SHARP WEAK TYPE ESTIMATES FOR RIESZ TRANSFORMS

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ABSTRACT. Let d be a given positive integer and let $\{R_j\}_{j=1}^d$ denote the collection of Riesz transforms on $\mathbb{R}^d$. For $1 < p < \infty$, we determine the best constant C_p such that the following holds. For any locally integrable function f on $\mathbb{R}^d$ and any $j \in \{1, 2, \ldots, d\}$,

$$||(R_j f)_+||_{L^{p,\infty}(\mathbb{R}^d)} \leq C_p ||f||_{L^{p,\infty}(\mathbb{R}^d)}.$$

A related statement for Riesz transforms on spheres is also established. The proofs exploit Gundy-Varopoulos representation of Riesz transforms and appropriate inequality for orthogonal martingales.

1. INTRODUCTION

In recent years, there has been a considerable interest in finding the exact values of norms of various singular integral operators and Fourier multipliers. One of the motivations for this direction of research comes from the fact that good estimates for the L^p norm of the Riesz transforms on $\mathbb{R}^d$ and the Beurling-Ahlfors operator on $\mathbb{C}$ have important consequences in the study of quasiconformal mappings and related nonlinear geometric PDEs (cf. [8], [12]). The purpose of this paper is to continue this line of research and investigate the action of Riesz transforms on weak spaces $L^{p,\infty}$.

Let us start with recalling some related results from the literature. The first paper we mention is that of Pichorides [19], who identified the norm of the Hilbert transform as an operator on $L^p(\mathbb{R})$, $1 < p < \infty$. Recall that the Hilbert transform $\mathcal{H}$ on the line is the operator defined by the principal value integral

$$(1.1) \qquad \mathcal{H}f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-y)}{y} \mathrm{d}y.$$

Pichorides' result asserts that

$$||\mathcal{H}||_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} = \cot\left(\frac{\pi}{2p^*}\right), \qquad 1 < p < \infty,$$

where $p^* = \max\{p, p/(p-1)\}$. This statement has been extended to the higher-dimensional setting by Iwaniec and Martin [13] and, independently, by Bañuelos and Wang [3]. Suppose that $d \geq 2$ is a given integer. The counterpart of the Hilbert transform in $\mathbb{R}^d$ is the collection of Riesz transforms $(R_j)_{j=1}^d$ (see e.g. Stein [21]).

This family of operators is given by

$$(1.2) \qquad R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x - y|^{d+1}} f(y) \mathrm{d}y, \qquad j = 1, 2, \ldots, d,$$

where the integrals, as in (1.1), are supposed to exist in the sense of Cauchy principal values. The aforementioned result of Iwaniec and Martin [13] and Bañuelos and Wang [3] is the identification of L^p norms of Riesz transforms:

$$||R_j||_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} = \cot\left(\frac{\pi}{2p^*}\right), \qquad 1 < p < \infty, \ 1 \leq j \leq d.$$

Thus, the norms do not change when the dimension d increases. This result has been extended in numerous directions; the literature on this subject is very large, we only mention here the sharp weak-type bounds of Davis [6] and Janakiraman [14], and optimal logarithmic estimates due to the author [17].

In the present paper we will be interested in the action of Riesz transforms on the spaces $L^{p,\infty}$, $1 < p < \infty$, equipped with the norm

$$||f||_{p,\infty} = \sup \frac{1}{|A|^{1-1/p}} \int_A |f(x)| \mathrm{d}x,$$

where the supremum is taken over all measurable subsets A of $\mathbb{R}^d$ satisfying $0 < |A| < \infty$. We should mention here that $|| \cdot ||_{p,\infty}$ is equivalent to the more common norm given by

$$|||f|||_{p,\infty} = \sup\left\{\lambda|\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}|^{1/p} \ : \ \lambda > 0\right\}.$$

However, we do prefer to work with $|| \cdot ||_{p,\infty}$, since it is more convenient for our purposes. To formulate our results, we need some notation. Throughout the paper, for any fixed $\lambda \geq 0$, the function $\Phi_\lambda : [0, \infty) \to [0, \infty)$ is given by $\Phi_\lambda(t) = (t - \lambda)_+$. For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$, the symbol f^* stands for the non-increasing rearrangement of f, defined by

$$f^*(t) = \inf\left\{s > 0 : |\{x : |f(x)| > s\}| \leq t\right\}.$$

Furthermore, $f^{**} : (0, \infty) \to \mathbb{R}$ will denote the integral mean of f^*, given by

$$f^{**}(r) = \frac{1}{r} \int_0^r f^*(t) \mathrm{d}t.$$

In other words, f^{**} is the Hardy-Littlewood maximal function of f^*:

$$f^{**}(r) = \sup\left\{\frac{1}{|A|} \int_A f^*(t)\mathrm{d}t \ : \ A \subset (0, \infty), \ r \in A\right\}.$$

It is straightforward to see that the $L^{p,\infty}$-norm of f is related to f^{**} by the formula

$$||f||_{L^{p,\infty}(\mathbb{R}^d)} = \sup_{r>0} r^{1/p} f^{**}(r).$$

We are ready to state the first result of this paper.

Theorem 1.1. *Let $d \geq 1$ be a given integer and let $j \in \{1, 2, \ldots, d\}$.*
(i) For any integrable function $f : \mathbb{R}^d \to [0, 1]$ and any $r > 0$ we have

$$(1.3) \qquad r(R_j f)^{**}(r) \leq \frac{1}{\pi} \int_0^r \sinh^{-1}\left(\frac{2||f||_{L^1(\mathbb{R}^d)}}{s}\right) ds.$$

(ii) For any $1 < p < \infty$ we have

$$(1.4) \qquad ||(R_j f)_+||_{L^{p,\infty}(\mathbb{R}^d)} \leq \frac{2^{1-1/p}}{\pi} \int_0^\infty \frac{r^{-1/p}}{\sqrt{1+r^2}} dr \cdot ||f||_{L^{p,\infty}(\mathbb{R}^d)}$$

and

$$(1.5) \qquad ||R_j f||_{L^{p,\infty}(\mathbb{R}^d)} \leq \frac{2^{2-1/p}}{\pi} \int_0^\infty \frac{r^{-1/p}}{\sqrt{1+r^2}} dr \cdot ||f||_{L^{p,\infty}(\mathbb{R}^d)}.$$

The inequalities (1.3) and (1.4) are sharp.

Next, we turn to the case when the underlying measure space is the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, $d \geq 2$. There are many ways of defining Riesz transforms in this setting: see Arcozzi and Li [2] for an overview of the possibilities. We will work with two types R^c and R^b, the so-called *cylindrical Riesz transforms* and *Riesz transforms of ball type.* For the necessary definitions, we refer the reader to Section 4 below, and only mention here that for $d = 2$, both operators reduce to the Hilbert transform on the unit circle $\mathbb{S}^1 = [0, 2\pi)$, given by the singular integral

$$\mathcal{H}^{\mathbb{T}} f(x) = \frac{1}{\pi} \, \text{p.v.} \int_0^{2\pi} \frac{f(t)}{t-x} dt.$$

For $f : \mathbb{S}^{d-1} \to \mathbb{R}$, its non-increasing rearrangement f^*, integral mean f^{**} and weak norm $||f||_{L^{p,\infty}(\mathbb{S}^{d-1})}$ are defined by analogous formulas as in $\mathbb{R}^d$. We will establish the following statement for the directional Riesz transforms $R_{\ell m}^c$, $R_{\ell m}^b$ with respect to the (x_ℓ, x_m)-plane.

