SHARP WEIGHTED INEQUALITIES FOR HARMONIC MAXIMAL OPERATORS

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Abstract. The paper contains the proof of sharp weighted L^p inequalities for the harmonic maximal function in the dyadic context. The argumentation exploits the Bellman function technique: the estimates follow from the existence of certain special functions enjoying appropriate size conditions and concavity. The results hold true in the more general setting of probability spaces equipped with a tree-like structure.

1. Introduction

Our motivation comes from the question about boundedness of a certain important maximal operator arising in harmonic analysis, closely related to the classical dyadic maximal function. Let us start with the necessary notation and definitions. The dyadic maximal operator on $\mathbb{R}^d$, denoted by M, acts on locally integrable functions $f:\mathbb{R}^d\to\mathbb{R}$ by the formula

$$Mf(x)=\sup\left\{\langle|f|\rangle_Q : x\in Q,\, Q\subset\mathbb{R}^d \text{ is a dyadic cube}\right\}.$$

Here $\langle f\rangle_Q$ stands for $\frac{1}{|Q|}\int_Q f\mathrm{d}x$, the average of f over Q, and $|Q|$ denotes the Lebesgue measure of Q. This is a fundamental object in analysis and the theory of PDEs, and its boundedness in various function spaces has been intensively investigated and applied in various settings: see e.g. [5, 6, 7, 8, 9, 16] for an overview, consult also the references therein. We will be interested in a slightly different object, the so-called dyadic *harmonic* maximal operator $\mathcal{M}$ on $\mathbb{R}^d$, which is defined by the identity

$$\mathcal{M}f(x)=\sup\left\{\langle|f|^{-1}\rangle_Q^{-1} : x\in Q,\, Q\subset\mathbb{R}^d \text{ is a dyadic cube}\right\}.$$

Here and below, we use the convention $1/0=\infty$ and $1/\infty=0$. The joint behavior of M and $\mathcal{M}$ is similar to that of the arithmetic and the harmonic averages

$$\frac{|x_1|+|x_2|+\ldots+|x_n|}{n},\qquad \left(\frac{|x_1|^{-1}+|x_2|^{-1}+\ldots+|x_n|^{-1}}{n}\right)^{-1},$$

where x_1, x_2, …, x_n are arbitrary real numbers. In particular, we have the pointwise estimate $Mf\geq\mathcal{M}f$ on $\mathbb{R}^d$.

The harmonic maximal operators appeared for the first time in the works [2, 3, 4] in a slightly different form: the authors studied there the so-called minimal operator

$$\mathfrak{M}f(x)=\inf\left\{\langle|f|\rangle_Q \,:\, x\in Q,\, Q \text{ a dyadic cube}\right\},$$

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which is linked to $\mathcal{M}$ via the identity $\mathcal{M}f = \mathfrak{M}(|f|^{-1})^{-1}$. In a sense, the minimal operator controls f on the set where the function is small (while M controls f where the function is large). The minimal operator was used to study the fine structure of A_p weights in [2], further applications to weighted norm inequalities and differentiation theory can be found in [4].

The purpose of this paper is to study a certain class of two-weight L^p estimates for $\mathcal{M}$. Here and below, the word 'weight' refers to a nonnegative, integrable function on $\mathbb{R}^d$. Any weight w gives rise to a new measure on $\mathbb{R}^d$: with no risk of confusion, this measure is also denoted by w, and it is defined by $w(A) = \int_A w\mathrm{d}x$. The associated weighted L^p space, $0 < p < \infty$, is

$$L^p(w) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^d} |f|^p w \mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Let us discuss a few important weighted estimates for the dyadic maximal operator, which serve as the motivation for our research below. A classical result of Muckenhoupt [10] asserts that for $1 < p < \infty$, the dyadic maximal function is bounded as an operator on $L^p(w)$ if and only if the weight w belongs to the dyadic A_p class. The latter means that the A_p characteristic of w, given by

$$[w]_{A_p} := \sup \langle w \rangle_Q \langle w^{-1/(p-1)} \rangle_Q^{p-1},$$

is finite (the supremum is taken over all dyadic cubes in $\mathbb{R}^d$). There are numerous extensions and generalizations of this statement. For example, one can ask about the dependence of the norm $\|M\|_{L^p(w)\to L^p(w)}$ on the size of the characteristic $[w]_{A_p}$. More precisely, for a given $1 < p < \infty$, the problem is to find the least number $\alpha = \alpha(p)$ such that

$$\|Mf\|_{L^p(w)} \leq C_p [w]_{A_p}^{\alpha(p)} \|f\|_{L^p(w)}$$

for some C_p depending only on p. This problem was solved in the nineties by Buckley [1], who showed that the optimal exponent $\alpha(p)$ is equal to $1/(p-1)$. This result was further improved by Osękowski: the paper [15] contains, for any $1 < p < \infty$ and any $c \geq 1$, the identification of the optimal constant $C_{p,c}$ such that

$$\|M\|_{L^p(w)\to L^p(w)} \leq C_{p,[w]_{A_p}}. \tag{1.1}$$

In this paper we will study a related question for the dyadic harmonic maximal operator in the two-weight context. It follows from [4] that for any fixed $0 < p < \infty$, the operator $\mathcal{M}$ is bounded as an operator from $L^p(v)$ to $L^p(u)$ if and only if the pair (u, v) of weights satisfies

$$[u, v]_{A_{-p}} := \sup_Q \langle u \rangle_Q \langle v^{1/(p+1)} \rangle_Q^{-p-1} < \infty$$

(with the convention $0 \cdot 0^{-p-1} = 0$). Motivated by (1.1), one may ask about the optimal bound for $\|\mathcal{M}\|_{L^p(v)\to L^p(u)}$ in terms of $[u, v]_{A_{-p}}$. This interesting question is answered in the theorem below. This is one of our main results.

