SHARP MAXIMAL INEQUALITIES FOR CONTINUOUS-PATH SEMIMARTINGALES

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Abstract. Let $\alpha \geq 0$ be a fixed number and let X, Y be continuous-path semimartingales such that Y is α-differentially subordinate to X and X is either a nonnegative supermartingale, or a nonnegative submartingale. We introduce a method which enables us to derive the best constants in the inequality between the first moments of Y and the maximal function of X. This generalizes the previous results of Burkholder and the author. As an application, we obtain sharp versions of some maximal estimates for stochastic integrals and Itô processes.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, filtered by a nondecreasing right-continuous family $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$. Assume in addition, that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be adapted real-valued right-continuous semimartingales with limits from the left. Let $[X, Y]$ stand for the quadratic covariance process of X and Y (see e.g. [12] for details) and let $X^* = \sup_{t\geq 0} |X_t|$ be the maximal function of X. We will also use the notation $X^*_t = \sup_{0\leq s\leq t} |X_s|$ for the truncated maximal function of X. We say that Y is *differentially subordinate* to X, if the process $([X, X]_t - [Y, Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. This notion was originally introduced in the discrete-time setting by Burkholder and the above continuous-time definition is due to Wang [19] and Bañuelos and Wang [4]. As an example, take a semimartingale X, let $H = (H_t)_{t\geq 0}$ be a predictable process taking values in $[-1, 1]$ and assume that Y is the Itô integral of H with respect to X:

$$Y_t = \int_0^t H_s dX_s, \qquad t \geq 0.$$

Then Y is differentially subordinate to X: this is an immediate consequence of the equality $[X, X]_t - [Y, Y]_t = \int_0^t (1 - H_s^2) d[X, X]_s$. The differential subordination implies many interesting martingale inequalities in the discrete-time: see e.g. [5],

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[6], [7] and for some recent results in this direction, [15] and [18]. As shown by Wang in [19], there is a method of transferring such estimates to the continuous-time case. The obtained inequalities can be applied to the study of Riesz transforms and Beurling-Ahlfors operator (see [2], [3], [4] and [13]) as well as to a wider class of Fourier multipliers, see [1].

Let us state here the famous result of Burkholder (see [5] for discrete and [19] for continuous time version): if X and Y are martingales and Y is differentially subordinate to X, then we have the sharp estimate

$$||Y||_p \leq (p^* - 1)||X||_p, \qquad 1 < p < \infty, \tag{1.1}$$

where $p^* = \max\{p, p/(p-1)\}$. Here $||X||_p = \sup_{t\geq 0} ||X_t||_p$ denotes the p-th moment of X and by sharpness we mean that for any $\gamma < p^* - 1$ there is a pair X, Y as above such that $||Y||_p > \gamma||X||_p$. This result can be extended to a wider class of processes. For any general semimartingales X and Y, let

$$X = X_0 + M + A, \qquad Y = Y_0 + N + B \tag{1.2}$$

be the corresponding Doob-Meyer decompositions (which, in general, may not be unique). Fix a nonnegative number α. Following Wang [19] and the author [16], we say that Y is α-differentially subordinate to X (in short, Y is α-subordinate to X), if Y is differentially subordinate to X and there are decompositions (1.2) such that the process $(\alpha|A|_t - |B|_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of t. Here $|A|_t$ denotes the total variation of A on the interval $[0, t]$, $t \geq 0$. To give an example, suppose that $W = (W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion and X, Y are Itô processes of the following form. For any $t \geq 0$,

$$\begin{aligned} X_t &= X_0 + \int_{0+}^t \phi_s \mathrm{d}W_s + \int_{0+}^t \psi_s \mathrm{d}s, \\ Y_t &= Y_0 + \int_{0+}^t \zeta_s \mathrm{d}W_s + \int_{0+}^t \xi_s \mathrm{d}s, \end{aligned} \tag{1.3}$$

where ϕ, ψ, ζ and ξ are predictable processes satisfying the usual assumptions (see e.g. [14]). Now, we have that if $|\zeta_s| \leq |\phi_s|$ and $|\xi_s| \leq \alpha|\psi_s|$ for all s, then Y is α-subordinate to X. This domination allows to generalize the inequality (1.1) to the submartingale setting. Precisely, it was shown in [16] that if X is a nonnegative submartingale and Y is α-subordinate to X, for some $\alpha \geq 0$, then

$$||Y||_p \leq (p^*_\alpha - 1)||X||_p, \qquad 1 < p < \infty, \tag{1.4}$$

where $p^*_\alpha = \max\{(\alpha+1)p, p/(p-1)\}$. The inequality is sharp provided $\alpha \leq 1$. The original discrete-time version of this result was established by Choi in [10].

For $p = 1$, the inequalities (1.1) and (1.4) do not hold with any finite constant and there is a natural question about their substitutes in this limit case. It turns out that the first moment of Y can be compared to the first moment of the maximal function of X. In [8], Burkholder introduced a general method of proving inequalities of this type in the case when Y is a stochastic integral with respect to X. Though formulated in the martingale setting, the technique can be easily extended so that it works for wider classes of processes. Roughly speaking, the method translates the problem of proving a given inequality to that of finding a special function, which has certain convex-type properties. Using this tool, Burkholder established the following interesting result.

Theorem 1.1. *If X is a martingale and Y is an Itô integral, with respect to X, of a certain predictable process with values in $[-1,1]$, then*

$$||Y||_1 \le \gamma ||X^*||_1. \tag{1.5}$$

Here $\gamma = 2.536\ldots$ is the unique solution of the equation

$$\gamma - 3 = -\exp\left(\frac{1-\gamma}{2}\right).$$

The inequality is sharp.

In the present paper we impose the assumption that the dominating process X has continuous paths. As shown in Lemma 2.1 below, the subordination implies that Y also has this property. It turns out that this additional condition on the trajectories does not change the constants in (1.1) and (1.4) (see [5] and [11]). However, it *does* affect the constant γ appearing in (1.5). To be more precise, let us state here a result, proved by the author in [17].

Theorem 1.2. *If X and Y are continuous-path martingales such that Y is differentially subordinate to X, then we have*

$$||Y||_1 \le \sqrt{2}||X^*||_1. \tag{1.6}$$

The constant $\sqrt{2}$ is the best possible even in the setting of stochastic integrals.

The objective of this paper is to study corresponding inequalities when X is a nonnegative super- or submartingale. Our main results are the following.

Theorem 1.3. *Let α be a fixed nonnegative number. Assume that X is a nonnegative continuous-path supermartingale and Y is α-subordinate to X. Then*

$$||Y||_1 \le \beta ||X^*||_1, \tag{1.7}$$

where $\beta = \beta(possup) = \alpha + 1 + ((2\alpha+1)e)^{-1}$. The constant is the best possible, even if we restrict ourselves to Itô processes of the form (1.3).

Theorem 1.4. *Let α be a fixed nonnegative number. Assume that X is a nonnegative continuous-path submartingale and Y is α-subordinate to X. Then*

$$||Y||_1 \le \beta ||X^*||_1, \tag{1.8}$$

where $\beta = \beta(possub) = \alpha + \left(\frac{2\alpha+2}{2\alpha+1}\right)^{1/2}$. The constant is the best possible, even if we restrict ourselves to Itô processes of the form (1.3).

As an application, we obtain a sharp result for stochastic integrals. Note that if H is predictable and $[-1,1]$-valued, and $Y_t = \int_0^t H_s dX_s$, then Y is 1-subordinate to X. Consequently, the previous theorems immediately yield the following corollary.

Theorem 1.5. *Assume that X is a nonnegative continuous-path semimartingale and Y is an Itô integral as above. Then*

(i) if X is a supermartingale, then

$$||Y||_1 \le (2 + (3e)^{-1})||X^*||_1,$$

(ii) if X is a submartingale, then

$$||Y||_1 \le \left(1 + \frac{2\sqrt{3}}{3}\right)||X^*||_1.$$

Both inequalities are sharp.

The paper is organized as follows. In the next section we present a method which enables us to establish sharp maximal inequalities for continuous super- or submartingales. Namely, we show that validity of a given moment estimate is equivalent to the existence of upper solutions to certain corresponding nonlinear problems. Then, in Section 3 and Section 4 we apply this method and establish Theorem 1.3 and Theorem 1.4 there.

2. On the method of proof

Burkholder's technique, invented in [8], is an effective tool of studying maximal inequalities for stochastic integrals. However, in the present paper we deal with α-subordination and, furthermore, the processes we consider have continuous paths. This raises a question about the refinement of the method so that it works properly in this new setting. We will address it now.

Throughout this section, α is a fixed nonnegative number and D denotes the set $[0,\infty)\times\mathbb{R}\times(0,\infty)$. We are interested in the best constant β in the inequality

$$||Y||_1 \leq \beta ||X^*||_1, \tag{2.1}$$

provided that one of the following conditions hold:

(sup) X is a nonnegative continuous-path supermartingale and Y is α-subordinate to X,

(sub) X is a nonnegative continuous-path submartingale and Y is α-subordinate to X.

Our starting point is to show that α-subordination implies regularity of the trajectories of Y.

