm{d}t)$$

is precisely the periodic Hilbert transform of the function $f_n : x \mapsto f(nx)$, $|x| \leq \pi$. Consequently, by (2.1), we may write

$$\begin{aligned} |\{x \in (-\pi n, \pi n] : |g_n(x)| > \lambda\}| &= 2\pi n \, m\big(\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}} f_n(x)| > \lambda\}\big) \\ &\geq 2\pi n \int_{\mathbb{T}} (|\mathcal{H}^{\mathbb{T}} f_n(x)| - \lambda)_+ m(\mathrm{d}x) \\ &= \int_{-\pi n}^{\pi n} (|g_n(x)| - \lambda)_+ \mathrm{d}x. \end{aligned}$$

Now we let $n \to \infty$; using some routine limiting arguments, we get

$$|\{x \in \mathbb{R} : \mathcal{H}^{\mathbb{R}} f(x) > \lambda\}| \geq \int_{\mathbb{R}} (|\mathcal{H}^{\mathbb{R}} f(x) - \lambda)_+ \mathrm{d}x.$$

It remains to repeat the reasoning from the periodic case to obtain, for any $t > 0$,

$$\begin{aligned} &(\mathcal{H}^{\mathbb{R}} f)^{**}(t) - (\mathcal{H}^{\mathbb{R}} f)^*(t) \\ &\quad \leq \frac{1}{|\{x \in \mathbb{R} : |\mathcal{H}^{\mathbb{R}} f(x)| > \lambda\}|} \int_{\{|\mathcal{H}^{\mathbb{R}} f| > \lambda\}} \big[|\mathcal{H}^{\mathbb{R}} f(x)| - \lambda\big] \mathrm{d}x \leq 1. \end{aligned}$$ □

Sharpness. As we have shown in the previous section, for any $c \in (0,1)$ and λ sufficiently close to 0, there is a function $\varphi : \mathbb{T} \to [-1,1]$ such that $\int_{\mathbb{T}} \varphi \mathrm{d}m = 0$ and

$$\frac{1}{|\{x \in \mathbb{T} : |\mathcal{H}^{\mathbb{T}}\varphi(x)| > \lambda\}|} \int_{\{|\mathcal{H}^{\mathbb{T}}\varphi| > \lambda\}} |\mathcal{H}^{\mathbb{T}}\varphi(x)| \mathrm{d}x \geq c - \lambda. \tag{3.1}$$

We will expand this function onto the real line. We will use Davis' argument from [6]. For the sake of clarity, we have divided the reasoning into three parts.

1. A conformal mapping and its properties. Let H denote the closed upper halfplane of $\mathbb{C}$ and consider the conformal mapping $K(z) = -(1-z)^2/4z$. This function maps the halfdisc $\mathbb{D} \cap H$ onto H, and the boundary of $\mathbb{D} \cap H$ onto $\mathbb{R}$. Let L be the inverse of K. Then L maps $[0,1]$ onto the halfcircle $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$, and $\mathbb{R} \setminus [0,1]$ onto $(-1,1)$. Specifically, for $x \in [0,1]$ we have $L(x) = \exp(2i \arcsin(\sqrt{x}))$, while for $x \notin [0,1]$,

$$L(x) = \begin{cases} 1 - 2x - 2\sqrt{x^2 - x} & \text{if } x < 0, \\ 1 - 2x + 2\sqrt{x^2 - x} & \text{if } x > 1. \end{cases}$$

We will also need the property

$$L(z) \to 0 \qquad \text{as } z \to \infty. \tag{3.2}$$

Next, for a positive integer n, let d_n be the density of $L^n([0,1])$ on $\mathbb{T}$ with respect to m, i.e. for any $-\pi < \alpha < \beta < \pi$,

$$\int_\alpha^\beta d_n(e^{i\theta})\, m(\mathrm{d}\theta) = \left|\left\{r \in [0,1] : L^n(r) \in \{e^{i\theta} : \alpha < \theta < \beta\}\right\}\right|.$$

Then it is easy to prove that

$$d_n \to 1 \text{ uniformly on } \mathbb{T}, \tag{3.3}$$

see Lemma 3 in [6].

