# SHARP INEQUALITIES FOR RIESZ TRANSFORMS

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Abstract. We establish the following sharp local estimate for the family $\{R_j\}_{j=1}^d$ of Riesz transforms on $\mathbb{R}^d$. For any Borel subset $A$ of $\mathbb{R}^d$ and any function $f:\mathbb{R}^d\to\mathbb{R}$,

$$\int_A |R_jf(x)|\mathrm{d}x \leq C_p||f||_{L^p(\mathbb{R}^d)}|A|^{1/q}, \qquad 1<p<\infty.$$

Here $q=p/(p-1)$ is the harmonic conjugate to $p$,

$$C_p=\left[\frac{2^{q+2}\Gamma(q+1)}{\pi^{q+1}}\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{q+1}}\right]^{1/q}, \qquad 1<p<2,$$

and

$$C_p=\left[\frac{2^{q+2}\Gamma(q+1)}{\pi^{q}}\sum_{k=0}^\infty \frac{1}{(2k+1)^{q}}\right]^{1/q}, \qquad 2\leq p<\infty.$$

This enables us to determine the precise values of the weak-type constants for Riesz transforms for $1<p<\infty$. The proof rests on appropriate martingale inequalities, which are of independent interest.

## 1. Introduction.

The purpose of this paper is to establish a class of sharp inequalities for Riesz transforms in $\mathbb{R}^d$. These objects are fundamental examples of Calderón-Zygmund singular integral operators and are given by

$$R_jf(x)=\frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\ \text{p.v.}\int_{\mathbb{R}^d}\frac{x_j-y_j}{|x-y|^{d+1}}f(y)\mathrm{d}y$$

for $j=1,\,2,\,\ldots,\,d$ (see e.g. Stein [21]). In the particular case $d=1$, this family has only one element, the so-called Hilbert transform $\mathcal{H}$ on $\mathbb{R}$. There is an alternative definition of $R_j$: it can be defined as the Fourier multiplier with the symbol $i\xi_j/|\xi|$, $\xi\in\mathbb{R}^d\setminus\{0\}$, i.e., we have the identity

$$\widehat{R_jf}(\xi)=i\frac{\xi_j}{|\xi|}\hat{f}(\xi), \qquad \text{for } \xi\in\mathbb{R}^d\setminus\{0\}. \tag{1.1}$$

The problem of studying various norms of these objects and their extensions is classical and has interested many mathematicians. The celebrated result of M. Riesz [20] states that the Hilbert transform $\mathcal{H}$ is a bounded operator on $L^p(\mathbb{R})$ if and only if $1<p<\infty$. Pichorides [19] and Cole (unpublished; see [9]) identified

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the precise values of these norms:

$$||\mathcal{H}||_{L^p(\mathbb{R})\to L^p(\mathbb{R})} = E_p := \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } p \geq 2. \end{cases} \tag{1.2}$$

Using the so-called method of rotations, Iwaniec and Martin [13] extended this result to the $d$-dimensional setting. They proved that for $1 < p < \infty$, any function $f \in L^p(\mathbb{R}^d)$ and any $j = 1, 2, \ldots, d$ we have

$$||R_j f||_{L^p(\mathbb{R}^d)} \leq E_p ||f||_{L^p(\mathbb{R}^d)}, \tag{1.3}$$

and that the constant $E_p$ cannot be decreased for any fixed $j$, $p$ and $d$. An alternative, probabilistic proof of the estimate (1.3), based on a sharp inequality for orthogonal martingales, was given by Bañuelos and Wang in [3].

The motivation for the results obtained in this paper comes from the question about (1.3) in the limit case $p = 1$. Riesz transforms are not bounded on $L^1$, but there are several important substitutes for (1.3). For example, Kolmogorov [15] proved the weak-type $(1,1)$ estimate

$$||\mathcal{H}f||_{L^{1,\infty}(\mathbb{R})} \leq c_1 ||f||_{L^1(\mathbb{R})}$$

for some universal constant $c_1 < \infty$. Here, for $1 \leq p < \infty$ and any Borel function $f : \mathbb{R}^d \to \mathbb{R}$, we define the weak $p$-th norm of $f$ by

$$||f||_{L^{p,\infty}(\mathbb{R}^d)} = \sup_{\lambda > 0} \left[\lambda^p \left|\{x \in \mathbb{R}^d : |f(x)| \geq \lambda\}\right|\right]^{1/p}.$$

The optimal value of $c_1$ was found by Davis [7] to be equal to

$$\frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots} \simeq 1.34\ldots.$$

This result was further extended by Janakiraman [14], who identified the best constant in the corresponding weak-type $(p,p)$ bound for $1 \leq p \leq 2$. Namely, he showed that

$$||\mathcal{H}f||_{L^{p,\infty}(\mathbb{R})} \leq c_p ||f||_{L^p(\mathbb{R})},$$

where the optimal $c_p$ is equal to

$$c_p = \left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\left|\frac{2}{\pi}\log|t|\right|^p}{t^2+1}\mathrm{d}t\right)^{-1/p}.$$

The question about the best weak-type constant in the range $p > 2$ remains open. Another open problem concerns the sharp analogues of the above estimates for Riesz transforms. In fact, it is not even known whether Riesz transforms satisfy the weak-type $(1,1)$ estimate with a constant which does not depend on the dimension.

We will establish another version of (1.3), which is related to the action of $R_j$ from $L^p(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$. This in turn provides some information on the weak-type constants for Riesz tranforms, under a proper renorming of the space $L^{p,\infty}(\mathbb{R}^d)$.

Our main result can be stated as follows. Throughout the paper, we use the convention that for any $1 < p < \infty$, the number $q$ stands for the harmonic conjugate to $p$, i.e., $q = p/(p-1)$.

**Theorem 1.1.** *Fix $1 < p < \infty$, let $f \in L^p(\mathbb{R}^d)$ and assume that $A$ is a Borel subset of $\mathbb{R}^d$. Then for any $j = 1, 2, \ldots, d$ we have*

$$\int_A |R_j f(x)|dx \leq C_p ||f||_{L^p(\mathbb{R}^d)} |A|^{1/q}, \tag{1.4}$$

*where*

$$C_p = \left[ \frac{2^{q+2}\Gamma(q+1)}{\pi^{q+1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{q+1}} \right]^{1/q}, \qquad 1 < p < 2,$$

*and*

$$C_p = \left[ \frac{2^{q+2}\Gamma(q+1)}{\pi^{q}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{q}} \right]^{1/q}, \qquad 2 \leq p < \infty.$$

*The constant is the best possible for each $p$, $d$ and $j$.*

This yields the following sharp weak-type $(p,p)$ inequality for Riesz transforms. It is well-known, see e.g. Grafakos [10], that the quantity

$$|||f|||_{L^{p,\infty}(\mathbb{R}^d)} = \sup_{0<|A|<\infty} |A|^{-1/q} \int_A |f(x)|\, \mathrm{d}x$$

defines a norm on $L^{p,\infty}(\mathbb{R}^d)$ for $1 < p < \infty$. Thus the above theorem immediately gives the following.