Theorem 1.2. *Let $1 \leq \ell < m \leq d$ be fixed and let $R \in \{R_{\ell m}^c, R_{\ell m}^b\}$. Then for any $1 < p < \infty$ we have*

$$(1.6) \qquad ||(Rf)_+||_{L^{p,\infty}(\mathbb{S}^{d-1})} \leq \frac{2^{1-1/p}}{\pi} \int_0^\infty \frac{r^{-1/p}}{\sqrt{1+r^2}} dr \cdot ||f||_{L^{p,\infty}(\mathbb{S}^{d-1})}$$

and

$$(1.7) \qquad ||Rf||_{L^{p,\infty}(\mathbb{S}^{d-1})} \leq \frac{2^{2-1/p}}{\pi} \int_0^\infty \frac{r^{-1/p}}{\sqrt{1+r^2}} dr \cdot ||f||_{L^{p,\infty}(\mathbb{S}^{d-1})}.$$

The inequality (1.6) is sharp for $d = 2$.

A few words about the proof and the organization of the paper. We will use a probabilistic approach to Riesz transforms, which has its roots at the works of Varopoulos and Gundy. To be more precise, we will first establish an appropriate martingale version of the inequality (1.3): this is done in Section 2. Then, in Section 3, we deduce (1.3), (1.4) and (1.5), using the stochastic representation of Riesz transforms [10]. In Section 4, we use a similar approach to deduce (1.6) and (1.7). The final part of the paper is devoted to the optimality of the constants appearing in Theorems 1.1 and 1.2.

2. A MARTINGALE INEQUALITY

The results of this paper depend heavily on an appropriate martingale inequalities. Let us start with introducing the necessary probabilistic background and notation. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-σ-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let X, Y be two adapted real-valued martingales with

right-continuous trajectories that have limits from the left. The symbol $[X, Y]$ will stand for the quadratic covariance process of X and Y, see e.g. Dellacherie and Meyer [7] for details. The martingales X, Y are said to be orthogonal if the process $[X, Y]$ is constant with probability 1. Following Bañuelos and Wang [3] and Wang [23], we say that Y is differentially subordinate to X, if the process $([X, X]_t - [Y, Y]_t)_{t \geq 0}$ is nonnegative and nondecreasing as a function of t.

The differential subordination implies many interesting inequalities comparing the sizes of X and Y. The literature on this subject is quite extensive, we refer the interested reader to the survey [5] by Burkholder, the paper of Wang [23] and the monograph [18]. Here we only mention one result, due to Bañuelos and Wang [23], which will be needed in our further considerations. We use the notation $||X||_p = \sup_{t \geq 0} ||X_t||_p$ for $1 \leq p \leq \infty$.

Theorem 2.1. *Suppose that X, Y are orthogonal martingales such that Y is differentially subordinate to X. Then for any $1 < p < \infty$,*

$$||Y||_p \leq \cot\left(\frac{\pi}{2p^*}\right) ||X||_p$$

(here, as previously, $p^* = \max\{p, p/(p-1)\}$). The constant is the best possible.

For $\lambda \geq 0$, we define the convex function $\Phi_\lambda : [0, \infty) \to \mathbb{R}$ by $\Phi(t) = (t - \lambda)_+$. The main result of this section is the following.

Theorem 2.2. *Suppose that X, Y are orthogonal martingales such that X takes values in $[0, 1]$, Y is differentially subordinate to X and $Y_0 \equiv 0$. Then for any $\lambda \geq 0$ we have*

$$(2.1) \qquad \mathbb{E}\Phi_\lambda(|Y_\infty|) \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Phi_\lambda\left(\left|\frac{1}{\pi}\log|s|\right|\right)\sin(\pi||X||_1)}{s^2 + 1 + 2s\cos(\pi||X||_1)}\, ds.$$

The inequality is sharp: there is a nontrivial pair X, Y for which both sides are equal.

Here Y_∞ stands for the pointwise limits of Y as $t \to \infty$; the existence of this limit follows immediately from the boundedness of X and Y in L^2 (see Theorem 2.1 above).

The proof of Theorem 2.2 will be based on the existence of certain special harmonic functions. Let $H = \mathbb{R} \times (0, \infty)$ denote the upper half-space and let $S = [0, 1] \times \mathbb{R}$ stand for the vertical strip in $\mathbb{R}^2$. For a given $\lambda \geq 0$, define the auxiliary function $\mathcal{U}_\lambda : H \to \mathbb{R}$ by the Poisson integral

$$\mathcal{U}_\lambda(\alpha, \beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta \Phi_\lambda\left(\left|\frac{1}{\pi}\log|t|\right|\right)}{(\alpha - t)^2 + \beta^2}\, dt.$$

Obviously, $\mathcal{U}_\lambda$ is harmonic on H and satisfies

$$(2.2) \qquad \lim_{(\alpha,\beta)\to(z,0)} \mathcal{U}_\lambda(\alpha, \beta) = \Phi_\lambda\left(\frac{1}{\pi}|\log|z||\right) \qquad \text{for } z \neq 0.$$

Consider a conformal mapping $\varphi(z) = -e^{-i\pi z}$ or, in real coordinates,

$$\varphi(x, y) = \left(-e^{\pi y}\cos(\pi x), e^{\pi y}\sin(\pi x)\right).$$

This function maps $(0, 1) \times \mathbb{R}$ onto H. Define U_λ in the interior of the strip S by the formula

$$(2.3) \qquad U_\lambda(x, y) = \mathcal{U}_\lambda(\varphi(x, y)).$$

The function U_λ is harmonic on $(0,1) \times \mathbb{R}$ and, by (2.2), can be extended to a continuous function on the whole S by setting $U_\lambda(0,y) = U_\lambda(1,y) = \Phi_\lambda(|y|)$.

Some further properties of U_λ are investigated in the lemma below.

Lemma 2.3. *(i) For $0 < x < 1$ and $y \in \mathbb{R}$ we have*

$$(2.4) \qquad U_\lambda(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\Phi_\lambda\left(\left|\frac{1}{\pi}\log|s| + y\right|\right) \sin(\pi x)}{(s + \cos(\pi x))^2 + \sin^2(\pi x)} \, ds$$

Thus, the right-hand side of (2.1) is equal to $U_\lambda(\|X\|_1, 0)$.

(ii) The partial derivative $U_{\lambda xx}$ is nonpositive in the interior of S.

(iii) We have the majorization

$$(2.5) \qquad U_\lambda(x,y) \geq \Phi_\lambda(|y|) \qquad \textit{for any } (x,y) \in S.$$

Proof. (i) It suffices to substitute $t = se^{\pi y}$ in the integral defining $\mathcal{U}_\lambda$.

(ii) By the harmonicity of U_λ inside the strip S, it suffices to show that $U_{\lambda yy} \geq 0$. But this is clear, in view of (2.4): for any $s \in \mathbb{R}$, the function $y \mapsto \Phi_\lambda\left(\left|\frac{1}{\pi}\log|s| + y\right|\right)$ is convex.

(iii) By (i), all we need is to show the majorization for $x \in \{0,1\}$. However, when $x = 0$ or $x = 1$, then both sides of (2.5) are equal. $\qquad \square$

In the proof of Theorem 2.2 we shall require the following auxiliary fact, which appears (in a slightly different form) as Corollary 1 in Bañuelos and Wang [4].

Lemma 2.4. *Suppose that X, Y are real-valued orthogonal martingales such that Y is differentially subordinate to X. Then Y has continuous paths and is orthogonal and differentially subordinate to X^c, the continuous part of X.*

We are ready to prove the martingale inequality.