Lemma 2.1. *Suppose that (sup) or (sub) holds. Then Y has continuous paths.*

Proof. Let $X = X_0 + M + A$, $Y = Y_0 + N + B$ be the decompositions coming from α-subordination (in fact, the first of them is unique, but we will not need this). We have that M and A are continuous. Therefore, it is clear that B also has this property: otherwise the process $\alpha|A| - |B|$ would not be nondecreasing. Let N^c be the unique continuous local martingale part N^c of N, satisfying

$$[N,N]_t = [N^c,N^c]_t + \sum_{0<s\leq t}|\Delta N_s|^2 = [N,N]_t^c + \sum_{0<s\leq t}|\Delta N_s|^2$$

for all $t\geq 0$. Since $[M,M]$ is continuous and $[M,M]-[N,N]$ is nondecreasing, the jump part $\sum_{0<s\leq t}|\Delta N_s|^2$ must vanish for all t. This proves the claim. □

Now we turn to the inequality (2.1). Clearly, this estimate is equivalent to saying that

$$\mathbb{E}V_\beta(X_t, Y_t, X_t^*\vee\varepsilon) \leq 0 \qquad \text{for all } \varepsilon>0 \text{ and } t\geq 0. \tag{2.2}$$

Here $V_\beta : D\to\mathbb{R}$ is given by the formula $V_\beta(x,y,z) = |y| - \beta z$. The key object in the study of this problem is the class of special functions, denoted by $\mathcal{U} = \mathcal{U}^{sup}(V_\beta)$. It consists of those functions $U : D\to\mathbb{R}$, which satisfy the conditions (2.3)–(2.7) below:

$$U \geq V_\beta \qquad \text{on } D, \tag{2.3}$$

$$U(x,y,z_1) \leq U(x,y,z_2) \qquad \text{if } (x,y,z_i)\in D \text{ and } z_1\geq z_2\geq x, \tag{2.4}$$

$$U(x,\cdot,z): y \mapsto U(x,y,z) \qquad \text{is convex for any fixed } 0 \le x \le z,\ z > 0. \tag{2.5}$$

The next property is that for all $\varepsilon \in \{-1,1\}$, λ_1, $\lambda_2 \in (0,1)$, $x \in [0,z]$, $y \in \mathbb{R}$, $z > 0$ and t_1, $t_2 \in [-x, z-x]$ such that $\lambda_1 + \lambda_2 = 1$ and $\lambda_1 t_1 + \lambda_2 t_2 = 0$,

$$U(x,y,z) \ge \lambda_1 U(x+t_1, y+\varepsilon t_1, z) + \lambda_2 U(x+t_2, y+\varepsilon t_2, z). \tag{2.6}$$

The final condition is that

$$U(x,y,z) \ge U(x-d, y \pm \alpha d, z) \qquad \text{if } x \le z \text{ and } 0 < d \le x, \tag{2.7}$$

In the submartingale setting, we consider an analogous class $\mathcal{U} = \mathcal{U}^{sub}(V_\beta)$, where the properties (2.3)–(2.6) remain unchanged and (2.7) is replaced by

$$U(x,y,z) \ge U(x+d, y \pm \alpha d, z) \qquad \text{if } 0 \le x < x+d \le z. \tag{2.8}$$

Before we proceed, let us comment on the condition (2.6). It simply means that for fixed $z > 0$, the function $U(\cdot,\cdot,z)$ is *diagonally concave* on $[0,z] \times \mathbb{R}$, that is, concave along any line segment of slope ± 1 contained in $[0,z] \times \mathbb{R}$.

For any function U on D, let $b(U) \in (-\infty, \infty]$ be given by

$$b(U) = \sup\{U(x,y,z) : |y| \le x \le z\}.$$

Finally, for any class $\mathcal{C}$ of real functions defined on D, let

$$b(\mathcal{C}) = \inf\{b(U) : U \in \mathcal{C}\},$$

with the convention $\inf \emptyset = \infty$. The main result of this section can be stated as follows.

Theorem 2.2. *(i) The inequality* (2.2) *holds for any X, Y satisfying (sup) if and only if $b(\mathcal{U}^{sup}(V_\beta)) \le 0$.*

(ii) The inequality (2.2) *holds for any X, Y satisfying (sub) if and only if $b(\mathcal{U}^{sub}(V_\beta)) \le 0$.*

This theorem will be proved in a sequence of lemmas below. For the reader's convenience, let us sketch the main idea. First we deal with the implication from the right to the left. Namely, we show that any function $U \in \mathcal{U}$, where $\mathcal{U} \in \{\mathcal{U}^{sup}(V_\beta), \mathcal{U}^{sub}(V_\beta)\}$, can be appropriately approximated by a C^∞ element of the class. Next we prove that if U is smooth, then the conditions from the definition of $\mathcal{U}$ can be rewritten as some differential inequalities. Finally, to establish (2.2), we apply Ito's formula to U and use these differential estimates to control the finite variation terms. To get the reverse implication of Theorem 2.2, we provide certain non-explicit formulas for elements of $\mathcal{U}$. It turns out that they yield finite functions if (2.2) is valid.

Let us turn to the rigorous proof.

Lemma 2.3. *Let $\mathcal{U} \in \{\mathcal{U}^{sup}, \mathcal{U}^{sub}\}$ and let $U \in \mathcal{U}$ be an arbitrary function. Then for any $\kappa > 0$ there is $\overline{U} = \overline{U}^\kappa \in \mathcal{U}$, which is of class C^∞ and satisfies $b(\overline{U}) \le b(U) + \kappa$.*

Proof. The reasoning does not depend on whether $\mathcal{U} = \mathcal{U}^{sup}$ or $\mathcal{U} = \mathcal{U}^{sub}$. Let $g : \mathbb{R}^3 \to [0,\infty)$ be a C^∞ function, supported on a ball of center $(0,0,0)$ and radius 1, satisfying $\int_{\mathbb{R}^3} g = 1$ and

$$g(r,s,t) = g(r,-s,t) = g(r,s,-t) \qquad \text{for all } r,\ s \text{ and } t. \tag{2.9}$$

Let $\delta = \kappa/(5\beta) > 0$ and let $\overline{U} = \overline{U}^\kappa : D \to \mathbb{R}$ be given by

$$\overline{U}(x,y,z) = \int_{[-1,1]^3} U(x+2\delta - r\delta, y - s\delta, z + 5\delta - t\delta) g(r,s,t)\mathrm{d}r\mathrm{d}s\mathrm{d}t + \kappa.$$

We will show that this function has the desired properties. Clearly, $\overline{U}$ if of class C^∞. To prove that $\overline{U}$ satisfies (2.3), we use the fact that U enjoys this majorization and hence, by the symmetry condition (2.9),

$$\overline{U}(x,y,z) \geq y - \beta(z+5\delta) + \delta \int_{[-1,1]^3} (-s+\beta t) g(r,s,t)\mathrm{d}r\mathrm{d}s\mathrm{d}t + 5\beta\delta = y - \beta z.$$

The inequality $\overline{U}(x,y,z) \geq -y - \beta z$ is established in the same manner. The remaining properties (2.4), (2.5), (2.6) and (2.7) follow immediately from the definition of $\overline{U}$. Finally, note that if $|y| \leq x \leq z$, then

$$|y - s\delta| \leq x + 2\delta - r\delta \leq z + 5\delta - t\delta,$$

for any $r,s,t \in [-1,1]$. Thus, for such x, y, z,

$$\overline{U}(x,y,z) \leq \int_{[-1,1]^3} b(U) g(r,s,t)\mathrm{d}r\mathrm{d}s\mathrm{d}t + \kappa = b(U) + \kappa.$$

The proof is complete. □

Lemma 2.4. *If $U \in \mathcal{U}$ is of class C^∞, then we have the following.*

(i) For any $x > 0$ and $y \in \mathbb{R}$,

$$U_z(x,y,x) \leq 0. \tag{2.10}$$

(ii, $\mathcal{U} = \mathcal{U}^{sup}(V_\beta)$) For any $(x,y,z) \in D$, $0 < x \leq z$, we have

$$U_x(x,y,z) - \alpha|U_y(x,y,z)| \geq 0. \tag{2.11}$$

(ii, $\mathcal{U} = \mathcal{U}^{sub}(V_\beta)$) For any $(x,y,z) \in D$, $0 < x \leq z$, we have

$$U_x(x,y,z) + \alpha|U_y(x,y,z)| \leq 0. \tag{2.12}$$

(iii) For all $(x,y,z) \in D$ with $0 < x \leq z$ there is $c = c(x,y,z) \geq 0$ such that if h, $k \in \mathbb{R}$, then

$$U_{xx}(x,y,z)h^2 + 2U_{xy}(x,y,z)hk + U_{yy}(x,y,z)k^2 \leq c(k^2 - h^2). \tag{2.13}$$

Proof. The property (i) follows from (2.4), while (ii) is an immediate consequence of the corresponding condition (2.7) or (2.8). To show (iii), note that by (2.5),

$$U_{yy}(x,y,z) \geq 0 \qquad \text{if } 0 < x \leq z,\ y \in \mathbb{R}. \tag{2.14}$$

By (2.6), for any fixed z the function $U(\cdot,\cdot,z)$ is concave along any line segment of slope ± 1, contained in $[0,z] \times \mathbb{R}$. Therefore,

$$U_{xx}(x,y,z) \pm 2U_{xy}(x,y,z) + U_{yy}(x,y,z) \leq 0 \qquad \text{for } 0 < x \leq z,\ y \in \mathbb{R}.$$

In particular this implies that $U_{xx}(x,y,z) + U_{yy}(x,y,z) \leq 0$ and hence, by (2.14),

$$U_{xx}(x,y,z) \leq 0 \qquad \text{if } 0 < x \leq z,\ y \in \mathbb{R}. \tag{2.15}$$

Consequently, if $h,\, k\in\mathbb{R}$, then

$$\begin{aligned}
&U_{xx}(x,y,z)h^2+2U_{xy}(x,y,z)hk+U_{yy}(x,y)k^2\\
&\qquad\leq U_{xx}(x,y,z)h^2-(U_{xx}(x,y,z)+U_{yy}(x,y,z))\frac{h^2+k^2}{2}+U_{yy}(x,y)k^2\\
&\qquad=\frac{U_{yy}(x,y,z)-U_{xx}(x,y,z)}{2}(k^2-h^2).
\end{aligned}$$

Hence, by (2.14) and (2.15), (iii) holds and we are done. □

Now we are able to show the first part of Theorem 2.2.