**Theorem 1.2.** *For any $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$, $d \geq 1$ and $j = 1, 2, \ldots, d$ we have*

$$|||R_j f|||_{L^{p,\infty}(\mathbb{R}^d)} \leq C_p ||f||_{L^p(\mathbb{R}^d)} \tag{1.5}$$

*and the constant $C_p$ is the best possible.*

Some remarks about the constants $C_p$ are in order. First, we see the same phenomenon as in (1.3): the best constants in (1.4) and (1.5) do not depend on the dimension. Next, for $1 < p < 2$, the series in the definition of $C_p$ is the famous $\beta$ function (see page 807 in Abramowitz and Stegun [1]), related to Bernoulli and Euler polynomials, and Catalan's constant. On the other hand, for $2 \leq p < \infty$ the constant $C_p$ can be rewritten using Riemann's zeta function:

$$C_p = \left[\pi^{-q}(2^{q+1} - 2)\Gamma(q+1)\zeta(q)\right]^{1/q}.$$

A few words about the proof and the organization of the paper. Theorem 1.1 will be established with the use of probabilistic methods. More precisely, we will present the proof of a sharp estimate for continuous-time martingales which can be regarded as a dual to (1.4). This is done in the next section. In Section 3 we establish (1.4) and in the final part of the paper we address the problem of the sharpness of this estimate.

## 2. A martingale inequality

2.1. **Background and the formulation of the result.** Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, equipped with $(\mathcal{F}_t)_{t\geq 0}$, a nondecreasing family of sub-$\sigma$-fields of $\mathcal{F}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X$, $Y$ be two adapted real-valued martingales with right-continuous trajectories that have limits from the left. The symbol $[X,Y]$ will stand for the quadratic covariance process of $X$ and $Y$, see e.g. Dellacherie and Meyer [8] for details. The martingales $X$, $Y$ are said to be *orthogonal* if the process $[X, Y]$ is constant with probability 1. Following Bañuelos and Wang [3] and Wang [23], we say that $Y$ is *differentially subordinate* to $X$, if the process $([X, X]_t - [Y, Y]_t)_{t\geq 0}$ is nonnegative and nondecreasing as a function of $t$. Differential subordination of $Y$ to $X$ implies many interesting inequalities which have found applications in many areas of mathematics; see e.g. [2]-[6], [16]-[18], [22] and [23].

Here is our main probabilistic result. For any $1 \leq p \leq \infty$, we use the notation $||X||_p = \sup_{t\geq 0} ||X_t||_p$ for the $p$-th norm of the martingale $X$.

**Theorem 2.1.** *Assume that $X$, $Y$ are orthogonal martingales such that $Y$ is differentially subordinate to $X$ and $Y_0 = 0$. Then for any $1 < q < \infty$ we have*

$$||Y||_q \leq C_p ||X||_1^{1/q} ||X||_\infty^{1/p} \tag{2.1}$$

*and the constant $C_q$ cannot be improved.*

To prove this statement, we will exploit the properties of a certain special superharmonic function defined on the strip $S = (-1, 1) \times \mathbb{R}$. We consider the cases $q \geq 2$ and $1 < q < 2$ separately.

2.2. **Proof of Theorem 2.1 for $q \geq 2$.** This case is a little easier. Let $H = \mathbb{R} \times (0, \infty)$ denote the upper halfplane and let $\mathcal{U} = \mathcal{U}_q : H \to \mathbb{R}$ be given by the Poisson integral

$$\mathcal{U}(\alpha, \beta) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\beta \left| \frac{2}{\pi} \log |t| \right|^q}{(\alpha - t)^2 + \beta^2} \mathrm{d}t - C_p^q.$$

Of course, the function $\mathcal{U}$ is harmonic on $H$; furthermore, it satisfies

$$\lim_{(\alpha,\beta)\to(z,0)} \mathcal{U}(\alpha, \beta) = \left(\frac{2}{\pi}\right)^q |\log |z||^q - C_p^q \qquad \text{for } z \neq 0. \tag{2.2}$$

Consider a conformal mapping $\varphi(z) = ie^{-i\pi z/2}$, or, in real coordinates,

$$\varphi(x, y) = \left(e^{\pi y/2} \sin\left(\frac{\pi}{2}x\right), e^{\pi y/2} \cos\left(\frac{\pi}{2}x\right)\right), \qquad \text{for } (x, y) \in S = (-1, 1) \times \mathbb{R}.$$

One easily verifies that $\varphi$ maps the strip $S$ onto $H$. Define $U = U_q$ on $S$ by

$$U(x, y) = \mathcal{U}(\varphi(x, y)). \tag{2.3}$$

The function $U$ is harmonic on $S$ and, by (2.2), can be extended to the continuous function on the closure $\overline{S}$ of $S$ by $U(\pm 1, y) = |y|^q$. It is easy to check that

$$U(x, y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\left| \frac{2}{\pi} \log |s| + y \right|^q \cos(\frac{\pi}{2}x)}{(s - \sin(\frac{\pi}{2}x))^2 + \cos^2(\frac{\pi}{2}x)} \mathrm{d}s - C_p^q \tag{2.4}$$

for $(x, y) \in S$. Further properties of $U$ are investigated in the lemma below.

**Lemma 2.2.** *(i) We have $U(0, 0) = 0$.*

*(ii) The function $U$ satisfies $U(x, y) = U(-x, y)$ on $\overline{S}$.*

*(iii) For any $(x, y) \in S$ we have $U_{yy}(x, y) \geq 0$.*

*(iv) We have*

$$U(x, y) \geq |y|^q - C_p^q |x| \quad \textit{for all } (x, y) \in \overline{S}. \tag{2.5}$$

*Proof.* (i) We have

$$\begin{aligned}
U(0,0)=\mathcal{U}(0,1) &= \frac{2^{q+1}}{\pi^{q+1}}\int_0^\infty \frac{|\log t|^q}{t^2+1}\mathrm{d}t - C_p^q\\
&= \frac{2^{q+1}}{\pi^{q+1}}\int_{-\infty}^\infty \frac{|s|^q e^s}{e^{2s}+1}\mathrm{d}s - C_p^q\\
&= \frac{2^{q+2}}{\pi^{q+1}}\int_0^\infty s^q e^{-s}\sum_{k=0}^\infty (-e^{-2s})^k \mathrm{d}s - C_p^q\\
&= \frac{2^{q+2}}{\pi^{q+1}}\Gamma(q+1)\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{q+1}} - C_p^q = 0.
\end{aligned}$$

(ii) This follows immediately from the substitution $s := -s$ in (2.4).