Proof of Theorem 2.2. Fix $t \in (0, \infty)$ and introduce the process $Z = (X, Y)$. Since U_λ is of class C^∞ in the interior of S, we may apply Itô's formula to obtain

$$U_\lambda(Z_t) = U_\lambda(Z_0) + I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + I_4,$$

where

$$I_1 = \int_{0+}^{t} U_{\lambda x}(Z_{s-})\mathrm{d}X_s + \int_{0+}^{t} U_{\lambda y}(Z_{s-})\mathrm{d}Y_s,$$

$$I_2 = 2\int_{0+}^{t} U_{\lambda xy}(Z_{s-})\mathrm{d}[X^c, Y]_s,$$

$$I_3 = \int_{0+}^{t} U_{\lambda xx}(Z_{s-})\mathrm{d}[X,X]_s^c + \int_{0+}^{t} U_{\lambda yy}(Z_{s-})\mathrm{d}[Y,Y]_s,$$

$$I_4 = \sum_{0 < s \leq t} \left\{ U_\lambda(Z_s) - U_\lambda(Z_{s-}) - U_{\lambda x}(Z_{s-})\Delta X_s \right\}.$$

Here ΔX_s denotes the jump of X at time s. Note that we have used above the equalities $Y_{s-} = Y_s$ and $Y = Y^c$, which are due to the continuity of paths of Y. Let us analyze the above terms separately. First, note that $U_\lambda(Z_0) = U_\lambda(X_0, 0) = U_\lambda(\|X\|_1, 0)$ (since X is nonnegative). The term I_1 has zero expectation, since both stochastic integrals are martingales. Next, we have $I_2 = 0$, because of the

orthogonality of X^c and Y. The differential subordination together with Lemma 2.3 (ii) give

$$I_3 \leq \int_0^t U_{\lambda xx}(Z_{s-})d[X,X]_s^c + \int_0^t U_{\lambda yy}(Z_{s-})d[X,X]_s^c = 0.$$

Finally, each summand in I_4 is nonpositive, by the concavity of $U_\lambda(\cdot, y)$ for any fixed $y \in \mathbb{R}$ (again, apply Lemma 2.3 (ii)). Therefore, using the majorization of that lemma, we obtain the estimate

(2.6) $$\mathbb{E}\Phi_\lambda(|Y_t|) \leq \mathbb{E}U_\lambda(X_t, Y_t) \leq U_\lambda(||X||_1, 0)$$

and (2.1) is established, in view of Fatou's lemma. To see that this estimate is sharp, pick any pair X, Y of continuous-path orthogonal martingales satisfying $d[X,X] = d[Y,Y]$ and such that $X_\infty = \lim_{t\to\infty} X_t \in \{0,1\}$ with probability 1. Then the last two terms in (2.6) are equal for each t, and the first two become equal when we let $t \to \infty$ (in view of Lebesgue's dominated convergence theorem). This proves the claim. $\square$

3. Inequalities for Riesz transforms in $\mathbb{R}^d$

There is a well-known representation of Riesz transforms in terms of the so-called background radiation process, introduced by Gundy and Varopoulos in [10]. Let us briefly describe this connection. Throughout this section, d is a fixed positive integer. Suppose that X is a Brownian motion in $\mathbb{R}^d$ and let Y be an independent Brownian motion in $\mathbb{R}$ (both processes start from the appropriate origins). For any $y > 0$, introduce the stopping time $\tau(y) = \inf\{t \geq 0 : Y_t \in \{-y\}\}$. For sufficiently regular f (say, $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$), let $V_f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ stand for the Poisson extension of f to the upper half-space. That is,

$$V_f(x,y) := \mathbb{E}f\left(x + X_{\tau(y)}\right),$$

or, which is the same, V_f is the convolution of f with the Poisson kernel

$$P_y(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \frac{y}{(|x|^2 + y^2)^{(d+1)/2}}, \qquad x \in \mathbb{R}^d, \, y > 0.$$

For any $(d+1) \times (d+1)$ matrix A we define the martingale transform $A*f$ by

$$A*f(x,y) = \int_{0+}^{\tau(y)} A\nabla V_f(x + X_s, y + Y_s) \cdot d(X_s, Y_s).$$

Note that $A*f(x,y)$ is a random variable for each x, y. Now, for any $f \in C_0^\infty(\mathbb{R}^d)$, any $y > 0$ and any matrix A as above, define $\mathcal{T}_A^y f : \mathbb{R}^d \to \mathbb{R}$ through the bilinear form

(3.1) $$\int_{\mathbb{R}^d} \mathcal{T}_A^y f(x)g(x)\, dx = \int_{\mathbb{R}^d} \mathbb{E}\left[A*f(x,y)g(x + X_{\tau(y)})\right] dx,$$

where g runs over $C_0^\infty(\mathbb{R}^d)$. Less formally, $\mathcal{T}^y f$ is given as the following conditional expectation with respect to the measure $\tilde{\mathbb{P}} = \mathbb{P}\otimes dx$ (dx denotes Lebesgue's measure on $\mathbb{R}^d$): for any $z \in \mathbb{R}^d$,

$$\mathcal{T}_A^y f(z) = \tilde{\mathbb{E}}\left[A*f(x,y)|x + X_{\tau(y)} = z\right].$$

See Gundy and Varopoulos [10] for the rigorous statement of this equality. The interplay between the operators $\mathcal{T}_A^y$ and Riesz transforms is explained in the following theorem, consult [10] or Gundy and Silverstein [9].

Theorem 3.1. *Let $A^j = [a^j_{\ell m}]$, $j = 1, 2, \ldots, d$ be the $(d+1) \times (d+1)$ matrices given by*

$$a^j_{\ell m} = \begin{cases} 1 & \text{if } \ell = d+1, \, m = j, \\ -1 & \text{if } \ell = j, \, m = d+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}^y_{A^j} f \to R_j f$ almost everywhere as $y \to \infty$.

We shall require the following auxiliary fact, see Lemma 3.2 in [17].

Lemma 3.2. *Let $f \in C^\infty_0(\mathbb{R}^d)$ and $A = A^j$ for some j. Then (3.1) holds for all $g \in L^q(\mathbb{R}^d)$, $1 < q < \infty$.*

We are ready to establish the inequalities of Theorem 1.1.

Proof of (1.3). Pick $j \in \{1, 2, \ldots, d\}$ and suppose that $A = A^j$ is the matrix of Theorem 3.1. Fix $\lambda > 0$ and an integrable function $f : \mathbb{R}^d \to [0, 1]$. For any $x \in \mathbb{R}^d$ and $y > 0$, denote

(3.2)
$$g(x) = \begin{cases} \dfrac{\mathcal{T}^y_A f(x)}{|\mathcal{T}^y_A f(x)|} 1_{\{|T^y_A f(x)| \geq \lambda\}} & \text{if } \mathcal{T}^y_A f(x) \neq 0, \\ 0 & \text{if } \mathcal{T}^y_A f(x) = 0. \end{cases}$$

Now, fix x, y and consider the pair $\xi = \xi^{x,y} = (\xi_t)_{t \geq 0}$, $\zeta = \zeta^{x,y} = (\zeta_t)_{t \geq 0}$ of martingales given by

$$\xi_t = V_f(x + X_{\tau(y) \wedge t}, y + Y_{\tau(y) \wedge t})$$

$$= V_f(x, y) + \int_{0+}^{\tau(y) \wedge t} \nabla V_f(x + X_s, y + Y_s) \cdot \mathrm{d}(X_s, Y_s)$$

and

$$\zeta_t = \int_{0+}^{\tau(y) \wedge t} A \nabla V_f(x + X_s, y + Y_s) \cdot \mathrm{d}(X_s, Y_s),$$

for $t \geq 0$. Then the martingale ζ is differentially subordinate to ξ, since

$$[\xi, \xi]_t - [\zeta, \zeta]_t = |V_f(x, y)|^2 + \sum_{k \notin \{j, d+1\}} \int_{0+}^{\tau(y) \wedge t} \left| \frac{\partial V_f}{\partial x_k}(x + X_s, y + Y_s) \right|^2 \mathrm{d}s$$

is nonnegative and nondecreasing as a function of t. Furthermore, ξ and ζ are orthogonal, which is a direct consequence of the equality $\langle Ax, x \rangle = 0$, valid for all $x \in \mathbb{R}^d$. Indeed,

$$[\xi, \zeta]_t = \int_{0+}^{\tau(y) \wedge t} \langle A \nabla V_f(x + X_s, y + Y_s), \nabla V_f(x + X_s, y + Y_s) \rangle \mathrm{d}s = 0.$$