2. Expansion of φ. Let Φ denote the holomorphic extension of $\varphi + i\mathcal{H}^{\mathbb{T}}\varphi$ to the unit disc. Then Φ satisfies $\Phi(0) = 0$: indeed, $\operatorname{Re}\Phi(0) = 0$ is due to the condition $\int_{\mathbb{T}} \varphi \mathrm{d}m = 0$, while $\operatorname{Im}\Phi(0) = 0$ follows from the normalization property of the periodic Hilbert transform. Combining this with (3.2), we see that the analytic function $F_n = \Phi(L^n(z))$ ($n = 1, 2, \ldots$), given on the halfplane H, satisfies $\lim_{z \to \infty} F_n(z) = 0$. Put $f_n(x) = \operatorname{Re} F_n(x)$ for any $x \in \mathbb{R}$. This function is bounded in absolute value by 1, since so is φ. Furthermore, f_n is integrable when $n \geq 2$. Indeed, for any $x \notin [-1,1]$ we have

$$|f_n(x)| = |\operatorname{Re}\Phi(L^n(x))| \leq \kappa_1 |L^n(x)| \leq \kappa_2 |x|^{-n},$$

for some universal constants κ_1, κ_2. Thus, we may speak of $\mathcal{H}^{\mathbb{R}} f_n$. Furthermore, by the aforementioned property $\lim_{z \to \infty} F_n(z) = 0$, we have $\mathcal{H}^{\mathbb{R}} f_n = \operatorname{Im} F_n|_{\mathbb{R}}$.

3. Computations. If $x \notin [0,1]$, then $L(x) \in (-1,1)$ and hence $L^n(x) \to 0$ as $n \to \infty$. Consequently, we have

$$|\{x \notin [0,1] : |\mathcal{H}^{\mathbb{R}} f_n(x)| > \lambda\}| \to 0$$

and, by Lebesgue's dominated convergence theorem,

$$\int_{\{x\notin[0,1]:|\mathcal{H}^{\mathbb{R}}f_n(x)|>\lambda\}} |\mathcal{H}^{\mathbb{R}}f_n(x)|\mathrm{d}x \xrightarrow{n\to\infty} 0.$$

Next, observe that by (3.3),

$$\begin{aligned}|\{x\in[0,1]:|\mathcal{H}^{\mathbb{R}}f_n(x)|>\lambda\}| &= |\{x\in[0,1]:|\operatorname{Im}\Phi(L^n(x))|>\lambda\}|\\ &= |\{x\in[0,1]:|\mathcal{H}^{\mathbb{T}}\varphi(L^n(x))|>\lambda\}|\\ &\xrightarrow{n\to\infty} m(\{x\in\mathbb{T}:|\mathcal{H}^{\mathbb{T}}\varphi(x)|>\lambda\})\end{aligned}$$

and

$$\begin{aligned}\int_{\{x\in[0,1]:|\mathcal{H}^{\mathbb{R}}f_n(x)|>\lambda\}} |\mathcal{H}^{\mathbb{R}}f_n(x)|\mathrm{d}x &= \int_{\{x\in[0,1]:|\mathcal{H}^{\mathbb{T}}\varphi(L^n(x))|>\lambda\}} |\mathcal{H}^{\mathbb{T}}\varphi(L^n(x))|\mathrm{d}x\\ &\xrightarrow{n\to\infty} \int_{\mathbb{T}} |\mathcal{H}^{\mathbb{T}}\varphi(x)|m(\mathrm{d}x).\end{aligned}$$

Let us put all the above facts together and combine them with (3.1). We get that for an arbitrary $\eta<1$ we have

$$\frac{1}{|\{x\in\mathbb{R}:|\mathcal{H}^{\mathbb{R}}f_n(x)|>\lambda\}|}\int_{\{|\mathcal{H}^{\mathbb{R}}f_n|>\lambda\}} |\mathcal{H}^{\mathbb{R}}f_n(z)|\mathrm{d}z \geq \eta(c-\lambda),$$

provided n is sufficiently large. Now, set $t=|\{x\in\mathbb{R}:|\mathcal{H}^{\mathbb{R}}f(x)|>\lambda\}|$. Arguing as above, we prove that

$$\big|\{x\notin[0,1]:|\mathcal{H}^{\mathbb{R}}f_n(x)|>2\lambda\}\big| \to 0$$

and

$$\big|\{x\in[0,1]:|\mathcal{H}^{\mathbb{R}}f_n(x)|>2\lambda\}\big| \to m(\{x\in\mathbb{T}:|\mathcal{H}^{\mathbb{T}}\varphi(x)|>2\lambda\})<t.$$

This shows that if n is sufficiently large, then $(\mathcal{H}^{\mathbb{R}}f_n)^*(t)\leq 2\lambda$. Hence, for large n,

$$||\mathcal{H}^{\mathbb{R}}f_n||_{W(\mathbb{R})} \geq (\mathcal{H}^{\mathbb{R}}f_n)^{**}(t)-(\mathcal{H}^{\mathbb{R}}f_n)^*(t)\geq \eta(c-\lambda)-2\lambda.$$

The latter constant can be made arbitrarily close to 1, by choosing appropriate values for the parameters η, c and λ. This proves that the constant 1 is indeed the best possible in (1.5). □

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