(iii) Simply apply (2.4) and the convexity of the function $y \mapsto |\log|s| + y|^q$ for any $s$.

(iv) By (iii) and the harmonicity of $U$, we have $U_{xx} \leq 0$ on $S$. Thus, by (i), it suffices to verify the majorization for $x \in \{0,1\}$. If $x = 1$, then both sides of (2.5) are equal. For $x = 0$, the bound becomes

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{\left|\frac{2}{\pi}\log|s| + y\right|^q}{s^2+1}\mathrm{d}s \geq |y|^q + C_p^q = \frac{1}{\pi}\int_{\mathbb{R}} \frac{|y|^q + \left|\frac{2}{\pi}\log|s|\right|^q}{s^2+1}\mathrm{d}s.$$

However, since $q \geq 2$, for any $y \in \mathbb{R}$ and any $s$ we have

$$\left|\frac{2}{\pi}\log|s| + y\right|^q + \left|\frac{2}{\pi}\log|s| - y\right|^q \geq 2|y|^q + 2\left|\frac{2}{\pi}\log|s|\right|^q. \tag{2.6}$$

Thus, dividing throughout by $s^2+1$ and integrating over $\mathbb{R}$ with respect to $s$ yields the desired bound, because of the equality

$$\int_{\mathbb{R}} \frac{\left|\frac{2}{\pi}\log|s| + y\right|^q}{s^2+1}\mathrm{d}s = \int_{\mathbb{R}} \frac{\left|\frac{2}{\pi}\log|s| - y\right|^q}{s^2+1}\mathrm{d}s,$$

which can be verified by substituting $s := 1/s$ in the integral on the right. $\square$

Recall the following well-known fact (see e.g. [8] for details). For any semimartingale $X$ there exists a unique continuous local martingale part $X^c$ of $X$ satisfying

$$[X,X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\Delta X_s|^2$$

for all $t \geq 0$. Here $\Delta X_s = X_s - X_{s-}$ denotes the jump of $X$ at time $s$. Furthermore, we have that $[X^c, X^c] = [X,X]^c$, the pathwise continuous part of $[X,X]$. Here is Lemma 2.1 from [4].

**Lemma 2.3.** *If $X$ and $Y$ are semimartingales, then $Y$ is differentially subordinate and orthogonal to $X$ if and only if $Y^c$ is differentially subordinate and orthogonal to $X^c$, $|Y_0| \leq |X_0|$ and $Y$ has continuous paths.*

We are ready to prove the martingale inequality.

*Proof of* (2.1). With no loss of generality, we may assume that $||X||_\infty = 1$. Let $t \in (0,\infty)$ and $(Z_s)_{s\geq 0} = ((X_s, Y_s))_{s\geq 0}$. Since $U$ is of class $C^\infty$ on $S$, we may apply Itô's formula to obtain

$$U(Z_t) = U(Z_0) + I_1 + \frac{1}{2}I_2 + \frac{1}{2}I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_{0+}^t U_x(Z_{s-})\mathrm{d}X_s + \int_{0+}^t U_y(Z_{s-})\mathrm{d}Y_s,\\
I_2 &= 2\int_{0+}^t U_{xy}(Z_{s-})\mathrm{d}[X^c,Y]_s,\\
I_3 &= \int_{0+}^t U_{xx}(Z_{s-})\mathrm{d}[X,X]_s^c + \int_{0+}^t U_{yy}(Z_{s-})\mathrm{d}[Y,Y]_s,\\
I_4 &= \sum_{0<s\leq t}\big\{U(Z_s) - U(Z_{s-}) - U_x(Z_{s-})\Delta X_s\big\}.
\end{aligned}$$

Note that we have used above the equalities $Y_{s-} = Y_s$ and $Y = Y^c$, which are due to the continuity of paths of $Y$. By Lemma 2.2 (ii), (iv) and the harmonicity of $U$, we have that the function $U(\cdot,0)$ is concave and even on $[-1,1]$, and hence $U(Z_0) \leq U(0,0) = 0$. The term $I_1$ has zero expectation, since the stochastic integrals are martingales. We have $I_2 = 0$ because of the orthogonality of $X^c$ and $Y$. The differential subordination together with Lemma 2.2 (iii) give

$$I_3 \leq \int_0^t U_{xx}(Z_{s-})d[X,X]_s^c + \int_0^t U_{yy}(Z_{s-})d[X,X]_s^c = 0.$$

Finally, each summand in $I_4$ is nonpositive, by the concavity of $U(\cdot,y)$ for any fixed $y\in\mathbb{R}$ (see Lemma 2.2 (iii)). Therefore, by Lemma 2.2 (ii),

$$\mathbb{E}|Y_t|^q - C_p^q\mathbb{E}|X_t| \leq \mathbb{E}U(X_t,Y_t) \leq 0 \tag{2.7}$$

and (2.1) is established. The sharpness of this estimate will follow from the optimality of $C_p$ in (1.4): see the remark at the end of Section 4. □

2.3. **Proof of Theorem 2.1 for** $1 < q < 2$**.** Though the arguments may seem similar, there are several important differences. The formula for the special function used in the previous subsection does not work here, because the majorization (2.5) fails to hold: indeed, the bound (2.6) is no longer valid. Thus we need a different function; in fact, as we will see, the object we will construct will not be harmonic on $S$, but only superharmonic.

To introduce the proper function for $1 < q < 2$, we start, as previously, with defining the auxiliary function $\mathcal{U} = \mathcal{U}_q : H \to \mathbb{R}$. It is given essentially by the same formula as before:

$$\mathcal{U}(\alpha,\beta) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\beta\left|\frac{2}{\pi}\log|t|\right|^q}{(\alpha-t)^2+\beta^2}\mathrm{d}t$$

(the only difference is that the summand $-C_p^q$ is thrown out). Obviously, $\mathcal{U}$ is harmonic on $H$ and satisfies

$$\lim_{(\alpha,\beta)\to(z,0)}\mathcal{U}(\alpha,\beta) = \left(\frac{2}{\pi}\right)^q|\log|z||^q \qquad \text{for } z\neq 0. \tag{2.8}$$

The crucial fact is that we will not copy $\mathcal{U}$ onto $S$ as previously, but we will use two separate copies onto two half-strips $(-1,0)\times\mathbb{R}$ and $(0,1)\times\mathbb{R}$. Formally, consider a conformal mapping $\varphi(z) = -e^{-i\pi z}$, which maps $(0,1)\times\mathbb{R}$ onto $H$. Define $U = U_p$ on $(0,1)\times\mathbb{R}$ by

$$U(x,y) = \mathcal{U}(\varphi(x,y)) - C_p^q x. \tag{2.9}$$

The function $U$ is harmonic on $(0,1)\times\mathbb{R}$ and, by (2.8), can be extended to the continuous function on $[0,1]\times\mathbb{R}$ by $U(0,y)=|y|^q$, $U(1,y)=|y|^q-C_p^q$. Finally, extend $U$ to the whole $\overline{S}$ by $U(x,y)=U(-x,y)$. It is easy to check that

$$U(x,y)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\left|\frac{2}{\pi}\log|s|+2y\right|^q\sin(\pi x)}{(s+\cos(\pi x))^2+\sin^2(\pi x)}\mathrm{d}s-C_p^q|x| \tag{2.10}$$

for $(x,y)\in S$. Further properties of $U$ are investigated in the lemma below.