Therefore, by Theorem 2.1,

$$||\zeta_{\tau(y)}||^p_p = ||\zeta||^p_p \leq \cot^p\left(\frac{\pi}{2p^*}\right) ||\xi||^p_p = \cot^p\left(\frac{\pi}{2p^*}\right) ||\xi_{\tau(y)}||^p_p, \qquad 1 < p < \infty,$$

and integrating both sides with respect to $x \in \mathbb{R}^d$ gives

$$\int_{\mathbb{R}^d} \mathbb{E}|A*f(x, y)|^p \mathrm{d}x \leq \cot^p\left(\frac{\pi}{2p^*}\right) \int_{\mathbb{R}^d} \mathbb{E}|f(x + X_{\tau(y)})|^p \mathrm{d}x = \cot^p\left(\frac{\pi}{2p^*}\right) ||f||^p_{L^p(\mathbb{R}^d)},$$

by virtue of Fubini's theorem. Similarly, for any $h \in L^q(\mathbb{R}^d)$ ($q = p/(p-1)$ is the harmonic conjugate to p),

$$\int_{\mathbb{R}^d} \mathbb{E}|h(x + X_{\tau(y)})|^q \mathrm{d}x = ||h||_{L^q(\mathbb{R}^d)}^q.$$

Combining these estimates with (3.1) and Hölder's inequality yields

$$\left| \int_{\mathbb{R}^d} \mathcal{T}_A^y f(x)h(x)\,\mathrm{d}x \right| = \left| \int_{\mathbb{R}^d} \mathbb{E}\big[A * f(x,y)h(x + X_{\tau(y)})\big]\mathrm{d}x \right|$$

$$\le \cot^p\left(\frac{\pi}{2p^*}\right) ||f||_{L^p(\mathbb{R}^d)}||h||_{L^q(\mathbb{R}^d)}.$$

This implies $\mathcal{T}_A^y f \in L^p(\mathbb{R}^d)$ and hence $g \in L^p(\mathbb{R}^d)$, by Chebyshev's inequality (recall that g is given by (3.2)). Thus, by Lemma 3.2, we may write

$$\int_{\mathbb{R}^d} \Phi_\lambda\left(|\mathcal{T}_A^y f(x)|\right)\mathrm{d}x = \int_{\mathbb{R}^d} \left(|\mathcal{T}_A^y f(x)| - \lambda\right) 1_{\{|\mathcal{T}_A^y f(x)| \ge \lambda\}} \mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \mathcal{T}_A^y f(x)g(x) - \lambda|g(x)|\mathrm{d}x$$

$$= \int_{\mathbb{R}^d} \mathbb{E}\big[A * f(x,y)g(x + X_{\tau(y)}) - \lambda|g(x + X_{\tau(y)})|\big]\mathrm{d}x$$

$$\le \int_{\mathbb{R}^d} \mathbb{E}\big(|A * f(x,y)| - \lambda\big)|g(x + X_{\tau(y)})|\mathrm{d}x$$

$$\le \int_{\mathbb{R}^d} \Phi_\lambda(|A * f(x,y)|)\mathrm{d}x,$$

where in the last line we have used the fact that $|g|$ takes values in $\{0,1\}$. Now, by (2.1), we get that

$$\int_{\mathbb{R}^d} \mathbb{E}\Phi_\lambda\left(|A * f(x,y)|\right)\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\Phi_\lambda(|\zeta_\infty^{x,y}|)\mathrm{d}x$$

$$\le \int_{\mathbb{R}^d} U_\lambda(||\xi^{x,y}||_1, 0)\mathrm{d}x.$$

The function U_λ is nonnegative, so by Fubini's theorem, we may write

$$\int_{\mathbb{R}^d} U_\lambda(||\xi^{x,y}||_1, 0)\mathrm{d}x$$
$$= \int_{-\infty}^\infty \frac{\Phi_\lambda\left(\left|\frac{1}{\pi}\log|s|\right|\right)}{(s+1)^2} \int_{\mathbb{R}^d} \frac{\sin(\pi||\xi^{x,y}||_1)}{\pi||\xi^{x,y}||_1} \frac{(s+1)^2}{s^2 + 2s\cos(\pi||\xi^{x,y}||_1) + 1} \cdot ||\xi^{x,y}||_1 \mathrm{d}x\,\mathrm{d}s.$$

Since f is nonnegative, bounded by 1 and integrable, we have

$$\lim_{y \to \infty} \sup_{x \in \mathbb{R}^d} ||\xi^{x,y}||_1 = \lim_{y \to \infty} \sup_{x \in \mathbb{R}^d} V_f(x,y) = 0.$$

Indeed, with no loss of generality we may assume that f is compactly supported. Then it suffices to use of the following straightforward property of the Poisson kernel: for any positive numbers D and ε, there is a level y_0 such that if $y \ge y_0$, $x \in \mathbb{R}$ and E is a set of measure D, then $\int_E P_y(x)\mathrm{d}x < \varepsilon$.

Next, an application of Fubini's theorem gives

$$\int_{\mathbb{R}^d} ||\xi^{x,y}||_1 \mathrm{d}x = ||f||_{L^1(\mathbb{R}^d)}.$$

Thus, by Lebesgue's dominated convergence theorem, Fatou's lemma and Theorem 3.1, we get the bound

$$(3.3) \qquad \int_{\mathbb{R}^d} \Phi_\lambda(|R_j f(x)|) \, \mathrm{d}x \leq \int_{-\infty}^{\infty} \frac{\Phi_\lambda\left(\left|\frac{1}{\pi} \log |s|\right|\right)}{(s+1)^2} \mathrm{d}s \cdot ||f||_{L^1(\mathbb{R}^d)}.$$

However, we easily compute that

$$
\begin{aligned}
(3.4) \qquad \int_{-\infty}^{\infty} \frac{\Phi_\lambda\left(\left|\frac{1}{\pi} \log |s|\right|\right)}{(s+1)^2} \mathrm{d}s &= \int_0^\infty \frac{\left(\frac{1}{\pi} \log s - \lambda\right)_+}{(s+1)^2} \mathrm{d}s + \int_0^\infty \frac{\left(\frac{1}{\pi} \log s - \lambda\right)_+}{(s-1)^2} \mathrm{d}s \\
&= 2\int_1^\infty \frac{\left(\frac{1}{\pi} \log s - \lambda\right)_+}{(s+1)^2} \mathrm{d}s + 2\int_1^\infty \frac{\left(\frac{1}{\pi} \log s - \lambda\right)_+}{(s-1)^2} \mathrm{d}s \\
&= 2\int_0^\infty \frac{\left(\frac{u}{\pi} - \lambda\right)_+}{(e^u+1)^2} \mathrm{d}u + 2\int_0^\infty \frac{\left(\frac{u}{\pi} - \lambda\right)_+}{(e^u-1)^2} \mathrm{d}u \\
&= \frac{2}{\pi} \int_{\pi\lambda}^\infty \frac{\mathrm{d}u}{e^u+1} + \frac{2}{\pi} \int_{\pi\lambda}^\infty \frac{\mathrm{d}u}{e^u-1} \\
&= \frac{2}{\pi} \int_{\pi\lambda}^\infty \frac{\mathrm{d}u}{\sinh u}.
\end{aligned}
$$

Pick an arbitrary measurable subset E of $\mathbb{R}^d$ with $0 < |E| < \infty$ and decompose it into the union of

$$E^+ = E \cap \{x : |R_j f(x)| \geq \lambda\}, \qquad \text{and} \qquad E^- = E \cap \{x : |R_j f(x)| < \lambda\}.$$