Theorem 1.1. *Suppose that $0 < p < \infty$. Then for any pair (u, v) of weights on $\mathbb{R}^d$ satisfying $[u, v]_{A_{-p}} < \infty$, we have*

$$\|\mathcal{M}\|_{L^p(v)\to L^p(u)} \leq \frac{(p+1)^{1+1/p}}{p} [u, v]_{A_{-p}}^{1/p}. \tag{1.2}$$

The estimate is sharp: for any $0<p<\infty$, any $c>0$ and any $\varepsilon>0$ there is a pair (u,v) with $[u,v]_{A_{-p}}\leq c$ such that

$$\|\mathcal{M}\|_{L^p(v)\to L^p(u)}>\frac{(p+1)^{1+1/p}}{p}c^{1/p}-\varepsilon.$$

Actually, we will establish the above result in the context of probability spaces equipped with a tree-like structure [6]. Here is the precise definition.

Definition 1.2. Suppose that (X,μ) is a nonatomic probability space. A set $\mathcal{T}$ of measurable subsets of X will be called a tree if the following conditions are satisfied:

(i) $X\in\mathcal{T}$ and for every $Q\in\mathcal{T}$ we have $\mu(Q)>0$.

(ii) For every $Q\in\mathcal{T}$ there is a finite subset $C(Q)\subset\mathcal{T}$ containing at least two elements such that
 (a) the elements of $C(Q)$ are pairwise disjoint subsets of Q,
 (b) $Q=\bigcup C(Q)$.

(iii) $\mathcal{T}=\bigcup_{m\geq 0}\mathcal{T}^m$, where $\mathcal{T}^0=\{X\}$ and $T^{m+1}=\bigcup_{Q\in\mathcal{T}^m}C(Q)$.

(iv) We have $\lim_{m\to\infty}\sup_{Q\in\mathcal{T}^m}\mu(Q)=0$.

An important example, which links this definition with the preceding considerations, is the cube $X=[0,1)^d$ endowed with Lebesgue measure and the tree of its dyadic subcubes. Any probability space equipped with a tree gives rise to the corresponding harmonic maximal operator $\mathcal{M}_{\mathcal{T}}$, acting on functions $f:X\to\mathbb{R}$ by

$$\mathcal{M}_{\mathcal{T}}f(x)=\sup\left\{\langle |f|^{-1}\rangle_{Q,\mu}^{-1}:x\in Q,Q\in\mathcal{T}\right\}.$$

Here $\langle f\rangle_{Q,\mu}=\frac{1}{\mu(Q)}\int_Q f\mathrm{d}\mu$ is the average of f over Q with respect to μ. In analogy to the dyadic setting described above, if (u,v) is a pair of weights on X, we define

$$[u,v]_{A_{-p}}:=\sup_{Q\in\mathcal{T}}\langle u\rangle_{Q,\mu}\langle v^{1/(p+1)}\rangle_{Q,\mu}^{-p-1}<\infty.$$

Furthermore, for $0<p<\infty$, the weighted space $L^p(w)$ is given by

$$L^p(w)=\left\{f:X\to\mathbb{R}:\|f\|_{L^p(w)}=\left(\int_X|f|^p w\mathrm{d}\mu\right)^{1/p}<\infty\right\}.$$

Here is the probabilistic version of Theorem 1.1.

Theorem 1.3. *Suppose that (X,μ) is a probability space endowed with a tree structure $\mathcal{T}$. If $0<p<\infty$ and (u,v) is a pair of weights on X satisfying $[u,v]_{A_{-p}}<\infty$, then we have*

$$\|\mathcal{M}_{\mathcal{T}}\|_{L^p(v)\to L^p(u)}\leq\frac{(p+1)^{1+1/p}}{p}[u,v]_{A_{-p}}^{1/p}. \tag{1.3}$$

The estimate is sharp for each individual triple $(X,\mathcal{T},\mu)$. (Here the sharpness is understood as in Theorem 1.1 above).

By restricting to the dyadic context and some standard scaling arguments (which enable to pass from $[0,1)^d$ to $\mathbb{R}^d$), we see that (1.3) implies (1.2). These scaling arguments enable to extend the estimate to more general measure spaces with some additional tree-like structure, but we will not discuss this issue further here.

The remaining part of the paper is organized as follows. The inequality (1.3) is established in the next section with the use of Bellman function method (cf.

[11, 12, 13, 14]): we extract the validity of the estimate from the existence of a certain special function, enjoying appropriate size and concavity requirements. The final part of the paper is devoted to the sharpness of (1.2) and (1.3) for an arbitrary probability space equipped with a tree structure.