Lemma 2.5. *Let $\mathcal{U}\in\{\mathcal{U}^{sup}(V_\beta),\mathcal{U}^{sub}(V_\beta)\}$. The inequality $b(\mathcal{U})\leq 0$ implies* (2.2) *for all X, Y satisfying the corresponding assumption (sup) or (sub).*

Proof. We will only prove the assertion in the supermartingale setting; the submartingale case can be handled in a similar manner. Fix $\varepsilon>0$ and $t\geq 0$. We start with some reductions. First, let $X=X_0+M+A$ and $Y=Y_0+N+B$ be the Doob-Meyer decompositions of X and Y. By standard localizing procedure we may and do assume that M, N and the two stochastic integrals appearing in I_1 below are martingales. The second observation is that adding a small $\eta>0$ to X if necessary (this does not affect the subordination), we may restrict ourselves to those X, for which

$$\mathbb{P}(X_s>0 \text{ for all } s\geq 0)=1.$$

Introduce the process $Z_s=(X_s,Y_s,X_s^*\vee\varepsilon)$, $s\in[0,t]$. By the definition of $b(\mathcal{U})$ and Lemma 2.3, for any $\kappa>0$ there is a C^∞ function $U=U^\kappa\in\mathcal{U}$ satisfying $b(U)<b(\mathcal{U})+\kappa\leq\kappa$. The main step of the proof is to show that $\mathbb{E}U(Z_t)\leq b(U)$ for all t. To do this, apply Itô's formula to U and Z to obtain

$$U(Z_t)=I_0+I_1+I_2+I_3+\frac{1}{2}I_4, \tag{2.16}$$

where

$$\begin{aligned}
I_0&=U(Z_0)=U(X_0,Y_0,X_0\vee\varepsilon)\leq b(U),\\
I_1&=\int_{0+}^t U_x(Z_s)dM_s+\int_{0+}^t U_y(Z_s)dN_s,\\
I_2&=\int_{0+}^t U_x(Z_s)dA_s+\int_{0+}^t U_y(Z_s)dB_s,\\
I_3&=\int_{0+}^t U_z(Z_s)d(X^*\vee\varepsilon)_s,\\
I_4&=\int_{0+}^t U_{xx}(Z_s)d[X,X]_s+2\int_{0+}^t U_{xy}(Z_s)d[X,Y]_s+\int_{0+}^t U_{yy}(Z_s)d[Y,Y]_s.
\end{aligned}$$

Next we observe that $\mathbb{E}I_1=0$, by the properties of stochastic integrals. Furthermore, we have $I_2\leq 0$, by α-subordination and property (ii) from Lemma 2.4. Indeed, $-A$ and $-\alpha A-|B|$ are nondecreasing (recall that we study the supermartingale setting), so

$$I_2\leq\int_{0+}^t U_x(Z_s)dA_s+\int_{0+}^t|U_y(Z_s)|d|B|_s\leq\int_{0+}^t(U_x(Z_s)-\alpha|U_y(Z_s)|)dA_s\leq 0.$$

The next step is to observe that $I_3\leq 0$, in virtue of (2.10). To see this, note that the support of $d(X^*\vee\varepsilon)_s$ is concentrated on the set $\{s: X_s=X_s^*\geq\varepsilon\}$, on which U_z is nonpositive. It remains to deal with the term I_4; it is done by standard approximation argument, see e.g. Wang [19]. Specifically, let $0 \leq s_0 < s_1 \leq t$. For any $j \geq 0$, let $(\eta_i^j)_{1 \leq i \leq i_j}$ be a sequence of nondecreasing finite stopping times with $\eta_0^j \equiv s_0$, $\eta_{i_j}^j \equiv s_1$ satisfying the condition $\lim_{j\to\infty} \max_{1\leq i\leq i_j-1} |\eta_{i+1}^j - \eta_i^j| = 0$. Keeping j fixed, we apply, for each $i = 0, 1, 2, \ldots, i_j$, the property (iii) from Lemma 2.4 to $x = X_{s_0}$, $y = Y_{s_0}$, $z = X_{s_0}^* \vee \varepsilon$ and $h = h_i^j = X_{\eta_{i+1}^j} - X_{\eta_i^j}$, $k = k_i^j = Y_{\eta_{i+1}^j} - Y_{\eta_i^j}$. Summing the obtained $i_j + 1$ inequalities and letting $j \to \infty$ yields

$$\begin{aligned} U_{xx}(Z_{s_0})[X,X]_{s_0}^{s_1} + 2U_{xy}(Z_{s_0})[X,Y]_{s_0}^{s_1} + U_{yy}(Z_{s_0})[Y,Y]_{s_0}^{s_1} \\ \leq c(Z_{s_0})([Y,Y]_{s_0}^{s_1} - [X,X]_{s_0}^{s_1}) \leq 0, \end{aligned}$$

in virtue of differential subordination. Here we have used the notation $[S,T]_{s_0}^{s_1} = [S,T]_{s_1} - [S,T]_{s_0}$. Thus $I_4 \leq 0$, simply by approximating this term by discrete sums. Plugging all the above facts about I_i's into (2.16) gives $\mathbb{E}U(Z_t) \leq b(U)$. In view of (2.3) and the inequality $b(U) \leq \kappa$ (which is guaranteed by the choice of U), we obtain

$$\mathbb{E}V_\beta(X_t, Y_t, X_t^* \vee \varepsilon) \leq \kappa.$$

Since $\kappa > 0$ was arbitrary, (2.2) follows. $\square$

We turn to the reverse implication of Theorem 2.2. Let us first distinguish certain classes of processes. Namely, for a fixed $\alpha \geq 0$, $x \geq 0$ and $y \in \mathbb{R}$, let $\mathcal{I}_\alpha^{sup}(x,y)$ (respectively, $\mathcal{I}_\alpha^{sub}(x,y)$) denote the class of all pairs (X,Y) of bounded Itô processes of the form (1.3), satisfying the following properties:

a) $(X_0, Y_0) = (x, y)$,
b) X is a nonnegative supermartingale (resp., nonnegative submartingale),
c) $|\zeta_s| = |\phi_s|$ and $|\xi_s| = \alpha|\psi_s|$ for all $s \geq 0$.

Here the filtration may vary, as well as the probability space. Since X is bounded, the limit $X_\infty = \lim_{t\to\infty} X_t$ exists almost surely. It is not difficult to show that the condition c) implies that $Y_\infty = \lim_{t\to\infty} Y_t$ exists with probability 1 as well. If $\alpha = 1$, this follows from the escape inequalities of Theorem 5.2 and Theorem 9.2 in [9]; for other values of the parameter α the reasoning is similar.

Let us introduce the functions U^{sup}, $U^{sub} : D \to (-\infty, \infty]$ by the formulas

$$U^{sup}(x,y,z) = \sup\{\mathbb{E}V_\beta(X_\infty, Y_\infty, X^* \vee z) : (X,Y) \in \mathcal{I}_\alpha^{sup}(x,y)\} \tag{2.17}$$

and, similarly,

$$U^{sub}(x,y,z) = \sup\{\mathbb{E}V_\beta(X_\infty, Y_\infty, X^* \vee z) : (X,Y) \in \mathcal{I}_\alpha^{sub}(x,y)\}. \tag{2.18}$$

Now we are ready to complete the proof of Theorem 2.2. In fact, we state here a stronger fact, which will allow us to claim that the best constant in (2.2) is already the best possible if we restrict ourselves to Itô processes.

Lemma 2.6. *Let $\beta > 0$ be fixed.*

(i) Suppose that inequality (2.2) *holds for any $\varepsilon > 0$ and any Itô processes* (1.3) *which satisfy (sup). Then $U^{sup} \in \mathcal{U}^{sup}(V_\beta)$ and $b(U^{sup}) \leq 0$.*

(ii) Suppose that inequality (2.2) *holds for any $\varepsilon > 0$ and any Itô processes* (1.3) *which satisfy (sub). Then $U^{sub} \in \mathcal{U}^{sub}(V_\beta)$ and $b(U^{sub}) \leq 0$.*

Proof. As previously, we focus on the supermartingale setting; the reasoning in the submartingale case is the same. For simplicity, we will write $\mathcal{I}(x,y)$ instead of $\mathcal{I}^{sup}_\alpha(x,y)$, and U instead of U^{sup}.