By (3.3), we have

$$\int_{E^+} \Phi_\lambda(|R_j f(x)|) \mathrm{d}x \leq \int_{\mathbb{R}^d} \Phi_\lambda(|R_j f(x)|) \mathrm{d}x \leq \frac{2}{\pi} \int_{\pi\lambda}^\infty \frac{\mathrm{d}u}{\sinh u} \cdot ||f||_{L^1(\mathbb{R}^d)}$$

and, obviously,

$$\int_{E^-} \Phi_\lambda(|R_j f(x)|) \mathrm{d}x = 0.$$

Adding both above statements, we get an inequality which is equivalent to

$$(3.5) \qquad \int_E |R_j f(x)| \mathrm{d}x \leq \frac{2}{\pi} \int_{\pi\lambda}^\infty \frac{\mathrm{d}u}{\sinh u} \cdot ||f||_{L^1(\mathbb{R}^d)} + \lambda |E|.$$

Now optimize the right-hand side over λ. A direct computation of the derivative with respect to this parameter shows that the minimal value is attained for $\lambda = \sinh^{-1}\left(2||f||_1/|E|\right)/\pi$. For this choice of λ, we obtain the estimate

$$\int_E |R_j f(x)| \mathrm{d}x \leq \frac{1}{\pi} \int_0^{|E|} \sinh^{-1}\left(\frac{2||f||_{L^1(\mathbb{R}^d)}}{s}\right) \mathrm{d}s.$$

This is precisely the desired upper bound for $R_j f^{**}$. $\qquad\qquad \square$

Proof of (1.4) *and* (1.5). Clearly, it suffices to establish the first estimate, since the constant in (1.5) is twice bigger than that in (1.4). Pick $1 < p < \infty$, a function f on $\mathbb{R}^d$ satisfying $||f||_{L^{p,\infty}(\mathbb{R}^d)} \leq 1$, and a measurable subset E of $\mathbb{R}^d$ with $0 < |E| < \infty$. Riesz transform R_j is a Fourier multiplier with the symbol $-i\xi_j/|\xi|$, i.e., we have

the equality $\widehat{Rf}(\xi) = -i\xi_j\hat{f}(\xi)/|\xi|$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. Thus, by Parseval's identity, we have

$$
\begin{aligned}
\int_E R_j f(x)\mathrm{d}x &= \int_{\mathbb{R}^d} R_j f(x)\chi_E(x)\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \widehat{R_j f}(x)\widehat{\chi_E}(x)\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} f(x)R_j\chi_E(x)\mathrm{d}x \\
&\leq \int_0^\infty f^*(r)(R_j\chi_E)^*(r)\mathrm{d}r.
\end{aligned}
$$

Here in the last passage we have used the Hardy-Littlewood inequality $\int fg \leq \int f^*g^*$, which is actually valid on general measure spaces (see [11]). Now, fix $0 < m < M$ and use integration by parts, then (1.3) and then integration by parts again, to obtain

$$
\begin{aligned}
\int_m^M f^*(r)(R_j\chi_E)^*(r)\mathrm{d}r &= f^*(M)M(R_j\chi_E)^{**}(M) - f^*(m)m(R_j\chi_E)^{**}(m) \\
&\quad + \int_m^M r(R\chi_E)^{**}(r)\mathrm{d}(-f^*)(r) \\
&\leq f^*(M)M(R_j\chi_E)^{**}(M) - f^*(m)m(R_j\chi_E)^{**}(m) \\
&\quad + \int_m^M \frac{1}{\pi}\int_0^r \sinh^{-1}\left(\frac{2|E|}{s}\right)\mathrm{d}s\,\mathrm{d}(-f^*)(r) \\
&= I + II + \frac{1}{\pi}\int_m^M \sinh^{-1}\left(\frac{2|E|}{r}\right)f^*(r)\mathrm{d}r.
\end{aligned}
$$

Here

$$
I = f^*(M)M(R_j\chi_A)^{**}(M) - f^*(M)\cdot\frac{1}{\pi}\int_0^M \sinh^{-1}\left(\frac{2|E|}{s}\right)\mathrm{d}s \leq 0,
$$

in view of (1.3), and

$$
II = f^*(m)\cdot\frac{1}{\pi}\int_0^m \sinh^{-1}\left(\frac{2|E|}{s}\right)\mathrm{d}s - f^*(m)m(R_j\chi_E)^{**}(m),
$$

which is positive by (1.3). However, by de l'Hospital rule and the assumption $||f||_{L^{p,\infty}(\mathbb{R}^d)} \leq 1$, we get $\lim_{m\to\infty} II = 0$. Thus, letting $m \to 0$ and $M \to \infty$, we obtain

$$
\int_E R_j f(x)\mathrm{d}x \leq \frac{1}{\pi}\int_0^\infty \sinh^{-1}\left(\frac{2|E|}{r}\right)f^*(r)\mathrm{d}r.
$$

Therefore, integrating by parts and arguing as above, we get

$$
\int_E R_j f(x)\mathrm{d}x \leq \frac{2|E|}{\pi}\int_0^\infty \frac{f^{**}(r)}{\sqrt{4|E|^2 + r^2}}\mathrm{d}r.
$$

However, $r^{1/p}f^{**}(r) \leq 1$, so the substitution $r := 2r|E|$ in the integral above yields

$$
||(R_j f)_+||_{L^{p,\infty}(\mathbb{R}^d)} \leq \frac{2^{1-1/p}}{\pi}\int_0^\infty \frac{r^{-1/p}}{\sqrt{1+r^2}}\mathrm{d}r,
$$

since E was arbitrary. The claim is proved.   □

4. Inequalities for Riesz transforms on spheres

Now we will analyze the weak-type estimates for Riesz transforms on the unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ equipped with the standard Riemannian metric and normalized $SO(d)$ invariant measure. We will work with two non-equivalent notions of Riesz transforms on the spheres. These two possibilities arise from the fact that there are two natural ways to "fill in" $\mathbb{S}^{d-1}$ so that it is the boundary of an d-dimensional Riemannian manifold (cf. [2]). Let us analyze these two cases separately.

Firstly, one can express $\mathbb{S}^{d-1}$ as the boundary of the cylinder $\mathbb{S}^{d-1} \times [0, \infty)$, and this leads to the cylindrical Riesz transform R^c. For fixed $1 \le \ell < m \le d$, consider the differential operator $\mathcal{T}_{\ell m} = x_\ell \partial_m - x_m \partial_\ell$. If $x_\ell + i x_m = r e^{i\theta}$, then $\mathcal{T}_m = \partial/\partial\theta$ is the derivative with respect to the angular coordinate in the (x_ℓ, x_m) plane and hence is a well defined vector field on $\mathbb{S}^{d-1}$. We define the directional Riesz transform (of cylinder type) by

$$R^c_{\ell m} = \mathcal{T}_{\ell m} \circ (-\Delta_{\mathbb{S}^{d-1}})^{-1/2},$$

where $\Delta_{\mathbb{S}^{d-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. See Stein [20] for the detailed exposition of the subject.