2. Proof of (1.3)

Throughout this section, p is a given positive number and $(X, \mathcal{T}, \mu)$ is a fixed probability space with a tree structure. To keep the notation short, we will write $\langle f\rangle_Q$ instead of $\langle f\rangle_{Q,\mu}$: this should not lead to any confusion. For an arbitrary $c > 0$, we consider

$$\mathcal{D} = \mathcal{D}_{p,c} = \{(x, y, z) \in (0, \infty)^3 : x \leq cy^{p+1}\}$$

and let $B : \mathcal{D}_{p,c} \to \mathbb{R}$ be defined by

$$B(x, y, z) = xz^{-p} + cpz.$$

This function is a key tool in the proof of the following statement.

Theorem 2.1. *Suppose that a pair (u, v) of weights on X satisfies $[u, v]_{A_{-p}} \leq c$. Then for any $R \in \mathcal{T}$ we have*

$$\int_R \big(\mathcal{M}_{\mathcal{T}}(v^{-1/(p+1)}\chi_R)\big)^p u d\mu \leq (p + 1)[u, v]_{A_{-p}} \int_R v^{1/(p+1)} d\mu. \tag{2.1}$$

The constant $(p + 1)[u, v]_{A_{-p}}$ is the best possible.

Proof. It is convenient to split the argumentation into three parts.

Step 1. Since $R \in \mathcal{T}$, there is an integer m such that $R \in \mathcal{T}^m$. Consider the functional sequences $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$ and $(z_n)_{n\geq m}$ given by

$$x_n(\omega) = \langle u\rangle_{Q^n(\omega)}, \qquad y_n(\omega) = \left\langle v^{1/(p+1)}\right\rangle_{Q^n(\omega)}, \qquad z_n(\omega) = \min_{m\leq k\leq n} y_k(\omega),$$

where $Q^n(\omega)$ denotes the unique element of $\mathcal{T}^n$ which contains ω. There is a nice stochastic interpretation of these sequences: $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$ are martingales induced by u and $v^{1/(p+1)}$ (on the probability space $(R, \mu/\mu(R))$ with the filtration $(\sigma(\mathcal{T}^n))_{n\geq m}$), while $(z_n)_{n\geq m}$ is the 'minimal function' of $(y_n)_{n\geq m}$. Obviously, for any $n \geq m$ and any $Q \in \mathcal{T}^n$, the functions x_n, y_n and z_n are constant on Q and

$$\int_Q x_{n+1}\mathrm{d}\mu = \mu(Q)x_n|_Q, \qquad \int_Q y_{n+1}\mathrm{d}\mu = \mu(Q)y_n|_Q. \tag{2.2}$$

In addition, the sequence $(z_n)_{n\geq m}$ is nonincreasing and we have

$$\begin{aligned}\lim_{n\to\infty} z_n(\omega) &= \inf_{n\geq m} \left\langle v^{1/(p+1)}\right\rangle_{Q^n(\omega)} \\ &= \inf_{n\geq m} \left\langle v^{1/(p+1)}\chi_R\right\rangle_{Q^n(\omega)} \\ &= \inf_{n\geq 0} \left\langle v^{1/(p+1)}\chi_R\right\rangle_{Q^n(\omega)} = \mathcal{M}_{\mathcal{T}}(v^{-1/(p+1)}\chi_R)^{-1}(\omega)\end{aligned} \tag{2.3}$$

almost everywhere. Finally, by the definition of $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$, $(z_n)_{n\geq m}$ and the assumption $[u, v]_{A_{-p}} \leq c$, it follows at once that $(x_n, y_n, z_n) \in \mathcal{D}_{p,c}$.

Step 2. Now we will compose the sequences $(x_n)_{n\geq m}$, $(y_n)_{n\geq m}$ and $(z_n)_{n\geq m}$ with the Bellman function B introduced above. The purpose of this step is to prove that the sequence $\left(\int_R B(x_n,y_n,z_n)\mathrm{d}\mu\right)_{n\geq m}$ is nondecreasing. It follows from (2.2) that if $n\geq m$ and Q is an element of $\mathcal{T}^n$, then

$$\int_Q B(x_n,y_n,z_n)\mathrm{d}\mu=\mu(Q)B(x_n,y_n,z_n)|_Q=\int_Q B(x_{n+1},y_{n+1},z_n)\mathrm{d}\mu, \tag{2.4}$$

since the dependence of B on x (and y) is linear. Now, observe that

$$B(x_{n+1},y_{n+1},z_n)\geq B(x_{n+1},y_{n+1},z_{n+1}). \tag{2.5}$$

Indeed, if $z_n=z_{n+1}$, there is nothing to prove; on the other hand, if $z_n>z_{n+1}$, then necessarily $y_{n+1}=z_{n+1}<z_n$ (since $z_{n+1}=\min\{z_n,y_{n+1}\}$) and

$$\begin{aligned}B(x_{n+1},y_{n+1},z_n)-B(x_{n+1},y_{n+1},z_{n+1})&=\int_{z_{n+1}}^{z_n}B_z(x_{n+1},y_{n+1},s)\mathrm{d}s\\&=p\int_{z_{n+1}}^{z_n}\big(-x_{n+1}s^{-p-1}+c\big)\mathrm{d}s\\&\geq p\int_{z_{n+1}}^{z_n}\big(-x_{n+1}y_{n+1}^{-p-1}+c\big)\mathrm{d}s\geq 0,\end{aligned}$$

where the last estimate follows from the condition $[u,v]_{A_{-p}}\leq c$. This completes the proof of (2.5). Plugging this into (2.4) and summing over all $Q\in\mathcal{T}^n$ which are contained in R, we obtain the desired monotonicity of the sequence $\left(\int_R B(x_n,y_n,z_n)\mathrm{d}\mu\right)_{n\geq m}$.