**Lemma 2.4.** *(i) The right-hand partial derivative $U_x(0+,0)$ is equal to $0$.*
*(ii) For any $(x,y)\in S$ we have $U_{yy}(x,y)\geq 0$.*
*(iii) We have*

$$U(x,y)\geq |y|^q-C_p^q|x| \quad \textit{for all } (x,y)\in\overline{S}. \tag{2.11}$$

*(iv) For any $(x,y)\in S$ we have $U_{yyy}(x,y)\leq 0$.*
*(v) The function $U$ is superharmonic on $S$.*

*Proof.* (i) Directly from (2.10), we compute that

$$\begin{aligned}
U_x(0+,0)&=\int_{\mathbb{R}}\frac{\left|\frac{2}{\pi}\log|s|\right|^q}{(s+1)^2}\mathrm{d}s-C_p^q\\
&=\int_0^\infty\frac{\left|\frac{2}{\pi}\log s\right|^q}{(s+1)^2}\mathrm{d}s+\int_0^\infty\frac{\left|\frac{2}{\pi}\log s\right|^q}{(s-1)^2}\mathrm{d}s-C_p^q\\
&=\frac{2^{q+1}}{\pi^q}\left[\int_1^\infty\frac{(\log s)^q}{(s+1)^2}\mathrm{d}s+\int_0^\infty\frac{(\log s)^q}{(s-1)^2}\mathrm{d}s\right]-C_p^q\\
&=\frac{2^{q+1}q}{\pi^q}\left[\int_1^\infty\frac{(\log s)^{q-1}}{s(s+1)}\mathrm{d}s+\int_0^\infty\frac{(\log s)^{q-1}}{s(s-1)}\mathrm{d}s\right]-C_p^q\\
&=\frac{2^{q+2}q}{\pi^q}\int_1^\infty\frac{(\log s)^{q-1}}{s^2-1}\mathrm{d}s-C_p^q\\
&=\frac{2^{q+2}q}{\pi^q}\int_0^\infty\frac{t^{q-1}e^{-t}}{1-e^{-2t}}\mathrm{d}t-C_p^q\\
&=\frac{2^{q+2}q}{\pi^q}\sum_{k=0}^\infty\int_0^\infty t^{q-1}e^{-(2k+1)t}\mathrm{d}t-C_p^q\\
&=\frac{2^{q+2}}{\pi^q}\Gamma(q+1)\sum_{k=0}^\infty(2k+1)^{-q}-C_p^q=0.
\end{aligned}$$

(ii) This is shown word-by-word as in the case $q\geq 2$.

(iii) By (ii) and the harmonicity of $U$ on the half-strip $(0,1)\times\mathbb{R}$, we get that $U_{xx}\leq 0$ on this set and hence it suffices to show the majorization for $x\in\{0,1\}$. However, when $x=0$ or $x=1$, then both sides of (2.11) are equal.

(iv) By Fubini's theorem, we have

$$U_y(x,y)=\frac{2q}{\pi}\int_{-\infty}^\infty\frac{\sin(\pi x)\left|\frac{2}{\pi}\log|s|+2y\right|^{q-2}\left(\frac{2}{\pi}\log|s|+2y\right)}{(s+\cos(\pi x))^2+\sin^2(\pi x)}\mathrm{d}s.$$

Therefore, for $\varepsilon\in(0,y)$ we have

$$2U_y(x,y)-U_y(x,y-\varepsilon)-U_y(x,y+\varepsilon)=\frac{2^{q+1}q}{\pi}\int_{-\infty}^\infty\frac{f_{y,\varepsilon}\left(\frac{1}{\pi}\log|s|\right)\sin(\pi x)}{(s+\cos(\pi x))^2+\sin^2(\pi x)}\mathrm{d}s,$$

where

$$f_{y,\varepsilon}(h) = 2|y+h|^{q-2}(y+h) - |y-\varepsilon+h|^{q-2}(y-\varepsilon+h) - |y+\varepsilon+h|^{q-2}(y+\varepsilon+h).$$

Denote the latter integral by $I$. After splitting it into integrals over the nonpositive and nonnegative halfline, and substitution $s = \pm e^r$, we get

$$I = \int_{-\infty}^{\infty} f_{y,\varepsilon}\left(r/\pi\right) g^x(r)\mathrm{d}r,$$

where

$$g^x(r) = \frac{\sin(\pi x)e^r}{(e^r+\cos(\pi x))^2+\sin^2(\pi x)} + \frac{\sin(\pi x)e^r}{(e^r-\cos(\pi x))^2+\sin^2(\pi x)}.$$

Observe that $f_{y,\varepsilon}(h) \leq 0$ for $h \geq -y$ and that we have $f_{y,\varepsilon}(-y+h) = -f_{y,\varepsilon}(-y-h)$ for all $h$. Furthermore, $g^x$ is even and, for $r > 0$,

$$(g^x)'(r) = \frac{\sin(\pi x)e^r(1-e^r)}{[(e^r+\cos(\pi x))^2+\sin^2(\pi x)]^2} + \frac{\sin(\pi x)e^r(1-e^r)}{[(e^r+\cos(\pi x))^2+\sin^2(\pi x)]^2} \leq 0.$$

This implies $I \leq 0$ and, since $\varepsilon \in (0,x)$ was arbitrary, the function $U(x,\cdot) : y \mapsto U_y(x,y)$ is convex on $(0,\infty)$.