We turn to the second type of Riesz transform on $\mathbb{S}^{d-1}$ (cf. Korányi and Vági [15, 16]). Denote by $\mathcal{H}_k$ the space of spherical harmonics of degree k (see Stein [21]) and let

$$(4.1) \qquad \mathcal{E}_0 = \left\{ f : \mathbb{S}^{d-1} \to \mathbb{R} \ : \ f = \sum_{k=1}^{N} f_k, \ f_k \in \mathcal{H}_k, \ N = 1, 2, \dots \right\}$$

be the space of harmonic polynomials with null average on $\mathbb{S}^{d-1}$. For a fixed $f \in \mathcal{E}_0$, let H be the solution on the unit ball $\mathbb{B}^d$ of the Neumann problem with boundary data f, normalized so that $H(0) = 0$. Less formally, this can be expressed by the equation

$$\left(\frac{\partial}{\partial\nu} \right)^{-1} f = H|_{\mathbb{S}^{d-1}},$$

where ν is the outward pointing normal vector to $\mathbb{S}^{d-1}$. One easily extends $(\partial/\partial\nu)^{-1}$ to $L^2_0(\mathbb{S}^{d-1})$ by the following formula: if $f = \sum_{k\ge 1} f_k$ is the decomposition of f into spherical harmonics, then $(\partial/\partial\nu)^{-1} f = \sum_{k\ge 1} f_k/k$. We define the directional Riesz transforms of ball type by the formula

$$R^b_{\ell m} = \mathcal{T}_{\ell m} \circ \left(\frac{\partial}{\partial\nu} \right)^{-1}.$$

An important remark is in order. In general, if M is a complete Riemannian manifold equipped with the corresponding gradient ∇_M and the Laplace-Beltrami operator Δ_M, then one defines the associated Riesz transform by

$$R^M = \nabla_M \circ (-\Delta_M)^{-1/2}.$$

Since $\mathbb{S}^{d-1}$ is a Riemannian manifold, this gives rise to the question about the relation of the associated Riesz transform $R^{\mathbb{S}^{d-1}}$ to those introduced above. It turns out that this new operator is strictly related to the cylindrical case. To make

this interplay more apparent, observe the identity

$$|\nabla_{\mathbb{S}^{d-1}} f| = \left(\sum_{\ell < m} |\mathcal{T}_{\ell m} f|^2 \right)^{1/2},$$

valid for smooth functions $f : \mathbb{S}^{d-1} \to \mathbb{R}$. Consequently, if R^c denotes the vector $(R^c_{\ell m})_{1 \le \ell < m \le d}$, then $|R^{\mathbb{S}^{d-1}} f| = |R^c f|$, and thus the analysis of $R^{\mathbb{S}^{d-1}}$ reduces to that of R^c.

From Green's formula and elementary properties of the Laplace-Beltrami operator (cf. [20, 21], we infer that if R is a directional Riesz transform (of cylinder or ball type) and f, g are sufficiently regular functions on $\mathbb{S}^{d-1}$, then

$$\int_{\mathbb{S}^{d-1}} Rf(x) g(x) \mathrm{d}x = - \int_{\mathbb{S}^{d-1}} f(x) Rg(x) \mathrm{d}x.$$

This will allow us to carry out the appropriate duality argument (see the proof of (1.4) above).

Now we will describe the probabilistic representation of the above Riesz transforms. Let $B = (B^1, B^2, \dots, B^d)$ be the standard Brownian motion in $\mathbb{R}^d$, starting from 0, and let $\tau = \inf\{t \ge 0 : B_t \notin \mathbb{B}^d\}$ be the first exit time of B from the unit ball. Note that B_τ has the uniform distribution on $\mathbb{S}^{d-1}$. Let A be a continuous function on the closed unit ball, with values in the class of $d \times d$ matrices. This function gives rise to the following operation on stochastic integrals. For a given $f \in C^\infty(\mathbb{S}^{d-1})$, let F be its Poisson extension to $\mathbb{B}^d$ and put

$$A * F = \left(\int_0^{\tau \wedge t} A(B_s) \nabla_{\mathbb{R}^d} F(B_s) \cdot \mathrm{d}B_s \right)_{t \ge 0}.$$

We define the A-transform of f by the conditional expectation

$$T_A f(x) = \mathbb{E}[A * F | B_\tau = x], \qquad x \in \mathbb{S}^{d-1}.$$

The connection between the operators T_A and directional Riesz transforms is explained in the following statement, see Arcozzi [1].

Theorem 4.1. *For given $1 \le \ell < m \le d$, a function $\varphi : [0,1] \to \mathbb{R}$ and $x \in \overline{\mathbb{B}}^d$, let $A_{\ell m}(x)$ be the matrix with entries*

$$A_{\ell m}^{ij}(x) = \begin{cases} \varphi(|x|^2) & \text{if } i = \ell, \ j = m, \\ -\varphi(|x|^2) & \text{if } i = m, \ j = \ell, \\ 0 & \text{otherwise.} \end{cases}$$

(i) *If $\varphi \equiv 1$, then $T_{A_{\ell m}} = R^b_{\ell m}$.*
(ii) *Suppose that $d \ge 3$ and let φ be defined by the formula*

$$\varphi(e^{-2t/(d-2)}) = \frac{\int_0^t I_0(s)\, ds}{e^t - 1}, \qquad t \ge 0,$$

where $I_0(z) = \sum_{j=0}^\infty (z/2)^{2j}/(j!)^2$, $z \in \mathbb{C}$, is the modified Bessel function of order 0. Then $T_{A_{\ell m}} = R^c_{\ell m}$.

We are ready to establish the bounds for Riesz transforms. We start with the following analogue of (1.3).

Theorem 4.2. *Let $1 \le \ell < m \le d$ be fixed and let $f : \mathbb{S}^{d-1} \to [0,1]$ be an integrable function. Then, for $R \in \{R_{\ell m}^c, R_{\ell m}^b\}$ and any $0 < r \le 1$,*

$$(4.2) \qquad r(Rf)^{**}(r) \le \frac{1}{\pi} \int_0^r \sinh^{-1}\left(\frac{\sin\left(\pi||f||_{L^1(\mathbb{S}^{d-1})}\right)}{\tan(\pi s/2)} \right) ds,$$

and the inequality is sharp for $d = 2$.

Proof. As we have already observed, the random variable B_τ is uniformly distributed on $\mathbb{S}^{d-1}$. Therefore, by conditional version of Jensen's inequality,

$$\int_{\mathbb{S}^{d-1}} \Phi_\lambda(|Rf(x)|)\mathrm{d}x = \mathbb{E}\Phi_\lambda(|T_{A_{\ell m}}f(B_\tau)|) \le \mathbb{E}\Phi_\lambda(|A_{\ell m} * F|).$$

However, the martingale $A * F$ is orthogonal and differentially subordinate to the martingale $(F(B_{\tau \wedge t}))_{t \ge 0}$. Consequently, by (2.1), we may write

$$\int_{\mathbb{S}^{d-1}} \Phi_\lambda(|Rf(x)|)\mathrm{d}x \le U_\lambda(||F(B_\tau)||_1, 0) = U_\lambda(||f||_{\mathbb{S}^{d-1}}, 0).$$

Now, for a given measurable subset E of $\mathbb{S}^{d-1}$, we write the decomposition $E = E^+ \cup E^-$, where

$$E^+ = E \cap \{x \in \mathbb{S}^{d-1} : |Rf(x)| \ge \lambda\}, \qquad E^- = E \cap \{x \in \mathbb{S}^{d-1} : |Rf(x)| < \lambda\}.$$

We have

$$\int_{E^+} \Phi_\lambda(|Rf(x)|)\mathrm{d}x \le \int_{\mathbb{S}^{d-1}} \Phi_\lambda(|Rf(x)|)\mathrm{d}x \le U_\lambda(||f||_{\mathbb{S}^{d-1}}, 0)$$

and, of course,

$$\int_{E^-} \Phi_\lambda(|Rf(x)|)\mathrm{d}x = 0.$$

Summing these facts, we obtain

$$\int_E |Rf(x)|\mathrm{d}x \le U_\lambda(||f||_{\mathbb{S}^{d-1}}, 0) + \lambda \cdot |E|.$$

Optimizing the right-hand side over $\lambda > 0$, we get the inequality

$$\int_E |Rf(x)|\mathrm{d}x \le \frac{1}{\pi} \int_0^{|E|} \sinh^{-1}\left(\frac{\sin\left(\pi||f||_{L^1(\mathbb{S}^{d-1})}\right)}{\tan(\pi s/2)} \right) ds,$$

which is (4.2). We postpone the proof of the sharpness to the next section. $\square$

Proof of (1.6) *and* (1.7). As previously, it suffices to establish the first estimate. The proof is similar to that of (1.4). We fix $1 < p < \infty$, a function $f \in L^{p,\infty}(\mathbb{S}^{d-1})$ of norm one and a set $E \subset \mathbb{S}^{d-1}$ of positive measure. We start from observing that

$$\int_E Rf(x)\mathrm{d}x = \int_{\mathbb{S}^{d-1}} Rf(x)\chi_E(x)\mathrm{d}x$$

$$= -\int_{\mathbb{S}^{d-1}} f(x)R\chi_E(x)\mathrm{d}x \le \int_0^1 f^*(r)(R\chi_E)^*(r)\mathrm{d}r.$$