The first thing which needs to be checked is the finiteness of U. This is straightforward: if $(x,y,z)\in D$ and $(X,Y)\in\mathcal{I}(x,y)$, then $Y-y$ is α-subordinate to X; hence, by the triangle inequality and (2.2),

$$\mathbb{E}V_\beta(X_t,Y_t,X^*\vee z)\leq |y|+\mathbb{E}V_\beta(X_t,Y_t-y,X^*\vee z)\leq |y|.$$

Since $(X,Y)\in\mathcal{I}(x,y)$ and $t\geq 0$ were arbitrary, we obtain $U^{sup}(x,y)\leq|y|<\infty$. Let us now verify that U satisfies the properties from the definition of $\mathcal{U}^{sup}(V_\beta)$. To show (2.3), it suffices to note that the constant pair (x,y) belongs to the class $\mathcal{I}(x,y)$. The condition (2.4) follows directly from the definition of U and the fact that V_β also has this property. To prove (2.5), fix (x,y_1,z), $(x,y_2,z)\in D$, $\lambda\in(0,1)$ and let $y=\lambda y_1+(1-\lambda)y_2$. Take (X,Y) from $\mathcal{I}(x,y)$. Since (X,y_1-y+Y) and (X,y_2-y+Y) belong to $\mathcal{I}(x,y_1)$ and $\mathcal{I}(x,y_2)$, respectively, we have, by the triangle inequality,

$$\begin{aligned}\mathbb{E}V_\beta(X_\infty,Y_\infty,X^*\vee z)&\leq \lambda\mathbb{E}V_\beta(X_\infty,y_1-y+Y_\infty,X^*\vee z)\\&\quad+(1-\lambda)\mathbb{E}V_\beta(X_\infty,y_2-y+Y_\infty,X^*\vee z)\\&\leq \lambda U(x,y_1,z)+(1-\lambda)U(x,y_2,z).\end{aligned}$$

Now it suffices to take supremum over X and Y and (2.5) follows.

We turn to (2.6). In fact, we will prove the following stronger version, which allows one of t_i's to take values larger than $z-x$. Namely, for all $\varepsilon\in\{-1,1\}$, λ_1, $\lambda_2\in(0,1)$, $x\in[0,z]$, $y\in\mathbb{R}$, $z>0$ and $t_1<0<t_2$ such that $\lambda_1+\lambda_2=1$, $\lambda_1t_1+\lambda_2t_2=0$ and $x+t_1\geq 0$,

$$\begin{aligned}U(x,y,z)&\geq \lambda_1\big[U(x+t_1,y+\varepsilon t_1,z)-\beta(x+t_2-z)_+\big]\\&\quad+\lambda_2U(x+t_2,y+\varepsilon t_2,z).\end{aligned}\tag{2.19}$$

This more general statement will be needed in the proof of the optimality of the constant $\beta(possub)$. We will prove (2.19) only for $\varepsilon=1$, the argumentation for $\varepsilon=-1$ is similar. Let $x_i=x+t_i$, $y_i=y+t_i$ and take two pairs (X^1,Y^1) and (X^2,Y^2) from $\mathcal{I}(x_1,y_1)$ and $\mathcal{I}(x_2,y_2)$, respectively. Let ψ^i, ϕ^i, ζ^i and ξ^i denote the corresponding predictable processes in the decompositions of X^i and Y^i. We may and do assume that these processes are given on the same probability space equipped with the same filtration and are driven by the same Brownian motion W. Enlarging the probability space if necessary, we may assume that there is a Brownian motion B starting from x, which is independent of W. It will be used to "glue" (X^1,Y^1) and (X^2,Y^2) into one Itô process (X,Y). Precisely, introduce the stopping time $\tau=\inf\{t:B_t\in\{x_1,x_2\}\}$ and set

$$X_t=\begin{cases}B_t & \text{if } t\leq\tau,\\ X^i_{t-\tau} & \text{if } t>\tau \text{ and } B_\tau=x_i,\end{cases}$$

and

$$Y_t=\begin{cases}y-x+B_t & \text{if } t\leq\tau,\\ Y^i_{t-\tau} & \text{if } t>\tau \text{ and } B_\tau=x_i.\end{cases}$$

Then $(X,Y) \in \mathcal{I}(x,y)$, with

$$\phi_t = \begin{cases} 1 & \text{if } t \le \tau, \\ \phi^i_{t-\tau} & \text{if } t > \tau \text{ and } B_\tau = x_i, \end{cases} \qquad \psi_t = \begin{cases} 0 & \text{if } t \le \tau, \\ \psi^i_{t-\tau} & \text{if } t > \tau \text{ and } B_\tau = x_i \end{cases}$$

and

$$\zeta_t = \begin{cases} 1 & \text{if } t \le \tau, \\ \zeta^i_{t-\tau} & \text{if } t > \tau \text{ and } B_\tau = x_i, \end{cases} \qquad \xi_t = \begin{cases} 0 & \text{if } t \le \tau, \\ \xi^i_{t-\tau} & \text{if } t > \tau \text{ and } B_\tau = x_i. \end{cases}$$

We have, with probability 1,

$$Y_\infty = Y^1_\infty 1_{\{B_\tau = x_1\}} + Y^2_\infty 1_{\{B_\tau = x_2\}} \tag{2.20}$$

and, since $x_1 < x < x_2$,

$$\begin{aligned} (X^* \vee z)1_{\{B_\tau = x_1\}} &\le (X^{1*} \vee x_2 \vee z)1_{\{B_\tau = x_1\}} \\ &\le \big[(X^{1*} \vee z) + (x_2 - z)_+\big]1_{\{B_\tau = x_1\}}, \\ (X^* \vee z)1_{\{B_\tau = x_2\}} &= (X^{2*} \vee z)1_{\{B_\tau = x_2\}}. \end{aligned} \tag{2.21}$$

Therefore, we get

$$\begin{aligned} U(x,y,z) &\ge \mathbb{E}|Y_\infty| - \beta\mathbb{E}(X^* \vee z) \\ &\ge -\beta(x_2 - z)_+\mathbb{P}(B_\tau = x_1) + \sum_{i=1}^2 \big(\mathbb{E}|Y^i_\infty| - \beta\mathbb{E}(X^{i*} \vee z)\big)\mathbb{P}(B_\tau = x_i) \\ &= -\lambda_1\beta(x_2 - z)_+ + \sum_{i=1}^2 \lambda_i(\mathbb{E}|Y^i_\infty| - \beta\mathbb{E}(X^{i*} \vee z)). \end{aligned}$$

Now take supremum on the right-hand side over the classes $\mathcal{I}(x_1,y_1)$ and $\mathcal{I}(x_2,y_2)$ to obtain (2.19).

Next we establish (2.7). Take $(X,Y) \in \mathcal{I}(x-d, y-\alpha d)$ and consider the process (X',Y') defined by the formula

$$(X',Y')_t = \begin{cases} (x-t, y-\alpha t) & \text{if } t \le d, \\ (X_{t-d}, Y_{t-d}) & \text{if } t > d, \end{cases}$$

with respect to the filtration $(\mathcal{F}'_t) = (\mathcal{F}_{(t-d)\vee 0})$. It is easy to see that $(X',Y') \in \mathcal{I}(x,y)$, $(X_\infty, Y_\infty) = (X'_\infty, Y'_\infty)$ and $(X')^* \vee z = X^* \vee z$, so

$$\mathbb{E}V_\beta(X_\infty, Y_\infty, X^* \vee z) = \mathbb{E}V_\beta(X'_\infty, Y'_\infty, (X')^*_\infty) \le U(x,y,z).$$

Take supremum over (X,Y) to get $U(x-d, y-\alpha d, z) \le U(x,y,z)$. Finally, we will show that $b(U^{sup}) \le 0$. Suppose that the point $(x,y,z) \in D$ satisfies $|y| \le x \le z$. The first inequality implies that for any $(X,Y) \in \mathcal{I}(x,y)$, the process Y is α-subordinate to X, so

$$\mathbb{E}V_\beta(X_t, Y_t, X^*_t \vee z) \le 0.$$

In consequence, $b(U^{sup}) \le 0$ and the proof is complete. □

We end this section proving two additional properties of U^{sup} and U^{sub}. First, these functions are symmetric with respect to the variable y, that is,

$$U^{sup}(x,y,z) = U^{sup}(x,-y,z), \quad U^{sub}(x,y,z) = U^{sub}(x,-y,z) \qquad \text{on } D. \tag{2.22}$$

This follows immediately from the equality $V_\beta(x,y,z) = V_\beta(x,-y,z)$ and the fact that $(X,Y) \in \mathcal{I}(x,y)$ if and only if $(X,-Y) \in \mathcal{I}(x,-y)$. The second property is that for any $z > 0$ and $x \in [0, z]$, $U^{sup}(x, \cdot, z)$ and $U^{sub}(x, \cdot, z)$ are 1-Lipschitz. Indeed, if y_1, $y_2 \in \mathbb{R}$ and $(X, Y) \in \mathcal{I}_\alpha^{sup}(x, y_1)$, then $(X, Y - y_1 + y_2) \in \mathcal{I}_\alpha^{sup}(x, y_2)$, so, by the triangle inequality,

$$\mathbb{E}V_\beta(X_\infty, Y_\infty, X^* \vee z) \le |y_2 - y_1| + \mathbb{E}V_\beta(X_\infty, Y_\infty, X^* \vee z) \le |y_2 - y_1| + U^{sup}(x, y_2, z).$$

Taking supremum over (X, Y) gives

$$U^{sup}(x, y_1, z) - U^{sup}(x, y_2, z) \le |y_1 - y_2|$$

and, by symmetry, $U^{sup}(x, y_2, z) - U^{sup}(x, y_1, z) \le |y_1 - y_2|$, which yields the claim. The reasoning for U^{sub} is the same.