(v) We have that $U$ is symmetric with respect to $y$-axis, continuous on $\overline{S}$ and harmonic on the half-strips $(0,1)\times\mathbb{R}$, $(-1,0)\times\mathbb{R}$. Thus we will be done if we show that the one-sided derivative $U_x(0+,y)$ is nonpositive for $y \neq 0$. Since $U$ satisfies $U(x,y) = U(x,-y)$ on $S$ (substitute $s := 1/s$ in (2.10)), we may restrict ourselves to $y \geq 0$. Observe that the integral in (2.10) remains unchanged if we make the substitution $x := 1-x$. Consequently, its partial derivative with respect to $x$ vanishes at $x = 1/2$, and hence the further differentiation with respect to $y$ yields $U_{xy}(1/2,y) = 0$ for all $y$. Next, using (iv) and the harmonicity of $U_y$, we get $U_{xxy}(x,y) \geq 0$ for $y \geq 0$ and thus for any $(x,y) \in (0,1/2)\times(0,\infty)$ we have $U_{xy}(x,y) \leq 0$. It remains to use (i) to complete the proof. □

*Proof of* (2.1). As previously, we may assume that $||X||_\infty = 1$. In comparison with the case $q \geq 2$, we need to overcome the problem that $U$ is not of class $C^2$ (which makes Itô's formula unavailable). This can be done by an appropriate mollification argument. Let $g : \mathbb{R}^2 \to [0,\infty)$ be a $C^\infty$ radial function, supported on the unit ball and satisfying $\int_{\mathbb{R}^2} g = 1$. For any $\delta > 0$ define $U^\delta : \overline{S} \to \mathbb{R}$ by

$$U^\delta(x,y) = \int_{[-1,1]^2} U((1-\delta)x+\delta u, (1-\delta)y+\delta v)g(u,v)\mathrm{d}u\mathrm{d}v.$$

This function is superharmonic and inherits the concavity with respect to the variable $y$. Furthermore, we have $U^\delta(0,0) \leq U(0,0)$, because $U$ is superharmonic and $g$ is radial. Finally, by (2.11), we easily check that

$$U^\delta(x,y) \geq \big|(1-\delta)|y|-\delta\big|^q - C_p^q(1-\delta)|x| - C_p^q\delta.$$

Consequently, if we repeat the reasoning from the case $q \geq 2$, we obtain that for any $t \geq 0$,

$$\mathbb{E}\big|(1-\delta)|Y_t|-\delta\big|^q \leq C_p^q(1-\delta)\mathbb{E}|X_t| + C_p^q\delta \leq C_p^q||X||_1 + C_p^q\delta.$$

It suffices to let $\delta \to 0$, apply Fatou's lemma and take the supremum over $t$ to get the claim. □

## 3. Inequalities for Riesz transforms

There is a well-known representation of Riesz transforms in terms of the so-called background radiation process, introduced by Gundy and Varopoulos in [12]. Let us briefly describe this connection. Throughout this section, $d$ is a fixed positive integer. Suppose that $Z = (X, Y)$ is a Brownian motion in $\mathbb{R}^d \times \mathbb{R}$, starting from the origin. For any $y > 0$, introduce the stopping time $\tau(y) = \inf\{t \geq 0 : Y_t = -y\}$. If $f$ belongs to $\mathcal{S}(\mathbb{R}^d)$, the class of rapidly decreasing functions on $\mathbb{R}^d$, let $W_f : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ stand for the Poisson extension of $f$ to the upper half-space. That is,

$$W_f(x, y) := \mathbb{E} f\left(x + X_{\tau(y)}\right).$$

For any $(d + 1) \times (d + 1)$ matrix $A$ we define the martingale transform $A{*}f$ by

$$A{*}f(x, y) = \int_{0+}^{\tau(y)} A\nabla W_f((x, y) + Z_s) \cdot \mathrm{d}Z_s.$$

Note that $A * f(x, y)$ is a random variable for each $x$, $y$. Now, for any $f \in C_0^\infty$, any $y > 0$ and any matrix $A$ as above, define $\mathcal{T}_A^y f : \mathbb{R}^d \to \mathbb{R}$ through the bilinear form

$$\int_{\mathbb{R}^d} \mathcal{T}_A^y f(x) g(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathbb{E}\big[A{*}f(x, y) g(x + X_{\tau(y)})\big]\mathrm{d}x, \tag{3.1}$$

where $g$ runs over $C_0^\infty(\mathbb{R}^d)$. Less formally, $\mathcal{T}^y f$ is given as the following conditional expectation with respect to the measure $\tilde{\mathbb{P}} = \mathbb{P} \otimes \mathrm{d}x$ ($\mathrm{d}x$ denotes Lebesgue's measure on $\mathbb{R}^d$): for any $w \in \mathbb{R}^d$,

$$\mathcal{T}_A^y f(w) = \tilde{E}\left[A{*}f(x, y) | x + X_{\tau(y)} = w\right].$$

See Gundy and Varopoulos [12] for the rigorous statement of this equality. The interplay between the operators $\mathcal{T}_A^y$ and Riesz transforms is explained in the following theorem, consult [12] or Gundy and Silverstein [11].

**Theorem 3.1.** *Let $A^j = [a_{\ell m}^j]$, $j = 1, 2, \ldots, d$ be the $(d + 1) \times (d + 1)$ matrices given by*

$$a_{\ell m}^j = \begin{cases} 1 & \text{if } \ell = d + 1,\ m = j, \\ -1 & \text{if } \ell = j,\ m = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

*Then $\mathcal{T}_{A^j}^y f \to R_j f$ almost everywhere as $y \to \infty$.*

We turn to the dual version of Theorem 1.1.

**Theorem 3.2.** *For any $1 < q < \infty$, $f \in L^q(\mathbb{R}^d)$, $d \geq 1$ and $1 \leq j \leq d$ we have the sharp estimate*

$$||R_j f||_{L^q(\mathbb{R}^d)} \leq C_p ||f||_{L^1(\mathbb{R}^d)}^{1/q} ||f||_{L^\infty(\mathbb{R}^d)}^{1/p}. \tag{3.2}$$

*Proof.* Fix $j \in \{1, 2, \ldots, d\}$, $x \in \mathbb{R}$ and $y > 0$. By a standard density argument, it suffices to establish the estimate (3.2) for $f \in C_0^\infty(\mathbb{R}^d)$ satisfying $||f||_{L^\infty(\mathbb{R}^d)} \leq 1$. Consider the pair $\xi = (\xi_t)_{t\geq 0}$, $\eta = (\eta_t)_{t\geq 0}$ of martingales given by

$$\xi_t = W_f((x, y) + Z_{\tau(y)\wedge t}) = W_f(x, y) + \int_{0+}^{\tau(y)\wedge t} \nabla W_f((x, y) + Z_s) \cdot \mathrm{d}Z_s$$

and

$$\eta_t = \int_{0+}^{\tau(y)\wedge t} A^j \nabla W_f((x, y) + Z_s) \cdot \mathrm{d}Z_s,$$

for $t \geq 0$. Observe that $||\xi||_\infty \leq 1$ because $f$, and hence also $W_f$, are bounded in absolute value by 1. Then the martingale $\eta$ is differentially subordinate to $\xi$, since $[\xi,\xi]_t - [\eta,\eta]_t$ equals

$$|W_f(x,y)|^2 + \sum_{k\notin\{j,d+1\}} \int_{0+}^{\tau(y)\wedge t} \left|\frac{\partial W_f}{\partial x_k}((x,y)+Z_s)\right|^2 \mathrm{d}s$$

and is nonnegative and nondecreasing as a function of $t$. Furthermore, $\xi$ and $\eta$ are orthogonal, which is a direct consequence of the equality $\langle A^j x, x\rangle = 0$, valid for all $x \in \mathbb{R}^d$. Indeed, $[\xi,\eta]_t$ equals

$$\int_{0+}^{\tau(y)\wedge t} \langle A^j \nabla W_f((x,y)+Z_s), \nabla W_f((x,y)+Z_s)\rangle \mathrm{d}s = 0.$$