Step 3. We are ready for the proof of (2.1). Note that

$$\int_R x_n z_n^{-p}\mathrm{d}\mu\leq\int_R B(x_n,y_n,z_n)\mathrm{d}\mu\leq\int_R B(x_m,y_m,z_m)\mathrm{d}\mu,$$

where in the second passage we have used the previous step. But $R\in\mathcal{T}^m$, so the functions x_m, y_m and z_m are constant on R; actually, $z_m=y_m$, by the very definition of z_m. Since $x_m\leq cy_m^{p+1}$ (which is due to $[u,v]_{A_{-p}}\leq c$), we get $B(x_m,y_m,z_m)=x_my_m^{-p}+cpy_m\leq cy_m+cpy_m=c(p+1)y_m$ and hence

$$\int_R B(x_m,y_m,z_m)\mathrm{d}\mu\leq\mu(R)B(x_m,y_m,y_m)\big|_R\leq c(p+1)\int_R v^{1/(p+1)}\mathrm{d}\mu.$$

On the other hand, x_n is the conditional expectation of u on $\mathcal{T}^n$, so

$$\int_R x_nz_n^{-p}\mathrm{d}\mu=\int_R z_n^{-p}u\mathrm{d}\mu\xrightarrow{n\to\infty}\int_R(\mathcal{M}_{\mathcal{T}}(v^{-1/(p+1)}\chi_R))^pu\mathrm{d}\mu,$$

by virtue of (2.3) and Lebesgue's monotone convergence theorem (recall that the sequence z_n^{-1} is nondecreasing). Putting all the above facts together, we get the desired estimate (2.1). The sharpness of this inequality will follow immediately from the sharpness of (1.3). See Remark 2.3 below. □

The second ingredient is the following Carleson embedding theorem for negative exponents, which will also be proved by Bellman function method.

Theorem 2.2. *Suppose that (u,v) is a pair of weights on X, let K be a positive constant and assume that nonnegative numbers α_Q (indexed by $Q\in\mathcal{T}$) satisfy*

$$\frac{1}{\mu(R)}\sum_{Q\subseteq R}\alpha_Q\langle v^{1/(p+1)}\rangle_Q^{-p}\leq K\langle v^{1/(p+1)}\rangle_R \tag{2.6}$$

for all $R \in \mathcal{T}$. Then for any integrable and nonnegative function f on X we have

$$\sum_{Q\in\mathcal{T}} \alpha_Q \langle f\rangle_Q^{-p} \leq K\left(\frac{p+1}{p}\right)^p \int_X f^{-p} v\, d\mu. \tag{2.7}$$

Proof. We may assume that $K = 1$, by homogeneity. Furthermore, by Fatou's lemma and Lebesgue's monotone convergence theorem, we may assume that $f > 0$ μ-almost everywhere (replacing f with $f + \varepsilon$ if necessary, and letting $\varepsilon \downarrow 0$ at the very end). Consider the functional sequences $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$, $(z_n)_{n\geq 0}$, this time given by

$$x_n(\omega) = \langle f\rangle_{Q^n(\omega)}, \qquad y_n = \langle v^{1/(p+1)}\rangle_{Q^n(\omega)}$$

and

$$z_n(\omega) = \frac{1}{\mu(Q^n(\omega))} \sum_{Q\subseteq Q^n(\omega), Q\in\mathcal{T}} \alpha_Q \langle v^{1/(p+1)}\rangle_Q^{-p}.$$

Here $Q^n(\omega)$ is the same as in the proof of the previous theorem. Note that the condition $f > 0$ implies that $x_n > 0$ for each n. Furthermore, the assumption (2.6) implies that for any n we have

$$z_n \leq y_n. \tag{2.8}$$

Next, introduce the function

$$B(x, y, z) = x^{-p}\left(y - \frac{z}{p+1}\right)^{p+1}$$

defined for all $x > 0$ and all $y \geq z \geq 0$. This function is convex: it is easy to check that the Hessian D^2B is semipositive-definite. Therefore for any $x > 0$, $y \geq z \geq 0$ and any $h > -x$, $k > -y$ and $\ell > -z$ we have

$$\begin{aligned} &B(x+h, y+k, z+\ell) \\ &\quad\geq B(x,y,z) + \frac{\partial B}{\partial x}B(x,y,z)h + \frac{\partial B}{\partial y}B(x,y,z)k + \frac{\partial B}{\partial z}B(x,y,z)\ell. \end{aligned} \tag{2.9}$$