3. Proof of Theorem 1.3

We split this section into two parts. The first of them is devoted to the proof of (1.7), while the second concerns the optimality of the constant $\beta(possup)$. Throughout, α is a fixed nonnegative number.

3.1. **Proof of** (1.7). Let

$$\beta = \beta(possup) = \alpha + 1 + ((2\alpha + 1)e)^{-1}.$$

By Theorem 2.2, we see that it suffices to construct a function $U \in \mathcal{U}^{sup}(V_\beta)$ satisfying the condition $b(U) \le 0$. To do this, let us consider the following subsets of $[0, 1] \times \mathbb{R}$:

$$\begin{aligned} D_1 &= \left\{(x, y) : x \le \frac{\alpha + 1}{2\alpha + 1} \le x + |y|\right\}, \\ D_2 &= \left\{(x, y) : x > \frac{\alpha + 1}{2\alpha + 1},\ |y| - x \ge -\frac{\alpha + 1}{2\alpha + 1}\right\}, \\ D_3 &= ([0, 1] \times \mathbb{R}) \setminus (D_1 \cup D_2). \end{aligned}$$

First we introduce an auxiliary function $u : [0, 1] \times \mathbb{R} \to \mathbb{R}$, given by

$$u(x, y) = \begin{cases} \alpha x + |y| + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x + |y| - \frac{\alpha+1}{2\alpha+1}\right)\right] x - \beta & \text{on } D_1, \\ \alpha x + |y| + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x + |y| + \frac{\alpha+1}{2\alpha+1}\right)\right]\left(\frac{2\alpha+2}{2\alpha+1} - x\right) - \beta & \text{on } D_2, \\ \frac{2\alpha+1}{2\alpha+2}(|y|^2 - x^2) + (\alpha + 1)x + \frac{\alpha+1}{2(2\alpha+1)} - \beta & \text{on } D_3. \end{cases}$$

It can be easily verified that u is of class C^1 in the interior of the strip $[0, 1] \times \mathbb{R}$: simply check that the partial derivatives match at the common boundaries of D_1, D_2 and D_3. The desired function U is defined by

$$U(x, y, z) = (x \vee z)u\left(\frac{x}{x \vee z}, \frac{y}{x \vee z}\right), \qquad (x, y, z) \in D. \tag{3.1}$$

Lemma 3.1. *The function U satisfies the properties* (2.3)–(2.7) *and* $b(U) \le 0$.

Proof. We start with the property (2.5), which is equivalent to the convexity of $u(x, \cdot)$ for any fixed $x \in [0, 1]$. However, this is immediate from the formulas for u on D_1, D_2, D_3 and the fact that u is of class C^1. Next we turn to (2.6), which amounts to saying that u is diagonally concave on in its domain. Since u is of class C^1, this can we rephrased in the form

$$2|u_{xy}(x, y)| + \Delta u(x, y) \le 0 \tag{3.2}$$

for all (x,y) lying in the interior of D_1, D_2 or D_3. It is straightforward to check that in each of these three cases both sides are equal. Now let us verify the majorization (2.3). By homogeneity of U and V_β, it is equivalent to

$$u(x,y) \geq |y| - \beta \qquad \text{for } x \in [0,1],\ y \in \mathbb{R}. \tag{3.3}$$

By the diagonal concavity of u, it suffices to prove this inequality for $x \in \{0,1\}$. If $x = 0$ and $|y| \geq (\alpha+1)/(2\alpha+1)$, we get equality. For $x = 0$ and remaining y's, the estimate can be transformed into

$$\left(|y| - \frac{\alpha+1}{2\alpha+1}\right)^2 \geq 0.$$

If $x = 1$ and $|y| \geq \alpha/(2\alpha+1)$, then the inequality reads

$$\alpha + \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(|y| - \frac{\alpha}{2\alpha+1}\right)\right]\frac{1}{2\alpha+1} \geq 0,$$

which is obvious. Finally, for $x = 1$ and $|y| \leq \alpha/(2\alpha+1)$, (3.3) takes the form

$$\frac{2\alpha+1}{2\alpha+2}\left(|y| - \frac{\alpha+1}{2\alpha+1}\right)^2 + \alpha + \frac{1}{2\alpha+2} \geq 0,$$

which is also true. The next step is to prove (2.4). Clearly, it suffices to show that $U_z(x,y,z) \leq 0$ for $0 < x < z$ and $y \in \mathbb{R}$. In terms of u, this can be rewritten as

$$u(x,y) - xu_x(x,y) - yu_y(x,y) \leq 0 \qquad \text{on } (0,1) \times \mathbb{R}. \tag{3.4}$$

If $(x,y) \in D_1$, the inequality takes the form

$$x \cdot \frac{2\alpha+1}{\alpha+1}(x+|y|)\exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y| - \frac{\alpha+1}{2\alpha+1}\right)\right] - \beta \leq 0,$$

which follows from $x \leq 1 \leq \beta$ and an elementary bound $s\exp(-s+1) \leq 1$, applied to $s = (2\alpha+1)(x+|y|)/(\alpha+1)$. For $(x,y) \in D_2$, the desired estimate reads

$$\exp(-s-1)\left[s\left(\frac{2\alpha+2}{2\alpha+1} - x\right) + \frac{2\alpha+2}{2\alpha+1}\right] - \beta \leq 0, \tag{3.5}$$

where $s = (2\alpha+1)(-x+|y|)/(\alpha+1) \geq -1$. To prove it, note that if $s \leq 0$, then the left hand side does not exceed

$$\exp(-s-1)\left[\frac{s}{2\alpha+1} + \frac{2\alpha+2}{2\alpha+1}\right] - \beta,$$

which, as a function of s, is nonincreasing on $[-1,0]$ and equal to $1-\beta \leq 0$ for $s = -1$. On the other hand, if $s \geq 0$, the left hand side of (3.5) is not larger than

$$\frac{\alpha+1}{2\alpha+1}\exp(-s-1)(s+2) - \beta,$$

which is a nonincreasing function of $s \in [0,\infty)$, equal to $\frac{2\alpha+2}{2\alpha+1}e^{-1} - \beta < 0$. Finally, if $(x,y) \in D_3$, then (3.4) becomes

$$-\frac{2\alpha+1}{2\alpha+2}(y^2 - x^2) + \frac{\alpha+1}{2(2\alpha+1)} - \beta \leq 0.$$

It suffices to note that the left hand side attains its maximum for $x = 1$ and $y = 0$; this maximum is easily checked to be nonpositive.

Now let us prove (2.7). This condition is equivalent to $u_x(x,y)-\alpha|u_y(x,y)|\geq 0$ on $(0,1)\times\mathbb{R}$. A little calculation shows that $u_x(x,y)-\alpha|u_y(x,y)|$ equals

$$\begin{cases} \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y|-\frac{\alpha+1}{2\alpha+1}\right)\right]\left(1-\frac{2\alpha+1}{\alpha+1}x+\frac{2\alpha+1}{\alpha+1}\alpha x\right) & \text{on } D_1,\\ \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y|+\frac{\alpha+1}{2\alpha+1}\right)\right](2\alpha+1)(1-x) & \text{on } D_2,\\ -\frac{2\alpha+1}{\alpha+1}(x+|y|\alpha)+\alpha+1 & \text{on } D_3 \end{cases}$$

and all the expressions are easily seen to be nonnegative. Finally, we will prove that $b(U)\leq 0$: this is equivalent to

$$u(x,y)\leq 0 \qquad \text{for } |y|\leq x.$$

However, $u(x,y)\leq u(x,x)\leq u(1,1)=0$: here in the first passage we have used the convexity of $u(x,\cdot)$ and the equality $u(x,x)=u(x,-x)$, while in the second we have exploited the diagonal concavity of u and the inequality

$$\lim_{x\uparrow 1}[u_x(x,x)+u_y(x,x)]=\alpha+1-e^{-1}\geq 0.$$

The proof is complete. $\square$

Before we proceed to the optimality of the constant $\beta(possub)$, let us make here an important and interesting remark. There is a natural question whether there is only one function U satisfying the requirements of Lemma 3.1. Another problem one may think of is whether U coincides with the function U^{sup} defined by (2.17). The answer to both these questions is negative. The reason is very simple: U^{sup} satisfies $U^{sup}(0,y)=|y|-\beta$ for all $y\in\mathbb{R}$, since the class $\mathcal{I}_\alpha^{sup}(0,y)$ consists of only one pair $(X,Y)\equiv(0,y)$. However, one can say more, namely, the explicit formula for U^{sup} is the following. Let D_1, D_2, D_3 be as above and consider the sets

$$D_3^-=D_3\cap\left\{(x,y): x\leq\frac{\alpha+1}{2\alpha+1}\right\}, \qquad D_3^+=D_3\setminus D_3^-.$$