Thus, by (2.1), we have

$$\mathbb{E}|\eta_{\tau(y)}|^q = \sup_{t\geq 0} \mathbb{E}|\eta_t|^q \leq C_p^q ||\xi||_1.$$

Integrating this estimate with respect to $x \in \mathbb{R}^d$ and using Fubini's theorem, we get

$$\int_{\mathbb{R}^d} \mathbb{E}|A{*}f(x,y)|^q \mathrm{d}x \leq C_p^q ||f||_{L^1(\mathbb{R}^d)}.$$

Combining this with (3.1), we get

$$\begin{aligned}\int_{\mathbb{R}^d} T^y_{A^j} f(x) g(x)\mathrm{d}x &\leq \left(\mathbb{E}\int_{\mathbb{R}^d} |A{*}f(x,y)|^q \mathrm{d}x\right)^{1/q} \left(\mathbb{E}\int_{\mathbb{R}^d} |g(x+X_{\tau(y)})|^p \mathrm{d}x\right)^{1/p}\\ &\leq C_p ||f||^{1/q}_{L^1(\mathbb{R}^d)} ||g||_{L^p(\mathbb{R}^d)}.\end{aligned}$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, the above estimate yields

$$||T^y_{A^j} f||_{L^q(\mathbb{R}^d)} \leq C_p ||f||^{1/p}_{L^1(\mathbb{R}^d)}.$$

It suffices to let $y \to \infty$ and apply the assertion of Theorem 3.1 and Fatou's lemma. The sharpness of the estimate will follow from the optimality of the constant $C_p$ in (1.4). See the end of Section 4. □

*Proof of* (1.4). Fix $f \in L^p(\mathbb{R}^d)$ and put $g = 1_A R_j f/|R_j f|$ ($g = 0$ if the denominator is zero). By Parseval's identity and (1.1), we get

$$\begin{aligned}\int_A |R_j f(x)|\mathrm{d}x &= \int_{\mathbb{R}^d} R_j f(x) g(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^d} \widehat{R_j f}(x)\,\widehat{g}(-x)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^d} \widehat{f}(x)\,\widehat{R_j g}(-x)\,\mathrm{d}x\\ &= -\int_{\mathbb{R}^d} f(x)\,R_j g(x)\,\mathrm{d}x\\ &\leq ||f||_{L^p(\mathbb{R}^d)} ||R_j g||_{L^q(\mathbb{R}^d)}\\ &\leq C_p ||f||_{L^p(\mathbb{R}^d)} ||g||^{1/q}_{L^1(\mathbb{R}^d)}.\end{aligned} \tag{3.3}$$

Here in the latter passage we have used (3.2) and the fact that $g$ takes values in $[-1,1]$. It suffices to note that $||g||_{L^1(\mathbb{R}^d)} \leq |A|$ to complete the proof. □

## 4. Sharpness

4.1. **Two conformal mappings.** Let $D$ denote the open unit disc of $\mathbb{C}$ and let $K : D\cap H \to H$ be defined by $K(z) = -(1-z)^2/(4z)$ (here, as previously, $H$ stands for the upper half-plane). It is not difficult to verify that $K$ is conformal and hence so is its inverse $L$. Let us extend $L$ to the continuous function on $\overline{H}$. Consider another conformal map $F : \overline{D} \to \overline{S}$ (recall that $S$ is the strip $\{z \in \mathbb{C} : |\mathrm{Re} z| < 1\}$), given by

$$F(z) = \frac{2i}{\pi} \log \left[\frac{iz-1}{z-i}\right] - 1.$$

The following properties of $L$ and $F$ will be needed below. First, observe that $L$ maps the interval $[0,1]$ onto $\{e^{i\theta} : 0 \leq \theta \leq \pi\}$. More precisely, we have the following formula: if $x \in [0,1]$, then

$$L(x) = e^{i\theta}, \text{ where } \theta \in [0,\pi] \text{ is uniquely determined by } x = \sin^2(\theta/2). \tag{4.1}$$

In addition, $L$ maps the set $\mathbb{R} \setminus [0,1]$ onto the open interval $(-1,1)$; precisely, we have the identity

$$L(x) = \begin{cases} 1-2x-2\sqrt{x^2-x} & \text{if } x<0, \\ 1-2x+2\sqrt{x^2-x} & \text{if } x>1. \end{cases} \tag{4.2}$$

In particular, we easily check that for any $\delta > 0$, the function $L$ is bounded away from 1 outside any interval of the form $[-\delta, 1+\delta]$ and $|L(x)| = O(|x|^{-1})$ as $x \to \pm\infty$.

The function $F$ has the following properties:

$$F \text{ maps the unit circle onto the boundary of } S \tag{4.3}$$

and

$$F \text{ maps } [-1,1] \text{ onto itself.} \tag{4.4}$$

4.2. **Sharpness of** (1.4) **for** $d = 1$ **and** $1 < p \leq 2$**.** For any positive integer $n$, consider the conformal map $V_n : \overline{H} \to \overline{S}$ given by $V_n(z) = F(L^{2n}(z))$, and define $\varphi_n : \mathbb{R} \to \mathbb{R}$ by the formula $\varphi_n(x) = \mathrm{Re}\, V_n(x)$. Since $\lim_{z\to\infty} V_n(z) = 0$, we see that $\mathcal{H}\varphi_n(x) = \mathrm{Im}\, V_n(x)$ for $x \in \mathbb{R}$. Combining the above facts about $L$ and $F$, we observe that $\operatorname{Im} V_n(x) = 0$ for $x \in \mathbb{R} \setminus [0,1]$ and hence, using (4.1), we have