Now we will show that the sequence $(\int_X B(x_n, y_n, z_n)\mathrm{d}\mu)_{n\geq 0}$ enjoys a certain monotonicity property. To this end, fix $n \geq 0$, $Q \in \mathcal{T}^n$ and pairwise disjoint elements Q_1, Q_2, ..., Q_m of $\mathcal{T}^{n+1}$ whose union is Q. Put $x = x_n|_Q$, $y = y_n|_Q$ and $z = z_n|_Q$. Furthermore, for any $j = 1, 2, \ldots, m$, let h_j, k_j and ℓ_j be given by $x + h_j = x_{n+1}|_{Q_j}$, $y + k_j = y_{n+1}|Q_j$ and $z + \ell_j = z_{n+1}|Q_j$. It is easy to check that

$$\sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)} h_j = \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)} k_j = 0. \tag{2.10}$$

In addition,

$$\begin{aligned} z &= \frac{1}{\mu(Q)} \sum_{R\subseteq Q,\, R\in\mathcal{T}} \alpha_R \langle v^{1/(p+1)}\rangle_R^{-p} \\ &= \frac{\alpha_Q \langle v^{1/(p+1)}\rangle_Q^p}{\mu(Q)} + \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)} \cdot \frac{1}{\mu(Q_j)} \sum_{R\subseteq Q_j,\, R\in\mathcal{T}} \alpha_R \langle v^{1/(p+1)}\rangle_R^p \\ &= \frac{\alpha_Q \langle v^{1/(p+1)}\rangle_Q^p}{\mu(Q)} + \sum_{j=1}^m \frac{\mu(Q_j)}{\mu(Q)}(z + \ell_j), \end{aligned}$$

which amounts to saying that

$$\sum_{j=1}^{m} \frac{\mu(Q_j)}{\mu(Q)} \ell_j = -\frac{\alpha_Q \langle v^{1/(p+1)} \rangle_Q^{-p}}{\mu(Q)}. \tag{2.11}$$

Let us apply (2.9), with $h = h_j$, $k = k_j$ and $\ell = \ell_j$, multiply throughout by $\mu(Q_j)/\mu(Q)$ and sum the obtained estimates over j. By (2.10) and (2.11), we get

$$\sum_{j=1}^{n} \frac{\mu(Q_j)}{\mu(Q)} B(x+h_j, y+k_j, z+\ell_j) \geq B(x,y,z) - \frac{\partial B}{\partial z}(x,y,z) \cdot \frac{\alpha_Q \langle v^{1/(p+1)} \rangle_Q^{-p}}{\mu(Q)}.$$

However, we have

$$\frac{\partial B}{\partial z}(x,y,z) = -x^{-p} \left(y - \frac{z}{p+1} \right)^p \leq -\left(\frac{p}{p+1} \cdot \frac{y}{x} \right)^p,$$

where the latter bound follows from the estimate $z \leq y$. Therefore, the preceding estimate implies

$$\frac{1}{\mu(Q)} \int_Q B(x_{n+1}, y_{n+1}, z_{n+1}) \mathrm{d}\mu \geq \frac{1}{\mu(Q)} \int_Q B(x_n, y_n, z_n) \mathrm{d}\mu + \left(\frac{p}{p+1} \right)^p \frac{\alpha_Q \langle f \rangle_Q^{-p}}{\mu(Q)}.$$

Multiply both sides by $\mu(Q)$ and sum over all $Q \in \mathcal{T}^n$ to obtain

$$\int_X B(x_{n+1}, y_{n+1}, z_{n+1}) \mathrm{d}\mu \geq \int_X B(x_n, y_n, z_n) \mathrm{d}\mu + \left(\frac{p}{p+1} \right)^p \sum_{Q \in \mathcal{T}^n} \alpha_Q \langle f \rangle_Q^{-p}$$

and hence for each n we have

$$\begin{aligned} \int_X B(x_{n+1}, y_{n+1}, z_{n+1}) \mathrm{d}\mu &\geq \int_X B(x_0, y_0, z_0) \mathrm{d}\mu + \left(\frac{p}{p+1} \right)^p \sum_{Q \in \mathcal{T}^k,\, k \leq n} \alpha_Q \langle f \rangle_Q^p \\ &\geq \left(\frac{p}{p+1} \right)^p \sum_{Q \in \mathcal{T}^k,\, k \leq n} \alpha_Q \langle f \rangle_Q^p. \end{aligned}$$

It remains to note that

$$\int_X B(x_{n+1}, y_{n+1}, z_{n+1}) \mathrm{d}\mu \leq \int_X x_{n+1}^{-p} y_{n+1}^{p+1} \mathrm{d}\mu \leq \int_X f^{-p} v \mathrm{d}\mu,$$

where the last bound follows from Hölder's inequality: for any Q we have

$$\int_Q v^{1/(p+1)} \mathrm{d}\mu \leq \left(\int_Q f^{-p} v \mathrm{d}\mu \right)^{1/(p+1)} \left(\int_Q f \mathrm{d}\mu \right)^{p/(p+1)}. \quad \square$$