Then we have $U^{sup}(x,y,z)=(x\vee z)u^0(x/(x\vee z),y/(x\vee z))$, where $u^0=u$ on $D_1\cup D_2$ and

$$u^0(x,y)=\begin{cases} -x\log\left[\frac{2\alpha+1}{\alpha+1}(x+|y|)\right]+(\alpha+1)x+|y|-\beta & \text{on } D_3^-,\\ c_1+c_2(1-x+|y|)^{1/(\alpha+1)}(-(\alpha+1)(1-x)+|y|) & \text{on } D_3^+. \end{cases}$$

Here the parameters c_1 and c_2 are given by

$$c_1=-\frac{2\alpha(\alpha+1)}{(2\alpha+1)(\alpha+2)}-\frac{1}{(2\alpha+1)e}$$

and

$$c_2=\frac{(2\alpha+1)^{1/(\alpha+1)}\alpha^{\alpha/(\alpha+1)}}{\alpha+2}.$$

We omit further details in this direction.

3.2. **Sharpness.** Let $\beta>0$ be fixed and suppose that we have

$$||Y||_1\leq\beta||X^*||_1$$

for any nonnegative continuous X and any Y which is α-subordinate to X. Let U be the function given by (2.17) and let $u(x,y)=U(x,y,1)$ for $x\in[0,1]$ and $y\in\mathbb{R}$.

We will also use the notation $B(y) = u(1, y)$ for $y \in \mathbb{R}$. By Lemma 2.6, U belongs to $\mathcal{U}^{sup}(V_\beta)$ and satisfies $b(U) \leq 0$. In consequence, we have

$$u(1,1) \leq b(U) \leq 0, \tag{3.6}$$

$$u(x,y) \geq |y| - \beta \quad \text{for all } x \in [0,1],\ y \in \mathbb{R}, \tag{3.7}$$

$$u(x,y) \geq u(x-d, y-\alpha d) \quad \text{for } 0 \leq x-d \leq x \leq 1 \text{ and } y \in \mathbb{R} \tag{3.8}$$

and

$$u \quad \text{is diagonally concave.} \tag{3.9}$$

Furthermore, by (2.22), u satisfies

$$u(x,y) = u(x,-y) \qquad \text{for all } x \in [0,1] \text{ and } y \in \mathbb{R}. \tag{3.10}$$

We will show that the existence of $u : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfying these properties implies $\beta \geq \beta(\text{possup})$. We consider two cases.

The case $\alpha = 0$. Here the calculations are relatively simple. Take small $\delta > 0$ (in fact, $\delta \in (0,1)$ is enough) and use (3.8) with $x = 1$, $y \in \mathbb{R}$ and $d = \delta$ to obtain

$$B(y) = u(1,y) \geq u(1-\delta, y).$$

Next apply (3.9) to get

$$u(1-\delta, y) \geq \delta u(0, y+1-\delta) + (1-\delta)u(1, y-\delta) = \delta u(0, y+1-\delta) + (1-\delta)B(y-\delta).$$

Combine the two estimates above with the following consequence of (3.7):

$$u(0, y+1-\delta) \geq (y+1-\delta) - \beta.$$

As the result, we obtain

$$B(y) \geq \delta(y+1-\delta-\beta) + (1-\delta)B(y-\delta),$$

which can be rewritten in the form

$$B(y) - (y-\beta) \geq (1-\delta)\big[B(y-\delta) - (y-\delta-\beta)\big]. \tag{3.11}$$

Write this estimate twice, with $y = \delta$ and $y = 0$:

$$\begin{aligned} B(\delta) - (\delta - \beta) &\geq (1-\delta)(B(0)+\beta), \\ B(0) + \beta &\geq (1-\delta)(B(-\delta) - (-\delta-\beta)). \end{aligned}$$

But, by (3.10), B is an even function, so $B(-\delta) = B(\delta)$. Thus, combining the above two estimates yields

$$(B(0)+\beta)(2\delta - \delta^2) \geq 2\delta(1-\delta).$$

Dividing throughout by δ and letting $\delta \to 0$ gives

$$B(0) + \beta \geq 1. \tag{3.12}$$

Now we come back to (3.11). By induction, we get, for any integer N,

$$B(y) - (y-\beta) \geq (1-\delta)^N \big[B(y-N\delta) - (y-N\delta-\beta)\big].$$

Let $y = 1$ and $\delta = 1/N$. If we pass with N to infinity and use (3.12), we get

$$B(1) - (1-\beta) \geq e^{-1}(B(0)+\beta) \geq e^{-1}.$$

It suffices to apply (3.6) to get $\beta \geq 1 + e^{-1}$, as claimed.

The case $\alpha > 0$. Here the calculations are more involved. For $y \geq \alpha/(2\alpha+1)$, denote

$$C(y) = u\left(\frac{\alpha+1}{2\alpha+1}, y - \frac{\alpha}{2\alpha+1}\right).$$

It is convenient to split the proof into a few intermediate parts. Throughout, δ is a sufficiently small positive number.

Step 1. We will show that for any $y \geq \alpha/(2\alpha+1)$,

$$B(y) \geq \frac{\delta(2\alpha+1)}{\alpha} C(y+(\alpha+1)\delta) + \frac{\alpha - \delta(2\alpha+1)}{\alpha} B(y+(\alpha+1)\delta). \tag{3.13}$$

To prove this, note that by (3.8) we have

$$B(y) = u(1,y) \geq u(1-\delta, y+\alpha\delta)$$

and, by (3.9),

$$\begin{aligned} u(1-\delta, y+\alpha\delta) \geq & \frac{\delta(2\alpha+1)}{\alpha} u\left(\frac{\alpha+1}{2\alpha+1}, y - \frac{\alpha}{2\alpha+1} + (\alpha+1)\delta\right) \\ & + \frac{\alpha-\delta(2\alpha+1)}{\alpha} u(1, y+(\alpha+1)\delta) \\ = & \frac{\delta(2\alpha+1)}{\alpha} C(y+(\alpha+1)\delta) + \frac{\alpha-\delta(2\alpha+1)}{\alpha} B(y+(\alpha+1)\delta). \end{aligned}$$

Combining these two facts yields (3.13).

Step 2. Next we show that for $y \geq \alpha/(2\alpha+1)$,

$$\begin{aligned} C(y+(\alpha+1)\delta) \geq & \frac{\delta(2\alpha+1)}{2+\delta(2\alpha+1)}\left(y + \frac{1}{2\alpha+1} - \beta\right) \\ & + \frac{(\alpha+1)(2\alpha+1)\delta}{\alpha(2+\delta(2\alpha+1))} B(y) + \frac{2\alpha - (\alpha+1)(2\alpha+1)\delta}{\alpha(2+\delta(2\alpha+1))} C(y). \end{aligned} \tag{3.14}$$

The proof is similar to that of (3.13). By (3.9), we have

$$C(y+(\alpha+1)\delta) \geq \frac{\delta(2\alpha+1)}{2+\delta(2\alpha+1)} u\left(0, y+\frac{1}{2\alpha+1} + (\alpha+1)\delta\right) + \frac{2A}{2+\delta(2\alpha+1)},$$

where

$$A = u\left(\frac{\alpha+1}{2\alpha+1} + \frac{(\alpha+1)\delta}{2}, y - \frac{\alpha}{2\alpha+1} + \alpha\delta - \frac{(\alpha+1)\delta}{2}\right).$$

Furthermore, again by (3.9),

$$A \geq \frac{(\alpha+1)(2\alpha+1)\delta}{2\alpha} B(y) + \frac{2\alpha - (\alpha+1)(2\alpha+1)\delta}{2\alpha} C(y).$$

In addition, by (3.7),

$$u\left(0, y + \frac{1}{2\alpha+1} + (\alpha+1)\delta\right) \geq y + \frac{1}{2\alpha+1} + (\alpha+1)\delta - \beta.$$

Combining the three inequalities above gives (3.14).