$$\begin{aligned}
\int_{\mathbb{R}} |\mathcal{H}\varphi_n(x)|^q \,\mathrm{d}x &= \int_0^1 \left|\operatorname{Im} F(L^{2n}(x))\right|^q \mathrm{d}x \\
&= \frac{1}{2}\int_0^{\pi} \left|\operatorname{Im} F(e^{2in\theta})\right|^q \sin\theta \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{2n\pi} \left|\operatorname{Im} F(e^{i\theta})\right|^q \sin\left(\frac{\theta}{2n}\right)\frac{\mathrm{d}\theta}{2n} \\
&= \frac{1}{2}\int_0^{2\pi} \left|\operatorname{Im} F(e^{i\theta})\right|^q \sum_{k=0}^{n-1}\sin\left(\frac{k\pi}{n}+\frac{\theta}{2n}\right)\frac{\mathrm{d}\theta}{2n} \\
&= \frac{1}{2}\int_0^{2\pi} \left|\operatorname{Im} F(e^{i\theta})\right|^q \frac{\cos\left(\frac{\theta-\pi}{n}\right)}{2n\sin\left(\frac{\pi}{2n}\right)}\,\mathrm{d}\theta \qquad (4.5)\\
&\xrightarrow{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|\operatorname{Im} F(e^{i\theta})\right|^q \mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{2}{\pi}\log\left(\frac{\sin\theta}{1-\cos\theta}\right)\right|^q \mathrm{d}\theta \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \frac{\left|\frac{2}{\pi}\log t\right|^q}{t^2+1}\mathrm{d}t \\
&= C_p^q,
\end{aligned}$$

where in the last line we have applied Lemma 2.2 (i). Put $f_n = -|\mathcal{H}\varphi_n|^{q-2}\mathcal{H}\varphi_n$ and fix $\varepsilon > 0$. Using Parseval's identity and the above chain of inequalities, we derive that

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x &= -\int_{\mathbb{R}} f_n(x)\mathcal{H}\varphi_n(x)\,\mathrm{d}x \\
&= ||\mathcal{H}\varphi_n||_{L^q(\mathbb{R})}||f_n||_{L^p(\mathbb{R})} \qquad (4.6)\\
&\geq (C_p-\varepsilon)||f_n||_{L^p(\mathbb{R})}
\end{aligned}$$

provided $n$ is sufficiently large. Next, we have $|\varphi_n| \leq 1$ (since $|\operatorname{Re} F| \leq 1$), so

$$\begin{aligned}
\int_0^1 |\mathcal{H}f_n(x)|\mathrm{d}x &\geq \int_0^1 \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x \\
&= \int_{\mathbb{R}} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x - \int_{\mathbb{R}\setminus[0,1]} \mathcal{H}f_n(x)\varphi_n(x)\mathrm{d}x.
\end{aligned}$$

Now we shall prove that the last integral is smaller than $\varepsilon||f_n||_{L^p(\mathbb{R})}$ for sufficiently large $n$. To do this, it suffices to use Hölder's inequality and combine it with two facts: first, by the $L^p$-boundedness of the Hilbert transform, we have $||\mathcal{H}f_n||_{L^p(\mathbb{R})} \leq E_p||f_n||_{L^p(\mathbb{R})}$, and second,

$$\lim_{n\to\infty}\int_{\mathbb{R}\setminus[0,1]} |\varphi_n(x)|^q\mathrm{d}x = 0,$$

since $\varphi_n$ decreases to 0 sufficiently fast outside $[0,1]$ (see the remarks following (4.2)). Putting all the above things together, if we take $A = [0,1]$ and pick $n$ large enough, we obtain

$$\begin{aligned}\int_A |\mathcal{H}f_n(x)|\,\mathrm{d}x &> (C_p-\varepsilon)||f_n||_{L^p(\mathbb{R})} - \varepsilon||f_n||_{L^p(\mathbb{R})}\\ &= (C_p-2\varepsilon)||f_n||_{L^p(\mathbb{R})}|A|^{1/p}.\end{aligned}$$

Since $\varepsilon$ was arbitrary, the constant $C_p$ is indeed the best possible in (1.4).

4.3. **Sharpness of** (1.4) **for** $d=1$ **and** $2<p<\infty$**.** Here the argumentation is slightly more complicated. Pick $a\in(-1,0)$ close to $-1$; then $F(a)$ is also close to $-1$. The mapping $z\mapsto \frac{z-a}{az-1}$ is conformal from $D$ onto $D$ and hence the function $G:D\to[-(1+F(a))/2,(1-F(a))/2]\times\mathbb{R}$, given by

$$G(z)=\frac{1}{2}F\left(\frac{z-a}{az-1}\right)-\frac{1}{2}F(a),$$

is also conformal. Next, let $W_n(z)=G(L^{2n}(z))$ for $z\in\overline{H}$ and put $\psi_n(x)=\operatorname{Re}W_n(x)$, $x\in\mathbb{R}$. Then $\mathcal{H}\psi_n(x)=\operatorname{Im}W_n(x)$ and repeating the reasoning from (4.5), we see that

$$\lim_{n\to\infty}\int_{\mathbb{R}}|\mathcal{H}\psi_n(r)|^q\,\mathrm{d}r=\frac{1}{2\pi}\int_0^{2\pi}|\operatorname{Im}G(e^{i\theta})|^q\mathrm{d}\theta.$$

To compute the latter integral, note that

$$\operatorname{Im}G(z)=\operatorname{Im}\left[\frac{1}{2}F\left(\frac{e^{i\theta}-a}{ae^{i\theta}-1}\right)+\frac{1}{2}\right],$$

the function $z\mapsto\frac{1}{2}F\left(\frac{z-a}{az-1}\right)+\frac{1}{2}$ is conformal and maps $D$ onto $(0,1)\times\mathbb{R}$ and 0 to $\frac{1}{2}F(a)+\frac{1}{2}$. On the other hand, the function $(x,y)\mapsto U(x,y)+C_p^qx$ is harmonic on $(0,1)\times\mathbb{R}$ and equals $|y|^p$ for $x\in\{0,1\}$. This implies

$$\lim_{n\to\infty}\int_{\mathbb{R}}|\mathcal{H}\psi_n(r)|^q\,\mathrm{d}r=U\left(\frac{1}{2}+\frac{1}{2}F(a),0\right)+C_p^q\left(\frac{1}{2}+\frac{1}{2}F(a)\right).$$

Next, let $A_n=\left\{x\in\mathbb{R}:\psi_n(x)=\frac{1}{2}-\frac{F(a)}{2}\right\}$. It follows from (4.2) that if $x\notin[0,1]$, then $L^{2n}(x)<1$ and hence the equality in the definition of $A_n$ cannot hold. Therefore, the arguments presented in (4.5) give

$$\lim_{n\to\infty}|A_n|=\frac{1}{2\pi}\left|\left\{\theta\in[0,2\pi]:\operatorname{Re}G(e^{i\theta})=\frac{1}{2}-\frac{F(a)}{2}\right\}\right|.$$