Proof of (1.3). Take an arbitrary pair (u,v) with $[u,v]_{A_{-p}} < \infty$ and an arbitrary integrable function f. We may assume that f is nonnegative, since the passage from f to $|f|$ does not change the L^p norm of the function and may only increase the maximal function $\mathcal{M}_{\mathcal{T}} f$. Furthermore, by a simple approximation argument, we may assume that $\varphi = f^{-1}$ is measurable with respect to a σ-algebra generated by some generation $\mathcal{T}^N$. Then we have $\mathcal{M}_{\mathcal{T}} f = \max_{Q \in \mathcal{T}^n,\, n \leq N} \langle \varphi \rangle_Q^{-1} \chi_Q$ and hence for each $\omega \in X$ there is an element $Q = Q(\omega)$ belonging to $\bigcup_{n \leq N} \mathcal{T}^n$ such that $\mathcal{M}_{\mathcal{T}} f(\omega) = \langle \varphi \rangle_Q^{-1}$. Such a Q may not be unique: in such a case we pick the set belonging to $\mathcal{T}^n$ with n as small as possible.

For any $Q \in \mathcal{T}$, take $E(Q) = \{\omega \in Q : Q(\omega) = Q\}$ and put $\alpha_Q = u(E(Q))$. We will prove that the inequality (2.1) implies (2.6) with $K = (p+1)[u,v]_{A_{-p}}$. To this end, observe that for any R we have

$$\frac{1}{\mu(R)} \sum_{Q \subset R} \alpha_Q \langle v^{1/(p+1)} \rangle_Q^{-p} = \frac{1}{\mu(R)} \int_R \sum_{Q \subseteq R} \chi_{E(Q)} \langle v^{1/(p+1)} \rangle_Q^{-p} u \mathrm{d}\mu.$$

Notice that the sets $E(Q)$ are pairwise disjoint and $E(Q) \subset Q$; therefore, from the very definition of $\mathcal{M}_\mathcal{T}$, we have the pointwise bound $\sum_{Q \subseteq R} \chi_{E(Q)} \langle v^{1/(p+1)} \rangle_Q^{-p} \leq \mathcal{M}_\mathcal{T}(v^{-1/(p+1)} \chi_R)^p$ on R and hence (2.6) follows. Consequently, (2.7) is also true and applying it to the function φ (in the place of f) gives us precisely the desired weighted bound (1.3). □

Remark 2.3. The inequality (2.1) is sharp, for each individual probability space (X, μ) with a tree $\mathcal{T}$. Indeed, otherwise we would be able to improve the constant in the estimate (1.3); however, we will see in the next section that this is impossible.

3. Sharpness

3.1. **Sharpness of** (1.3)**.** Throughout this subsection, p and c are given positive parameters and $(X, \mathcal{T}, \mu)$ is a fixed probability space with a tree. We will show that for each $\varepsilon > 0$ there is a pair (u, v) of weights on X satisfying $[u, v]_{A_{-p}} \leq c$ and

$$\|\mathcal{M}_\mathcal{T}\|_{L^p(v) \to L^p(u)} > \frac{(p+1)^{1+1/p}}{p} c^{1/p} - \varepsilon.$$

It is convenient to split the reasoning into a few parts.

Step 1. Auxiliary geometrical facts and parameters. Pick $\tilde{c} \in (0, c)$ and δ, $\eta > 0$. If δ is chosen small enough, then the line ℓ passing through the points $K = (1 - \delta, \tilde{c}(1-\delta)^{p+1})$ and $L = (1, \tilde{c})$ lies below the curve $y = cx^{p+1}$. Fix such a δ and distinguish the point

$$M = \left(1 + \eta, \tilde{c}\left(1 + \eta \cdot \frac{1 - (1-\delta)^{p+1}}{\delta}\right)\right), \tag{3.1}$$

which is easily seen to lie on ℓ. See Figure 1 below. Note that if we let $\tilde{c} \to c$, then δ converges to 0.

Step 2. Construction. Recall the following technical fact (see [6]).

Lemma 3.1. *For every $Q \in \mathcal{T}$ and every $\beta \in (0,1)$ there is a subfamily $F(Q) \subset \mathcal{T}$ consisting of pairwise disjoint subsets of Q such that*

$$\mu\left(\bigcup_{R \in F(Q)} R\right) = \sum_{R \in F(Q)} \mu(R) = \beta \mu(Q).$$

We will apply this fact inductively, to construct a certain family $A_0 \supset A_1 \supset A_2 \supset \ldots$ of measurable subsets of X. We start with setting $A_0 = X$. Next, suppose that we have successfully constructed the set A_n. Assume in addition that this set can be expressed as a union of pairwise disjoint elements of $\mathcal{T}$, which will be called the *atoms* of A_n. (Note that such decomposition holds for $n = 0$: we have $A_0 = X \in \mathcal{T}$). For each atom Q of A_n, we use Lemma 3.1 with $\beta = \eta/(\eta + \delta)$, obtaining a subfamily $F(Q)$ of subsets of Q. Then we set $A_{n+1} = \bigcup_Q \bigcup_{Q' \in F(Q)} Q'$, where the first union is taken over all atoms Q of A_n. This set has the required decomposition property:

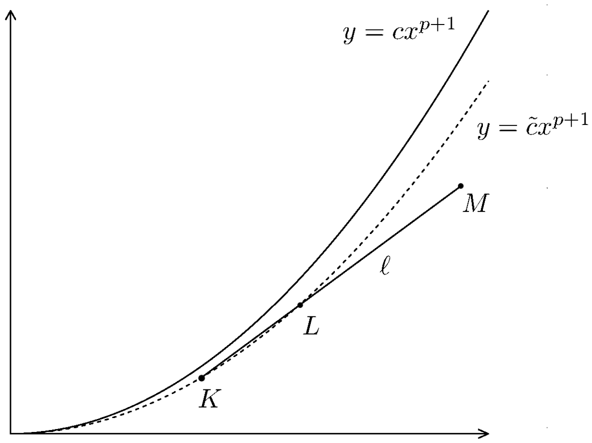

FIGURE 1. The crucial points and their geometric interpretation: $K = (1-\delta, \tilde{c}(1-\delta)^{p+1})$ and $L = (1, \tilde{c})$ lie on the curve $y = \tilde{c}x^{p+1}$, the point $M = \left(1+\eta, \tilde{c}\left(1+\eta \cdot \frac{1-(1-\delta)^{p+1}}{\delta}\right)\right)$ lies on the line ℓ.

obviously, it is a union of the family $\{F(Q) : Q \text{ is an atom of } A_n\}$, which consists of pairwise disjoint elements of $\mathcal{T}$. These elements are the atoms of A_{n+1}. The description of the induction step is complete.

It follows directly from the above construction that if Q is an atom of A_m, then for any $n \geq m$ we have

$$\mu(Q \cap A_n) = \mu(Q) \left(\frac{\eta}{\eta+\delta}\right)^{n-m}$$

and hence in particular,

$$\mu(Q \cap (A_n \setminus A_{n+1})) = \mu(Q) \left(\frac{\eta}{\eta+\delta}\right)^{n-m} \frac{\delta}{\eta+\delta}. \tag{3.2}$$

Recall the point M defined in (3.1) and denote its coordinates by M_x and M_y. Introduce the weights u, v on X by

$$u = M_y \sum_{n=0}^{\infty} (1-\delta)^{n(p+1)} \chi_{A_n \setminus A_{n+1}}, \qquad v = M_x^{p+1} \sum_{n=0}^{\infty} (1-\delta)^{n(p+1)} \chi_{A_n \setminus A_{n+1}}$$

and let $f : X \to \mathbb{R}$ be given by

$$f = \sum_{n=0}^{\infty} (1+r\delta)^{-n} \chi_{A_n \setminus A_{n+1}},$$

where r is an auxiliary parameter satisfying $-(p+1)/p < r < 0$.

Step 3. Verification of the condition $[u, v]_{A_{-p}} \leq c$. By (3.2), if Q is an atom of A_m, then

$$\begin{aligned} \langle u \rangle_Q &= M_y \sum_{n=m}^{\infty} (1-\delta)^{n(p+1)} \left(\frac{\eta}{\eta+\delta}\right)^{n-m} \frac{\delta}{\eta+\delta} \\ &= \frac{M_y \delta}{\eta+\delta-(1-\delta)^{p+1}\eta} \cdot (1-\delta)^{m(p+1)} = \tilde{c}(1-\delta)^{m(p+1)} \end{aligned} \tag{3.3}$$

and

$$\langle v^{1/(p+1)}\rangle_Q = M_x \sum_{n=m}^{\infty} (1-\delta)^n \left(\frac{\eta}{\eta+\delta}\right)^{n-m} \frac{\delta}{\eta+\delta} = (1-\delta)^m. \tag{3.4}$$

Now, suppose that R is an arbitrary element of $\mathcal{T}$. Then there is an integer m such that $R \subseteq A_{m-1}$ and $R \nsubseteq A_m$. We have

$$\langle u\rangle_R = \frac{1}{\mu(R)}\int_{R\setminus A_m} u\mathrm{d}\mu + \frac{1}{\mu(R)}\int_{R\cap A_m} u\mathrm{d}\mu.$$

But $u = M_y(1-\delta)^{(m-1)(p+1)}$ on $R\setminus A_m$; furthermore, by (3.3), applied to each atom Q of A_m contained in R, we get

$$\int_{R\cap A_m} u\mathrm{d}\mu = \mu(R\cap A_m)\cdot \tilde{c}(1-\delta)^{m(p+1)}.$$

Therefore, setting $\kappa := \mu(R\cap A_m)/\mu(R) \in [0,1]$, we rewrite the preceding equality in the form

$$\langle u\rangle_R = (1-\delta)^{(m-1)(p+1)}\Big[\kappa K_y + (1-\kappa)M_y\Big].$$

(In analogy to the above notation, K_y stands the second coordinate of the point K; the number K_x, which will appear below, is the first coordinate of this point). A similar calculation shows that

$$\langle v^{1/(p+1)}\rangle_R = (1-\delta)^{m-1}\Big[\kappa K_x + (1-\kappa)M_x\Big]$$

and therefore

$$\langle u\rangle_R \langle v^{1/(p+1)}\rangle_R^{-p-1} = \Big[\kappa K_y + (1-\kappa)M_y\Big]\Big[\kappa K_x + (1-\kappa)M_x\Big]^{-p-1}.$$

This number does not exceed c. Indeed, as κ ranges from 0 to 1, the point $\kappa K + (1-\kappa)M$ runs over the line segment KM which lies below the curve $y = c|x|^{p+1}$ (see Step 1). Since R was arbitrary, the inequality $[u,v]_{A_{-p}} \leq c$ follows.