Step 3. Multiply both sides of (3.14) by

$$\lambda_+ = \frac{2\alpha + 3 + \sqrt{(2\alpha+1)^2 - 4\delta(\alpha+1)(2\alpha+1)}}{2(\alpha+1)}$$

and add it to (3.13). After some lengthy but straightforward computations we get

$$\begin{aligned} C(y+(\alpha+1)\delta)-p_{\alpha,\delta}B(y+(\alpha+1)\delta) &\geq q_{\alpha,\delta}\big(C(y)-p_{\alpha,\delta}B(y)\big)\\ &\quad + r_{\alpha,\delta}\left(y+\frac{1}{2\alpha+1}-\beta\right), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} p_{\alpha,\delta} &= \frac{\alpha-\delta(2\alpha+1)}{\lambda_+\alpha-\delta(2\alpha+1)},\\ q_{\alpha,\delta} &= \frac{\lambda_+(2\alpha-(\alpha+1)(2\alpha+1)\delta)}{(2+\delta(2\alpha+1))(\lambda_+\alpha-\delta(2\alpha+1))},\\ r_{\alpha,\delta} &= \frac{\lambda_+\alpha(2\alpha+1)\delta}{(2+\delta(2\alpha+1))(\lambda_+\alpha-\delta(2\alpha+1))}. \end{aligned}$$

By induction, (3.15) gives that for any positive integer N,

$$C(y+N(\alpha+1)\delta)-p_{\alpha,\delta}B(y+N(\alpha+1)\delta)\geq I_1+I_2, \tag{3.16}$$

where

$$I_1=q_{\alpha,\delta}^N\big(C(y)-p_{\alpha,\delta}B(y)\big)$$

and

$$I_2=r_{\alpha,\delta}\sum_{k=0}^{N-1}q_{\alpha,\delta}^{N-1-k}\left(y+k(\alpha+1)\delta+\frac{1}{2\alpha+1}-\beta\right).$$

Now take $y_1>y_2\geq \alpha/(2\alpha+1)$, put $y=y_2$, $\delta=(y_1-y_2)/(N(\alpha+1))$ in (3.16), and let $N\to\infty$. One easily checks that then $p_{\alpha,\delta}\to 1/2$ and

$$q_{\alpha,\delta}=1-\delta(2\alpha+1)+o(\delta)=1-\frac{2\alpha+1}{\alpha+1}\frac{y_1-y_2}{N}+o(N^{-1}),$$

so

$$I_1\to\exp\left(-\frac{2\alpha+1}{\alpha+1}(y_1-y_2)\right)\left(C(y)-\frac{B(y)}{2}\right).$$

Furthermore, $r_{\alpha,\delta}=(2\alpha+1)\delta/2+o(\delta)$ and

$$\begin{aligned} I_2 =&\frac{(2\alpha+1)\delta}{2}\Bigg[\left(y_2+\frac{1}{2\alpha+1}-\beta\right)\frac{q_{\alpha,\delta}^N-1}{q_{\alpha,\delta}-1}\\ &\qquad + (\alpha+1)\delta\frac{(N-1)q_{\alpha,\delta}^{-2}-Nq_{\alpha,\delta}^{-1}+q_{\alpha,\delta}^{N-2}}{(q_{\alpha,\delta}-1)^2}\Bigg]+o(1)\\ \to&\frac{1}{2}\left[\exp\left(-\frac{2\alpha+1}{\alpha+1}(y_1-y_2)\right)-1\right]\left(-y_2+\beta+\frac{\alpha}{2\alpha+1}\right)+\frac{y_1-y_2}{2}. \end{aligned}$$

Putting all these facts into (3.16) and working a little bit yields

$$\begin{aligned} &C(y_1)-\frac{B(y_1)}{2}-\frac{y_1}{2}+\frac{\beta}{2}+\frac{\alpha}{2(2\alpha+1)}\\ &\quad\geq\exp\left[-\frac{2\alpha+1}{\alpha+1}(y_1-y_2)\right]\left(C(y_2)-\frac{B(y_2)}{2}-\frac{y_2}{2}+\frac{\beta}{2}+\frac{\alpha}{2(2\alpha+1)}\right). \end{aligned} \tag{3.17}$$

Step 4. Similarly, we multiply both sides of (3.14) by

$$\lambda_-=\frac{2\alpha+3-\sqrt{(2\alpha+1)^2-4\delta(\alpha+1)(2\alpha+1)}}{2(\alpha+1)},$$

add it to (3.13) and proceed as in the previous step. What we obtain is that for all $y_1 > y_2 \geq \alpha/(2\alpha+1)$,

$$C(y_1) - (\alpha+1)B(y_1) + \alpha y_1 + \frac{2\alpha^3}{2\alpha+1} + \alpha - \beta\alpha$$
$$\geq \exp\left[\frac{2\alpha+1}{2\alpha(\alpha+1)}(y_1 - y_2)\right]\left(C(y_2) - (\alpha+1)B(y_2) + \alpha y_2 + \frac{2\alpha^3}{2\alpha+1} + \alpha - \beta\alpha\right).$$

Note that this implies

$$C(y) - (\alpha+1)B(y) + \alpha y + \frac{2\alpha^3}{2\alpha+1} + \alpha - \beta\alpha \leq 0 \tag{3.18}$$

for all $y \geq \alpha/(2\alpha+1)$. Indeed, if this estimate is not valid for some y, then use the preceding inequality with $y_1 > y_2 = y$ and let $y_1 \to \infty$. As the result, we get that $C - (\alpha+1)B$ has exponential growth at infinity. However, this is impossible: both B and C are Lipschitz functions (see the end of Section 2).

Step 5. This is the final part. By (3.9) and then (3.6) we have

$$\begin{aligned} C\Big(\frac{\alpha}{2\alpha+1}\Big) &\geq \frac{\alpha+1}{2\alpha+1}B\Big(\frac{\alpha}{2\alpha+1}\Big) + \frac{\alpha}{2\alpha+1}u\Big(0, -\frac{\alpha+1}{2\alpha+1}\Big) \\ &\geq \frac{\alpha+1}{2\alpha+1}B\Big(\frac{\alpha}{2\alpha+1}\Big) + \frac{\alpha}{2\alpha+1}\Big(\frac{\alpha+1}{2\alpha+1} - \beta\Big). \end{aligned} \tag{3.19}$$

Combining this with (3.18), applied to $y = \alpha/(2\alpha+1)$, yields, after some manipulations,

$$B\Big(\frac{\alpha}{2\alpha+1}\Big) + \beta \geq \frac{1}{2(2\alpha+1)} + \frac{2\alpha+1}{\alpha}. \tag{3.20}$$

Now, use (3.17) with $y_1 = 1$ and $y_2 = \alpha/(2\alpha+1)$ and plug the two estimates above to get

$$\begin{aligned} C(1) - \frac{B(1)}{2} - \frac{1}{2} + \frac{\beta}{2} + \frac{\alpha}{2\alpha+1} &\geq e^{-1}\left(\frac{B(\frac{\alpha}{2\alpha+1}) + \beta}{2(2\alpha+1)} + \frac{\alpha(\alpha+1)}{(2\alpha+1)^2}\right) \\ &\geq (2e)^{-1}, \end{aligned}$$

or

$$C(1) - \frac{B(1)}{2} \geq \frac{1}{2} - \frac{\beta}{2} - \frac{\alpha}{2(2\alpha+1)} + (2e)^{-1}.$$

On the other hand, by (3.18) applied to $y = 1$, and by (3.6),

$$C(1) - \frac{B(1)}{2} \leq (\alpha + \frac{1}{2})B(1) - 2\alpha - \frac{2\alpha^3}{2\alpha+1} + \beta\alpha \leq -2\alpha - \frac{2\alpha^3}{2\alpha+1} + \beta\alpha.$$

Combining this with the previous inequality gives $\beta \geq \alpha + 1 + ((2\alpha+1)e)^{-1}$, as desired.

4. Proof of Theorem 1.4

4.1. **Proof of** (1.8)**.** First we introduce auxiliary parameters

$$\gamma = \left(\frac{2(\alpha+1)}{2\alpha+1}\right)^{1/2}, \qquad \overline{\gamma} = \gamma - \frac{1}{2\alpha+1}, \qquad \lambda = \frac{\gamma}{2}\exp\left(-1 + \frac{2}{\gamma}\right)$$

and let

$$\beta = \beta(\text{possub}) = \alpha + \gamma.$$

Consider the subsets D_1, D_2, D_3 of $[0,1]\times\mathbb{R}$, defined by

$$D_1 = \left\{(x,y) : x \leq \frac{\alpha}{2\alpha+1},\, x+|y| \geq \overline{\gamma}\right\},$$
$$D_2 = \left\{(x,y) : x > \frac{\alpha}{2\alpha+1},\, -x+|y| \geq \gamma - 1\right\},$$
$$D_3 = ([0,1]\times\mathbb{R}) \setminus (D_1 \cup D_2).$$

As previously, first we introduce an auxiliary function $u : [0,1]\times\mathbb{R} \to \mathbb{R}$. This time it is defined as follows. On the set D_1, put

$$u(x,y) = -\alpha x + |y| + \alpha + \lambda \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(x+|y| - \frac{\alpha}{2\alpha+1}\right)\right]\left(x + \frac{1}{2\alpha+1}\right) - \beta.$$

If $(x,y) \in D_2$, then set

$$u(x,y) = -\alpha x + |y| + \alpha + \lambda \exp\left[-\frac{2\alpha+1}{\alpha+1}\left(-x+|y| + \frac{\alpha}{2\alpha+1}\right)\right](1-x) - \beta.$$

Finally, on D_3, let

$$u(x,y) = \frac{|y|^2 - x^2 - 1}{2\gamma} - (\alpha - \frac{\alpha}{\gamma(2\alpha+1)})x.$$

One easily verifies that u is of class C^1 on $(0,1)\times\mathbb{R}$. The special function $U : D \to \mathbb{R}$ corresponding to (1.8) is given by the formula

$$U(x,y,z) = (x\vee z)u\left(\frac{x}{x\vee z}, \frac{y}{x \vee z}\right).$$

Lemma 4.1. *The function U belongs to $\mathcal{U}^{sub}(V_\beta)$ and satisfies $b(U) \leq 0$.*

Proof. This can be established exactly in the same manner as in the supermartingale setting. We omit the tedious calculations. □

In analogy to the previous section, one can ask about the explicit formula for the function U^{sub}. We have been unable to answer this question. What we could prove is that U^{sub} coincides with U defined above on the set

$$\left\{(x,y,z) \in D : \left(\frac{x}{x\vee z}, \frac{y}{x\vee z}\right) \in D_1 \cup D_2 \quad \text{or} \quad x + |y| = x \vee z\right\}.$$

Fortunately, we do not need U^{sub}: the function U is sufficient for our purposes.