However, $\operatorname{Re}G(e^{i\theta})$ takes values in the set $\{(-1-F(a))/2,(1-F(a))/2\}$ and by the mean-value property, $(2\pi)^{-1}\int_0^{2\pi}\operatorname{Re}G(e^{i\theta})\mathrm{d}\theta=G(0)$. This yields

$$\lim_{n\to\infty}|A_n|=\frac{1}{2}+\frac{F(a)}{2}.$$

Finally, put $f_n=-|\mathcal{H}\psi_n|^{q-2}\mathcal{H}\psi_n$ and pick $\varepsilon>0$. Since $\psi_n$ takes values in the interval $[-(1+F(a))/2,(1-F(a))/2]\subset[-1,1]$, we may write

$$\begin{aligned}\int_{A_n}|\mathcal{H}f_n(x)|\mathrm{d}x&\geq\int_{A_n}\mathcal{H}f_n(x)\psi_n(x)\mathrm{d}x\\&=\int_{\mathbb{R}}\mathcal{H}f_n(x)\psi_n(x)\mathrm{d}x-\int_{\mathbb{R}\setminus A_n}\mathcal{H}f_n(x)\psi_n(x)\mathrm{d}x.\end{aligned}$$

Arguing as in the case $1 < p \leq 2$, we show that

$$\int_{\mathbb{R}\setminus A_n} \mathcal{H}f_n(x)\psi_n(x)\mathrm{d}x \leq \varepsilon\left(\frac{1+F(a)}{2}\right)^{1/q} ||f_n||_{L^p(\mathbb{R})}$$

if $n$ is large enough. Furthermore, by the above considerations,

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}f_n(x)\psi_n(x)\,\mathrm{d}x &= \int_{\mathbb{R}} |\mathcal{H}\psi_n(x)|^q\,\mathrm{d}x \\ &\geq \left[U\left(\frac{1+F(a)}{2},0\right) + (C_p^q-\varepsilon)\left(\frac{1+F(a)}{2}\right)\right]^{1/q} ||f||_{L^p(\mathbb{R})}.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_{A_n} |\mathcal{H}f_n(x)|\mathrm{d}x \\ &\geq \left(\frac{1+F(a)}{2}\right)^{1/q}\left[\frac{2}{1+F(a)}U\left(\frac{1+F(a)}{2},0\right)+C_p^q-2\varepsilon\right]^{1/q}||f||_{L^p(\mathbb{R})} \\ &\geq (1-\varepsilon)\left[\frac{2}{1+F(a)}U\left(\frac{1+F(a)}{2},0\right)+C_p^q-2\varepsilon\right]^{1/q}||f||_{L^p(\mathbb{R})}|A_n|^{1/q}\end{aligned}$$

for sufficiently large $n$. Letting $\varepsilon \to 0$ and then $a \to -1$, we see, by Lemma 2.4 (i), that the constant $C_p$ is the best possible.

4.4. **The case $d > 1$.** Of course, it suffices to focus on Riesz transform $R_1$ only. Suppose that for some $C > 0$ we have

$$\int_A |R_1 f(x)|\mathrm{d}x \leq C||f||_{L^p(\mathbb{R}^d)}|A|^{1/w} \tag{4.7}$$

for all Borel subsets $A$ of $\mathbb{R}^d$ and all Borel functions $f : \mathbb{R}^d \to \mathbb{R}$. For $t > 0$, define the dilation operator $\delta_t$ as follows: for any function $g : \mathbb{R}\times\mathbb{R}^{d-1} \to \mathbb{R}$, we let $\delta_t g(\xi,\eta) = g(\xi, t\eta)$; for any $A \subset \mathbb{R}\times\mathbb{R}^{d-1}$, let $\delta_t A = \{(\xi, t\eta) : (\xi,\eta) \in A\}$. By (4.7), the operator $T_t := \delta_t^{-1}\circ R_1 \circ \delta_t$ satisfies

$$\begin{aligned}\int_A |T_t f(x)|\mathrm{d}x &= t^{d-1}\int_{\delta_t^{-1}A} |R_1\circ\delta_t f(x)|\mathrm{d}x \\ &\leq Ct^{d-1}||\delta_t f||_{L^p(\mathbb{R}^d)}|\delta_t^{-1}A|^{1/q} \\ &= C||f||_{L^p(\mathbb{R}^d)}|A|^{1/q}.\end{aligned} \tag{4.8}$$

Now fix $f \in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$. It is not difficult to check that the Fourier transform $\mathcal{F}$ satisfies the identity $\mathcal{F} = t^{d-1}\delta_t\circ\mathcal{F}\circ\delta_t$ and hence the operator $T_t$ satisfies the identity

$$\widehat{T_t f}(\xi,\eta) = i\frac{\xi}{(\xi^2+t^2|\eta|^2)^{1/2}}\widehat{f}(\xi,\eta)$$

for $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}^{d-1}$. By Lebesgue's dominated convergence theorem, we have

$$\lim_{t\to 0}\widehat{T_t f}(\xi,\eta) = \widehat{T_0 f}(\xi,\eta)$$

in $L^2(\mathbb{R}^d)$, where $\widehat{T_0 f}(\xi,\eta) = i\,\mathrm{sgn}\,(\xi)\widehat{f}$. Combining this with Plancherel's theorem, we conclude that there is a sequence $(t_n)_{n\geq 1}$ decreasing to 0 such that $T_{t_n} f$

converges to $T_0 f$ almost everywhere. Using Fatou's lemma and (4.8), we obtain

$$\int_A |T_0 f(x)| \mathrm{d}x \leq C||f||_{L^p(\mathbb{R}^d)} |A|^{1/q}. \tag{4.9}$$

Since $L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$, we easily verify that the above estimate holds true for all $f \in L^p(\mathbb{R}^d)$. Next, fix $\varepsilon > 0$. By the reasoning from the one-dimensional case, there is a Borel subset $B$ of $\mathbb{R}$ and $h \in L^p(\mathbb{R})$ such that

$$\int_B |\mathcal{H}h(x)| \mathrm{d}x > (C_p - \varepsilon)||h||_{L^p(\mathbb{R})} |B|^{1/p}. \tag{4.10}$$

Define $f : \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}$ by $f(\xi, \eta) = h(\xi) 1_{[0,1]^{d-1}}(\eta)$. We have $f \in L^p(\mathbb{R}^d)$ and $T_0 f(\xi, \eta) = \mathcal{H}h(\xi) 1_{[0,1]^{d-1}}(\eta)$, which is due to the identity

$$\widehat{T_0 f}(\xi, \eta) = i \mathrm{sgn}\,(\xi)\, \widehat{h}(\xi) \widehat{1_{[0,1]^{d-1}}}(\eta).$$

Plug this into (4.9) with the choice $A = B \times [0,1]^{d-1}$ to obtain

$$\int_B |\mathcal{H}h(\xi)| \mathrm{d}\xi \leq C||h||_{L^p(\mathbb{R}^d)} |B|^{1/q}.$$

This implies $C > C_p$ by virtue of (4.10) and the fact that $\varepsilon > 0$ was arbitrary. The proof is complete.

**Remark 4.1.** The optimality of the constant $C_p$ in (1.4) immediately implies the sharpness of (2.1) and (3.2). Indeed, if any of these estimates could be sharpened, this would yield an improvement of $C_p$ in (1.4): see the last passage in (3.3).

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