Step 4. Completion of the proof. In the same manner as above, one verifies that if Q is an atom of A_m, then

$$\langle f^{-1}\rangle_Q = \sum_{n=m}^{\infty}(1+r\delta)^n\left(\frac{\eta}{\eta+\delta}\right)^{n-m}\frac{\delta}{\eta+\delta} = \frac{(1+r\delta)^m}{1-r\eta}.$$

This immediately yields $\mathcal{M}_{\mathcal{T}} f \geq (1-r\eta)(1+r\delta)^{-m}$ on A_m and hence, by the definition of u, v and f, we obtain

$$(\mathcal{M}_{\mathcal{T}} f)^p u \geq \frac{(1-r\eta)^p M_y}{M_x^{p+1}} f^p v \qquad \text{on } A_m \setminus A_{m+1}.$$

The latter bound does not depend on m, so we can rewrite it uniformly as

$$(\mathcal{M}_{\mathcal{T}} f)^p u \geq \frac{(1-r\eta)^p M_y}{M_x^{p+1}} f^p v \qquad \text{on } X.$$

Consequently, the constant $(1-r\eta)^p M_y/M_x^{p+1}$ is the lower bound for the norm $\|\mathcal{M}_{\mathcal{T}}\|_{L^p(v)\to L^p(u)}$, as long as we have $\|f\|_{L^p(v)} < \infty$. Let us study the latter estimate. Note that

$$\int_X f^p v\mathrm{d}\mu = (1+\eta)^{p+1}\sum_{n=0}^{\infty}(1+r\delta)^{-np}(1-\delta)^{n(p+1)}\left(\frac{\eta}{\eta+\delta}\right)^n\frac{\delta}{\eta+\delta}$$

and observe that the ratio of the above geometric series is equal to

$$(1+r\delta)^{-p}(1-\delta)^{p+1}\cdot\frac{\eta}{\eta+\delta}\leq 1-pr\delta-(p+1)\delta+o(\delta).$$

Therefore for any r as above (i.e., satisfying $r>-(p+1)/p$), any $\eta>0$ and $\tilde{c}$ sufficiently close to c (so that δ is close enough to 0) we have $\|f\|_{L^p(v)}<\infty$. Rewrite the constant $(1-r\eta)^p M_y/M_x^{p+1}$ explicitly as

$$\frac{(1-r\eta)^p M_y}{M_x^{p+1}}=\frac{(1-r\eta)^p\cdot\tilde{c}\left(1+\eta\delta^{-1}\left(1-(1-\delta)^{p+1}\right)\right)}{(1+\eta)^{p+1}}.$$

Now, we choose η to be very large, then δ is made small, and finally, we pick r close to $-(p+1)/p$. Then the above expression can be made as close to $c(p+1)\left(\frac{p+1}{p}\right)^p$ as we wish. This establishes the desired sharpness.

3.2. **Sharpness of** (1.2)**.** Let $X=[0,1)^d$, $\mu=|\cdot|$ and let $\mathcal{T}$ be the dyadic tree. For given p, c and $\tilde{c}<c$, let u, v and f be the functions on X constructed in the previous subsection. We extend these functions to the whole $\mathbb{R}^d$ by setting $u\equiv\tilde{c}$, $v\equiv 1$ and $f\equiv 0$ on $\mathbb{R}^d\setminus[0,1)^d$. Then $[u,v]_{A_{-p}(\mathbb{R}^d)}\leq c$. Indeed, let Q be an arbitrary dyadic cube in $\mathbb{R}^d$. If $Q\subseteq[0,1)^d$, then we have $\langle u\rangle_Q\langle v^{1/(p+1)}\rangle_Q^{-p-1}\leq c$, as proved in Step 3 of the previous subsection. On the other hand, if Q is disjoint from $[0,1)^d$ or contains the unit cube properly, then $\langle u\rangle_Q\langle v^{1/(p+1)}\rangle_Q^{-p-1}=\tilde{c}<c$, by the very definition of u and v (by (3.3) and (3.4), the averages of u and v over $[0,1)^d$ are equal to $\tilde{c}$ and 1, respectively; these averages do not change if we pass from $[0,1)^d$ to Q). It remains to note that $\|\mathcal{M}f\|_{L^p(\mathbb{R}^d;u)}\geq\|\mathcal{M}_{\mathcal{T}}f\|_{L^p([0,1)^d;u)}$ and $\|f\|_{L^p(\mathbb{R}^d;v)}=\|f\|_{L^p([0,1)^d;v)}$ to get $\|\mathcal{M}\|_{L^p(\mathbb{R}^d;v)\to L^p(\mathbb{R}^d;u)}\geq\|\mathcal{M}_{\mathcal{T}}\|_{L^p(v)\to L^p(u)}$, which yields the claim.

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