4.2. **Sharpness.** As in the supermartingale case, one has to consider two possibilities: $\alpha = 0$ and $\alpha > 0$. We will focus on the second case, and leave the details of the first to the reader. So, fix positive α, β and assume that

$$||Y||_1 \leq \beta ||X^*||_1$$

for any nonnegative continuous submartingale X and any semimartingale Y which is α-subordinate to X. Let U be given by (2.18) and set $u(x,y) = U(x,y,1)$. Furthermore, let $B(y) = u(0,y)$ and $C(y) = u(\frac{\alpha}{2\alpha+1}, y - \frac{\alpha}{2\alpha+1})$ for $y \geq \alpha/(2\alpha+1)$. The function u satisfies (3.6), (3.7), (3.9) and the following analogue of (3.8):

$$(4.1) \qquad u(x,y) \geq u(x+d, y+\alpha d) \qquad \text{for } 0 \leq x \leq x+d \leq 1 \text{ and } y \in \mathbb{R}.$$

We split the proof into a few parts.

Step 1. Observe that the "reflected" function $\overline{u} : [0,1]\times\mathbb{R} \to \mathbb{R}$ given by $\overline{u}(x,y) = u(1-x,y)$ satisfies the conditions (3.6)–(3.10). In consequence, all the calculations from Subsection 3.2 are valid for this function. In particular, (3.18) yields

$$\overline{C}(y)-(\alpha+1)\overline{B}(y)+\alpha y+\frac{2\alpha^3}{2\alpha+1}+\alpha-\beta\alpha\leq 0,$$

where $\overline{B}$ and $\overline{C}$ are the corresponding restrictions of $\overline{u}$. Coming back to u, B, C just defined above, the latter inequality becomes

$$C(y)-(\alpha+1)B(y)+\alpha y+\frac{2\alpha^3}{2\alpha+1}+\alpha-\beta\alpha\leq 0. \tag{4.2}$$

Step 2. Here we will use a new argument. Applying (2.19) and the homogeneity of U, we get

$$\begin{aligned} u(1,0)&=U(1,0,1)\\ &\geq \frac{2\delta}{\gamma+2\delta}u\big(1-\frac{\gamma}{2},\frac{\gamma}{2}\big)-\frac{2\beta\delta^2}{\gamma+2\delta}+\frac{\gamma}{\gamma+2\delta}U(1+\delta,-\delta,1+\delta)\\ &=\frac{2\delta}{\gamma+2\delta}u\big(1-\frac{\gamma}{2},\frac{\gamma}{2}\big)+\frac{\gamma(1+\delta)}{\gamma+2\delta}u\big(1,\frac{\delta}{1+\delta}\big)-\frac{2\beta\delta^2}{\gamma+2\delta}. \end{aligned}\tag{4.3}$$

Moreover, by (3.9),

$$\begin{aligned} u\big(1,\frac{\delta}{1+\delta}\big)\geq{}&\frac{2\delta}{\gamma(1+\delta)+\delta}u\big(1-\frac{\gamma}{2}+\frac{\delta}{2(1+\delta)},\frac{\gamma}{2}+\frac{\delta}{2(1+\delta)}\big)\\ &+\frac{\gamma(1+\delta)-\delta}{\gamma(1+\delta)+\delta}\frac{1+2\delta}{1+\delta}u(1,0), \end{aligned}$$

and

$$\begin{aligned} u\big(1-\frac{\gamma}{2},\frac{\gamma}{2}\big)&\geq\frac{\gamma(2\alpha+1)}{2(\alpha+1)}C(\overline{\gamma})+\frac{2(\alpha+1)-\gamma(2\alpha+1)}{2(\alpha+1)}u(1,\gamma)\\ &\geq\frac{\gamma(2\alpha+1)}{2(\alpha+1)}C(\overline{\gamma})+\frac{2(\alpha+1)-\gamma(2\alpha+1)}{2(\alpha+1)}(\gamma-\beta), \end{aligned}\tag{4.4}$$

where the first passage above was allowed due to $\gamma<2(\alpha+1)/(2\alpha+1)$ and the second follows from (3.7). Plug these two estimates into (4.3) and combine the result with the following consequence of (3.9):

$$\begin{aligned} u(1-\frac{\gamma}{2}+\frac{\delta}{2(1+\delta)},\frac{\gamma}{2}+\frac{\delta}{2(1+\delta)})\geq{}&\frac{(\gamma(1+\delta)+\delta)(2\alpha+1)}{2(\alpha+1)(1+\delta)}C(\overline{\gamma})\\ &+\left(1-\frac{(\gamma(1+\delta)+\delta)(2\alpha+1)}{2(\alpha+1)(1+\delta)}\right)(\gamma-\beta). \end{aligned}$$

What we get is a rather complicated estimate of the form

$$u(1,0)\geq a_1C(\overline{\gamma})+a_2(\gamma-\beta)+a_3u(1,0)-\frac{2\beta\delta^2}{\gamma+2\delta}, \tag{4.5}$$

where the coefficients a_1, a_2 and a_3 depend on α and δ. We will not derive the explicit formulas for these; we will only need their asymptotic behavior as $\delta\to 0$:

$$a_1=\frac{2\alpha+1}{\alpha+1}\delta+o(\delta),\quad a_2=\frac{2(\alpha+1)-\gamma(2\alpha+1)}{\gamma(\alpha+1)}\delta+o(\delta)$$

and

$$a_3=1+\frac{\gamma-2}{\gamma}\delta+o(\delta).$$

These equations can be easily derived from the above estimates. Now, subtracting $u(1,0)$ from both sides of (4.5), dividing throughout by δ and letting $\delta \to 0$ gives

$$0 \geq \frac{2\alpha+1}{\alpha+1}C(\overline{\gamma}) + \frac{2(\alpha+1)-\gamma(2\alpha+1)}{\gamma(\alpha+1)}(\gamma-\beta) + \frac{\gamma-2}{\gamma}u(1,0). \tag{4.6}$$

Step 3. We use (3.9) to obtain

$$C(\overline{\gamma}) \geq \frac{\gamma(2\alpha+1)}{\gamma(2\alpha+1)+2\alpha}B(\overline{\gamma}) + \frac{2\alpha}{\gamma(2\alpha+1)+2\alpha}u(\frac{\gamma}{2}+\frac{\alpha}{2\alpha+1}, \frac{\gamma}{2}-\frac{\alpha+1}{2\alpha+1})$$

and

$$u(\frac{\gamma}{2}+\frac{\alpha}{2\alpha+1}, \frac{\gamma}{2}-\frac{\alpha+1}{2\alpha+1}) \geq \frac{2(\alpha+1)-\gamma(2\alpha+1)}{\gamma(2\alpha+1)}u(1-\frac{\gamma}{2},\frac{\gamma}{2}) + \frac{2\gamma(2\alpha+1)-2(\alpha+1)}{\gamma(2\alpha+1)}u(1,0).$$

Combining these two estimates and applying the lower bounds for $u(1-\frac{\gamma}{2},\frac{\gamma}{2})$ and $u(1,0)$ coming from (4.4) and (4.6), we obtain, after tedious, but straightforward computations,

$$\frac{(\alpha+1)(2\alpha+1)(2-\gamma)}{\alpha(2(\alpha+1)-\gamma(2\alpha+1))}(C(\overline{\gamma})-(\alpha+1)B(\overline{\gamma})) \geq (\gamma\alpha+\alpha+1)(\gamma-\beta).$$

However, $\gamma < 2(\alpha+1)/(2\alpha+1) < 2$ and, by (4.2),

$$C(\overline{\gamma})-(\alpha+1)B(\overline{\gamma}) \leq -\alpha\overline{\gamma} - \frac{2\alpha^3}{2\alpha+1} - \alpha + \beta\alpha.$$

Therefore, the preceding inequality yields

$$\beta \geq \gamma + \frac{2\alpha(2-\gamma)(\alpha+1)^2}{(\alpha+1)(2-\gamma)(2\alpha+1)+(\gamma\alpha+\alpha+1)(2(\alpha+1)-\gamma(2\alpha+1))},$$

or, after some calculation, $\beta \geq \gamma+\alpha$. This completes the proof.

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