Integrating by parts and noting that $\lim_{r \to 0} f^*(r)r(R\chi_E)^{**}(r) = 0$, we get

$$\int_0^1 f^*(r)(R\chi_E)^*(r)\mathrm{d}r = f^*(1)(R\chi_E)^{**}(1) + \int_0^1 r(R\chi_E)^{**}(r)\mathrm{d}(-f^*)(r).$$

By (4.2), the right-hand side does not exceed

$$\frac{f^*(1)}{\pi} \int_0^1 \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi s/2)}\right) \mathrm{d}s + \frac{1}{\pi} \int_0^1 \int_0^r \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi s/2)}\right) \mathrm{d}s d(-f^*)(r)$$

$$= \frac{1}{\pi} \int_0^1 f^*(r) \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi r/2)}\right) \mathrm{d}r$$

$$= -\frac{1}{\pi} \int_0^1 r f^{**}(r) \frac{\mathrm{d}}{\mathrm{d}r} \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi r/2)}\right) \mathrm{d}r$$

$$\leq -\frac{1}{\pi} \int_0^1 r^{1-1/p} \frac{\mathrm{d}}{\mathrm{d}r} \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi r/2)}\right) \mathrm{d}r$$

$$= \frac{p-1}{p\pi} \int_0^1 r^{-1/p} \sinh^{-1}\left(\frac{\sin\left(\pi|E|\right)}{\tan(\pi r/2)}\right) \mathrm{d}r$$

$$\leq \frac{p-1}{p\pi} \int_0^1 r^{-1/p} \sinh^{-1}\left(\frac{2|E|}{r}\right) \mathrm{d}r$$

$$= \frac{(2|E|)^{1-1/p}}{\pi} \int_0^{(2|E|)^{-1}} \frac{r^{-1/p}}{\sqrt{1+r^2}} \mathrm{d}r,$$

where the last three equalities follow again from integration by parts. Combining the above facts yields the inequality (1.6). □

5. Sharpness

5.1. **Sharpness of** (1.3) **and** (1.4), $d = 1$. In the one-dimensional case, if we take $f = \chi_E$ for some measurable $E \subset \mathbb{R}$ satisfying $0 < |E| < \infty$, then both sides of (1.3) are equal. This follows at once from the identity of Stein and Weiss [22]:

$$\left|\{x \in \mathbb{R} : |\mathcal{H}\chi_E(x)| > t\}\right| = \frac{2|E|}{\sinh(\pi t)}, \qquad t > 0.$$

We turn to (1.3). Fix $1 < p < \infty$, take $E = [-1, 1]$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = -\frac{p-1}{p}\left(\frac{|x|}{2|x^2-1|}\right)^{1/p} \operatorname{sgn} x.$$

Let us first compute the $L^{p,\infty}$-norm of f. Observe that

(5.1) $$f(x) = f(1/x) = -f(-x) = -f(-1/x).$$

Thus, for a given $t > 0$, we have

$$\{x : |f(x)| > t\} = (-x_t, -1/x_t) \cup (1/x_t, x_t)$$

for $x_t > 1$ satisfying $f(x_t) = -t$. The latter equality can be transformed into

$$\frac{x_t}{2(x_t^2-1)} = \left(\frac{pt}{p-1}\right)^p,$$

or $2(x_t - x_t^{-1}) = ((p-1)/pt)^p$. Consequently, we have that $|\{x : |f(x)| > t\}| = ((p-1)/pt)^p$, which implies that $f^*(t) = (p-1)t^{-1/p}/p$ and

$$f^{**}(r) = \frac{1}{r}\int_0^r f^*(t)\mathrm{d}t = r^{-1/p}, \qquad r > 0.$$

Thus, we have $||f||_{L^{p,\infty}(\mathbb{R})} = 1$. The next step in the analysis is to compare f and $\mathcal{H}\chi_E$, which is explicitly given by

$$\mathcal{H}\chi_E(x) = \frac{1}{\pi}\log\frac{|x+1|}{|x-1|}.$$

It is clear that these two functions have opposite signs $(f(x)\mathcal{H}\chi_E(x) < 0$ for almost all $x \in \mathbb{R})$ and are "equimonotone" in the sense that for any x_1, $x_2 \in \mathbb{R}$, we have $|f(x_1)| \le |f(x_2)|$ if and only if $|\mathcal{H}\chi_E(x_1)| \le |\mathcal{H}\chi_E(x_2)|$. The latter property follows from (5.1), the analogous symmetry condition for $\mathcal{H}\chi_E$ and the fact that on $[1,\infty)$ both functions are decreasing and nonnegative. Combining all the above facts gives

$$
\begin{aligned}
||(\mathcal{H}f)_+||_{L^{p,\infty}(\mathbb{R})} &\ge \frac{1}{2^{1-1/p}}\int_E \mathcal{H}f(x)\mathrm{d}x \\
&= \frac{1}{2^{1-1/p}}\int_\mathbb{R} \mathcal{H}f(x)\chi_E(x)\mathrm{d}x \\
&= -\frac{1}{2^{1-1/p}}\int_\mathbb{R} f(x)\mathcal{H}\chi_E(x)\mathrm{d}x \\
&= \frac{1}{2^{1-1/p}}\int_0^\infty f^*(x)(\mathcal{H}\chi_E)^*(x)\mathrm{d}x \\
&= \frac{p-1}{2^{1-1/p}p\pi}\int_0^\infty t^{-1/p}\sinh^{-1}(4/t)\mathrm{d}t \\
&= \frac{2^{1-1/p}}{\pi}\int_0^\infty \frac{t^{-1/p}}{\sqrt{1+t^2}}\mathrm{d}t.
\end{aligned}
$$

Thus, both sides of (1.4) are equal.

5.2. **Sharpness of** (1.3) **and** (1.4)**,** $d > 1$. Here the reasoning rests on an appropriate transference-type argument. We will focus on (1.4), the argument for the inequality (1.3) is similar. Clearly, it is enough to deal with the Riesz transform R_1 only. Suppose that for a fixed $1 < p < \infty$ and any $f \in L^{p,\infty}(\mathbb{R}^d)$ we have

$$||R_1 f||_{L^{p,\infty}(\mathbb{R}^d)} \le C||f||_{L^{p,\infty}(\mathbb{R}^d)},$$

that is, for any Borel set $E \subset \mathbb{R}^d$ of finite measure,

$$(5.2) \qquad \int_E |R_1 f(x)|\mathrm{d}x \le C||f||_{L^{p,\infty}(\mathbb{R}^d)}|E|^{1-1/p}.$$

For $t > 0$, define the dilation operator δ_t as follows: for any function $g : \mathbb{R}\times\mathbb{R}^{d-1} \to \mathbb{R}$, we let $\delta_t g(\xi,\zeta) = g(\xi,t\zeta)$; for any $A \subset \mathbb{R}\times\mathbb{R}^{d-1}$, let $\delta_t A = \{(\xi,t\zeta) : (\xi,\zeta) \in A\}$. By (5.2), the operator $T_t := \delta_t^{-1} \circ R_1 \circ \delta_t$ satisfies

$$
\begin{aligned}
\int_E T_t f(x)\mathrm{d}x &= t^{d-1}\int_{\delta_t^{-1}E} R_1 \circ \delta_t f(x)\mathrm{d}x \\
&\le Ct^{d-1}||\delta_t f||_{L^{p,\infty}(\mathbb{R}^d)}|\delta_t^{-1}E|^{1-1/p}.
\end{aligned}
$$

However, we easily derive that for any $r > 0$ we have $\delta_t f^*(r) = f^*(rt^{d-1})$ and hence $||\delta_t f||_{L^{p,\infty}(\mathbb{R}^d)} = t^{(1-d)/p}||f||_{L^{p,\infty}(\mathbb{R}^d)}$. Furthermore, $|\delta^{-1}E| = t^{1-d}|E|$, and thus the above inequality becomes

$$(5.3) \qquad \int_E T_t f(x)\mathrm{d}x \le C||f||_{L^{p,\infty}(\mathbb{R}^d)}|E|^{1-1/p}.$$

Now, the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-1}\delta_t \circ \mathcal{F} \circ \delta_t$ and hence the operator T_t is a Fourier multiplier such that

$$\widehat{T_t f}(\xi, \zeta) = -i\frac{\xi}{(\xi^2 + t^2|\zeta|^2)^{1/2}}\widehat{f}(\xi, \zeta), \qquad (\xi, \zeta) \in \mathbb{R} \times \mathbb{R}^{d-1},$$

for $f \in L^2(\mathbb{R}^d)$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{t \to 0} \widehat{T_t f}(\xi, \zeta) = \widehat{T_0 f}(\xi, \zeta)$$

in $L^2(\mathbb{R}^d)$, where

(5.4) $$\widehat{T_0 f}(\xi, \zeta) = -i\,\mathrm{sgn}\,(\xi)\widehat{f}(\xi, \zeta).$$

Combining this with Plancherel's theorem, we conclude that there is a sequence $(t_n)_{n \geq 1}$ decreasing to 0 such that $T_{t_n} f$ converges to $T_0 f$ almost everywhere. Thus, taking an arbitrary $f \in L^2(\mathbb{R}^d) \cap L^{p,\infty}(\mathbb{R}^d)$ and applying Fatou's lemma together with (5.3), we obtain

(5.5) $$\int_E T_0 f(x)\mathrm{d}x \leq C||f||_{L^{p,\infty}(\mathbb{R}^d)}|E|^{1-1/p},$$

which, by density arguments, extends to all $f \in L^{p,\infty}(\mathbb{R}^d)$. Now pick a Borel subset A of $\mathbb{R}$ satisfying $0 < |A| < \infty$, a function $g \in L^{p,\infty}(\mathbb{R})$, and apply (5.5) to the function $f(\xi, \zeta) = g(\xi)\chi_{[0,1]^{d-1}}(\zeta)$ and the set $E = A \times [0,1]^{d-1}$. We have $||f||_{L^{p,\infty}(\mathbb{R}^d)} = ||g||_{L^{p,\infty}(\mathbb{R})}$ and $T_0 f(\xi, \zeta) = \mathcal{H}g(\xi)\chi_{[0,1]^{d-1}}(\zeta)$, since, by (5.4),

$$\widehat{T_0 f}(\xi, \zeta) = -i\mathrm{sgn}\,(\xi)\,\widehat{g}(\xi)\widehat{\chi_{[0,1]^{d-1}}}(\zeta).$$

Thus, we obtain

$$\int_A \mathcal{H}g(\xi)\mathrm{d}\xi \leq C||g||_{L^{p,\infty}(\mathbb{R}^d)}|A|^{1-1/p},$$

which implies $C \geq \frac{2^{1-1/p}}{\pi}\int_0^\infty \frac{t^{-1/p}}{\sqrt{1+t^2}}\mathrm{d}t$ in view of the previous subsection. This completes the proof.

5.3. **Sharpness of** (1.6) **and** (4.2). Unfortunately, we have been able to prove the optimality of the bounds only in the case $d = 2$. If one takes a measurable subset E of $\mathbb{S}^1$, then for each $t > 0$ the distribution function of $\mathcal{H}^{\mathbb{T}}$ satisfies

$$\left|\{x \in \mathbb{S}^1 : |\mathcal{H}^{\mathbb{T}}\chi_E(x)| \geq t\}\right| = \frac{2}{\pi}\arctan\left(\frac{\sin\left(\pi|E|\right)}{\sinh(\pi t)}\right)$$

(cf. [22]), which implies that both sides of (4.2) are equal. To handle (1.6), we proceed as in the proof of the sharpness for the Hilbert transform on the line, with an aid of an additional limiting argument. Pick an arbitrary $\kappa > 1$. Then there is $\varepsilon > 0$ such that $\tan(\pi s/2) \leq \kappa\pi s/2$ for $0 < s < \varepsilon$. Let E be an arbitrary subset of $\mathbb{S}^1$ of positive measure. There is a function $f : \mathbb{S}^1 \to \mathbb{R}$ which satisfies

$$f^*(r) = \begin{cases} \frac{p-1}{p}r^{-1/p} & \text{if } r < \varepsilon, \\ 0 & \text{if } r \geq \varepsilon. \end{cases}$$

Furthermore, rearranging this function appropriately, we may assume that its modulus is "equimonotone" with $|\mathcal{H}^{\mathbb{T}}\chi_E|$; that is, for all $s, t \in \mathbb{S}^1$, we have $|f(s)| \leq |f(t)|$

if and only if $|\mathcal{H}^{\mathbb{T}}\chi_E(t)| \leq |\mathcal{H}^{\mathbb{T}}\chi_E(s)|$. Finally, we may change the sign of f on the appropriate set so that $f(x)\mathcal{H}^{\mathbb{T}}\chi_E(x) \leq 0$ for almost all $x \in \mathbb{S}^1$. Then

$$
\begin{aligned}
||(\mathcal{H}^{\mathbb{T}}f)_+||_{L^{p,\infty}(\mathbb{S}^1)} &\geq \frac{1}{|E|^{1-1/p}} \int_E \mathcal{H}^{\mathbb{T}}f(x)\mathrm{d}x \\
&= -\frac{1}{|E|^{1-1/p}} \int_{\mathbb{S}^1} f(x)\mathcal{H}^{\mathbb{T}}\chi_E(x)\mathrm{d}x \\
&= \frac{1}{|E|^{1-1/p}} \int_0^1 f^*(r)(\mathcal{H}^{\mathbb{T}}\chi_E)^*(r)\mathrm{d}r \\
&= \frac{p-1}{p|E|^{1-1/p}\pi} \int_0^\varepsilon r^{-1/p} \sinh^{-1}\left(\frac{\sin(\pi|E|)}{\tan(\pi r/2)}\right)\mathrm{d}r \\
&\geq \frac{p-1}{p|E|^{1-1/p}\pi} \int_0^\varepsilon r^{-1/p} \sinh^{-1}\left(\frac{\sin(\pi|E|)}{\kappa\pi r/2}\right)\mathrm{d}r,
\end{aligned}
$$

where in the last definition we have used the definition of ε (see the beginning of the proof). Thus, substituting $r := r|E|$ and noting that $||f||_{L^{p,\infty}(\mathbb{S}^1)} = 1$, we get the estimate

$$
\frac{||(\mathcal{H}^{\mathbb{T}}f)_+||_{L^{p,\infty}(\mathbb{S}^1)}}{||f||_{L^{p,\infty}(\mathbb{S}^1)}} \geq \frac{p-1}{p\pi} \int_0^{\varepsilon/|E|} r^{-1/p} \sinh^{-1}\left(\frac{\sin(\pi|E|)}{\kappa\pi r|E|/2}\right)\mathrm{d}r.
$$

Now, if we let $|E| \to 0$, then the right-hand side tends to

$$
\frac{p-1}{p\pi} \int_0^\infty r^{-1/p} \sinh^{-1}\left(\frac{2}{\kappa r}\right)\mathrm{d}r = \frac{2^{1-1/p}}{\kappa^{1-1/p}\pi} \int_0^\infty \frac{t^{-1/p}}{\sqrt{1+t^2}}\mathrm{d}t.
$$

by Lebesgue's monotone convergence theorem. Since $\kappa > 1$ was arbitrary, we see that the constant in (1.6) cannot be replaced by a smaller number.

Acknowledgment

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